Hamiltonian Cycles in Kneser Graphs∗

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Abstract. The Kneser graph \(K(n, k)\) is the graph whose vertices are all the subsets with \(k\) elements of a set that has \(n\) elements, and two vertices are joined by an edge if the corresponding pair of \(k\)-sets is disjoint. The odd graph \(O_k\) is the special case of the Kneser graph when \(n = 2k + 1\). A long-standing conjecture due to Lovász claims that \(O_k\) has a hamiltonian path for \(k \geq 1\). Previously, Lovász’s conjecture had been proved for all \(k \leq 13\). We have improved these values by showing that \(O_k\) has a hamiltonian path for \(14 \leq k \leq 17\). Additionally, we have established how close the odd graphs are to being hamiltonian: \(O_k\) has a closed spanning walk or trail in which every vertex appears at most twice.

1. Introduction

A spanning cycle in a graph is a hamiltonian cycle and a graph that contains such cycle is said to be hamiltonian. A hamiltonian path is a path that contains every vertex of the graph precisely once. Since its formulation by Hamilton in 1859, the hamiltonian cycle problem has been used in several practical applications such as the traveling salesman problem, or TSP for short: given a collection of cities and the cost of travelling between each pair of them, the TSP is to find the cheapest way of visiting all of the cities and returning to your starting point. Note that TSP is a variation of the hamiltonian cycle since each city is represented by a vertex in a graph.

Determining if a graph \(G\) has a hamiltonian cycle is an NP-Complete problem [Karp 1972], even if restricted to bipartite graphs [Krishnamoorthy 1975], planar 3-connected cubic graphs [Garey et al. 1976], or if a hamiltonian path is given as part of the instance [Papadimitriou and Steiglitz 1976]. The hamiltonian path problem is NP-Complete as well [Garey and Johnson 1979].

In this paper, we study hamiltonian cycles and paths in a graph class called Kneser graphs. Let \(n, k\) be integers such that \(n \geq k \geq 1\). The Kneser graph \(K(n, k)\) has as vertices the subsets of \(\{1, 2, \ldots, n\}\) that have cardinality \(k\). Two vertices are adjacent if their corresponding \(k\)-subsets are disjoint. The Kneser graph \(K(2k - 1, k - 1)\) is also called the odd graph \(O_k\) for \(k \geq 2\). For simplicity, we refer to \(O_k\) as \(K(2k + 1, k)\), \(k \geq 1\). With this definition, the graph \(O_1\) is a triangle, and \(O_2\) is the Petersen graph (see Figures 1 (a) e 1 (b)).

Kneser graphs have been extensively studied, especially because of their high degree of symmetry. Biggs mentions the following conjecture:

∗D.Sc. Thesis presented to COPPE/UFRJ under the supervision of professors Celina M. H. de Figueiredo, Luerbio Faria and Peter Horák. Supported by CNPq grants GD 141360/2007-0 and SWE 201648/2008-3.
**Conjecture 1** ([Biggs 1979]). The odd graph $O_k$ is hamiltonian for all $k > 2$.

The bipartite Kneser graph $B(n, k)$ has $\binom{n}{k} \cup \binom{n}{n-k}$ as its vertex set and its edges represent the inclusion between two such subsets. The vertex set of $B(n, k)$ can be seen as two (symmetric) layers of the $n$-dimensional cube. If we consider the two layers in the middle of the cube (see Figure 1 (d)), then the corresponding bipartite Kneser graph $B(2k+1, k)$ is called the middle-levels graph, and we denote it by $B_k$ (see examples in Figures 1 (c) and 1 (e)). We are now ready to state the conjecture that has been attributed to Dejter, Erdos, Trotter, and various other mathematicians, but was most probably originated with Havel.

**Conjecture 2** ([Havel 1983]). The middle-levels graph $B_k$ is hamiltonian for all $k \geq 1$.

In fact, both Conjectures 1 and 2 are strongly related to a notorious conjecture due to Lovász [Lovász 1970] that every connected vertex-transitive graph has a hamiltonian path. The odd graphs $O_k$ and the bipartite Kneser graphs $B_k$ form a well-studied family of connected, $(k+1)$-regular, vertex-transitive graphs. Therefore, the study of hamiltonian paths in these graphs may provide more evidence to support Lovász’s conjecture, or offer a counterexample for it.

However, a direct computation of hamiltonian paths or cycles in $O_k$ and $B_k$ is not feasible for large values of $k$, because $O_k$ has $\binom{2k+1}{k}$ vertices and $B_k$ has $2\binom{2k+1}{k}$ vertices (see Table 1 in Section 3). Previous verifications of Conjecture 2 for $k \leq 17$ [Shields and Savage 1999, Shields et al. 2009] and Conjecture 1 for $k \leq 13$ [Shields and Savage 2004] relied heavily on computational methods.

A $j$-factor of a graph $G$ is a $j$-regular spanning subgraph of $G$. For instance, an 1-factor is a perfect matching. A graph $G$ is $j$-factorable if $G$ is the union of disjoint $j$-factors. Two different 1-factorizations of $B_k$ were found in [Kierstead and Trotter 1988] and [Duffus et al. 1994] hoping that the union of two suitable 1-factors would provide a hamiltonian cycle of $B_k$. Unfortunately, it turned out not to be the case for the given two 1-factorizations. However, those 1-factorizations were used to find a 2-factorization of $O_k$ [Johnson and Kierstead 2004].

Biggs’ conjecture has been extensively studied, thanks to the motivation brought by Lovász’s conjecture. Due to the difficulty of proving Biggs’ conjecture, researchers try
to prove Havel’s conjecture instead, because it is expected to be a simpler problem, for $B_k$ is bipartite. However, both conjectures are still open. Hence, one option is to show these graphs are “close” to being hamiltonian, where the word “close” has been interpreted in several different ways. Firstly, long cycles in $B_k$ and $O_k$ have been sought. At the moment, the best result of this type is due to Johnson [Johnson 2004] who showed that $B_k$ contains a cycle of length $(1 - o(1)) |B_k|$ and $O_k$ contains a cycle of length $(1 - o(1)) |O_k|$ where the error term $o(1)$ is of the form $\frac{c}{\sqrt{n}}$ for some constant $c$. [Chen 2003] showed that the Kneser graph $K(n, k)$ and the bipartite Kneser graph $B(n, k)$ are hamiltonian for $n \geq 2k^2$. Note that, for fixed $n$, the smaller the parameter $k$ is, the denser both the Kneser graph $K(n, k)$ and the bipartite Kneser graph $B(n, k)$ are. Thus, the graphs $O_k$ and $B_k$ are the sparsest among all of these graphs. The density of a graph $G$ is the ratio $\frac{|E(G)|}{\left(\frac{n}{2}\right)^2}$.

Yet another interpretation of “close” to being hamiltonian is provided by [Jackson and Wormald 1990] where a hierarchy of graphs is introduced. A closed spanning walk where each vertex is traversed at most $q$ times is called a $q$-walk and a spanning tree of maximum degree $q$ is a $q$-tree. Thus, in this terminology, a hamiltonian cycle is a 1-walk, and a hamiltonian path is a 2-tree. The authors proved that any graph with a $q$-tree has a $q$-walk, and that a $q$-walk guarantees the existence of a $(q + 1)$-tree. These results give the following hierarchy among families of graphs:

1-walk (Hamiltonian cycle) $\implies$ 2-tree (Hamiltonian path) $\implies$ 2-walk
$\implies$ 3-tree $\implies$ 3-walk $\implies$ ...

The prism over a graph $G$ is the Cartesian product $G \square K_2$ of $G$ with the complete graph on two vertices. Hence, the prism over $G$ consists of two copies of $G$ with a 1-factor joining the corresponding vertices. It was shown in [Kaiser et al. 2007] that the property of having a hamiltonian prism is “sandwiched” between the existence of a 2-tree and the existence of a 2-walk. Thus,

2-tree $\implies$ Hamiltonian prism $\implies$ 2-walk

This means that graphs having a hamiltonian prism are close to being hamiltonian, even closer than graphs having a 2-walk. In [Horák et al. 2005] it is proved that for all $k \geq 1$, the prism over the middle-levels graph $B_k$ is hamiltonian. In the present Thesis, we have shown how close the odd graphs are to being hamiltonian:

**Theorem 3** ([Bueno and Horák 2009]). Denote a $q$-trail as a $q$-walk that does not repeat edges. The prism over the odd graph $O_k$, $k \geq 2$, $k$ even, is hamiltonian. For $k$ odd, $O_k$ has not only a 2-walk but also a 2-trail.

We have also found that some odd graphs have indeed a hamiltonian path:

**Theorem 4** ([Bueno et al. 2009, Bueno 2009]). The odd graph $O_k$ has a hamiltonian path for $14 \leq k \leq 17$.

We proved Theorem 4 without the direct aid of a computer. Instead, we use existing results on the middle-levels problem [Shields and Savage 1999, Shields et al. 2009], therefore further relating two fundamental problems: to find a hamiltonian path in the odd graph and to find a hamiltonian cycle in the middle-levels graph.

The present text is meant to be a brief introduction to the basic ideas underlying the proofs of the results contained in the Thesis. Obviously, it does not delve too
much into the details due to space constraints. For further details, we refer to the papers [Bueno et al. 2009, Bueno and Horák 2009].

2. On Hamiltonian Cycles in the Prism over the Odd Graphs

In order to prove the first part of Theorem 3, we constructed a spanning cubic subgraph $H$ of $O_k$ for $k$ even. Then we proved that $H$ is 3-connected. Since [Paulraj 1993] showed that every 3-connected cubic graph has a hamiltonian prism, $O_k$ has a hamiltonian prism for $k$ even.

For $k$ odd, we cannot apply the technique used when $k$ is even. In fact, for some odd values of $k$, under no circumstances it is possible to find a spanning cubic subgraph of $O_k$, because $O_k$ has an odd number of vertices. Through the two 1-factorizations of $B_k$ found by [Kierstead and Trotter 1988] and [Duffus et al. 1994], a 2-factorization of $O_k$ has been found by [Johnson and Kierstead 2004]. Using two of these 2-factorizations, we determine a spanning 4-regular subgraph $H'$ of $O_k$, for $k$ odd. Then we show that $H'$ is connected. Since every 4-regular connected graph is eulerian, $H'$ has a 2-trail.

To better compare hamiltonian prisms and 2-trails, the diagram in Figure 2 illustrates the hierarchy among some families of graphs close to being hamiltonian. Having a 2-walk is a necessary condition – but not sufficient – for having a hamiltonian prism, a 2-trail or a hamiltonian cycle. A hamiltonian prism implies a 2-walk but the converse does not hold in general, so a hamiltonian prism is slightly closer to being hamiltonian than a 2-walk. Since a 2-trail is a 2-walk that does not repeat edges, a 2-trail is slightly better than a 2-walk as well. Therefore a graph with a hamiltonian prism is as close to being hamiltonian as a graph with a 2-trail.

![Figure 2. Hierarchy among some families of graphs close to being hamiltonian.](image)

3. Hamiltonian Paths in Odd Graphs

Let $Z_n$ denote the set $\{1, \ldots, n\}$ with numbers taken modulo $n$, plus 1. We consider the vertices of $O_k$ and $B_k$ to be subsets of $Z_n$ and $n = 2k + 1$. We define two special $k$-subsets of $Z_n$, which are $r_1 = \{1, \ldots, k\}$ and $r_2 = \{2, 4, 6, \ldots, n - 1\}$.

Given a set $v \subseteq Z_n$, let $v + \delta$ denote the set $\{a + \delta : a \in v\}$ and $\overline{v}$ denote the complement of $v$ with respect to $Z_n$. We say that $u, v \subseteq Z_n$ satisfy $u \sim v$ if either (i) $u = v + \delta$ or (ii) $\overline{v} = v + \delta$ for some $\delta \in Z_n$. It is easy to verify that $\sim$ is an equivalence relation. We refer to the equivalence class of $v$ under $\sim$ as $\sigma(v)$. 
Given a graph $G$, we define the quotient graph $\overline{G}$ as the graph obtained from $G$ by identifying vertices that are equivalent according to $\sim$. More precisely, the vertices of $\overline{G}$ are the equivalence classes $\sigma(v)$ for $v \in V(G)$, and if $uv \in E(G)$ then $\sigma(u)\sigma(v) \in E(\overline{G})$. Note that if $uv \in E(G)$ satisfies $u \sim v$, then the vertex $\sigma(u) \in V(\overline{G})$ has a loop. The quotient graph $\overline{B}_k$ is called the reduced graph. The graphs $O_3$ and $\overline{O}_3$ are illustrated in Figure 3.

[Shields and Savage 1999] showed that each equivalence class $\sigma(v)$ of $\overline{B}_k$ consists of exactly $n = 2k + 1$ $k$-subsets and $n (k + 1)$-subsets. As a consequence, the reduced graph $\overline{B}_k$ has $2n$ times fewer vertices than $B_k$ (see Table 1). For example, $B_{17}$ has 9,075,135,300 vertices, while $\overline{B}_{17}$ has 129,644,790 vertices, which is 70 times smaller, but still quite large.

Furthermore, [Shields and Savage 1999] proved that the existence of a hamiltonian path in the reduced graph $\overline{B}_k$, starting at the vertex $\sigma(r_1)$ and ending at the vertex $\sigma(r_2)$ implies that $B_k$ is hamiltonian. We refer to a hamiltonian path starting at $\sigma(r_1)$ and ending at $\sigma(r_2)$ as a useful path. Using heuristics, [Shields and Savage 1999, Shields et al. 2009] determined useful paths in $\overline{B}_k$ for $1 \leq k \leq 17$.

We prove Theorem 4 by showing that if there is a useful path $P = (p_1, \ldots, p_m)$ in $\overline{O}_k$, then there is a hamiltonian path in $O_k$. We use some interesting properties about $\overline{O}_k$, all of them proved in [Bueno et al. 2009, Bueno 2009] and exhibited in $\overline{O}_3$ (Figure 3). First, we notice that the quotient graphs $\overline{O}_k$ and $\overline{B}_k$ are equal. Moreover, if there is an edge $\sigma(u)\sigma(v)$ in $\overline{O}_k$, then there is a perfect matching between the vertices of $\sigma(u)$ and the vertices of $\sigma(v)$ in $O_k$. Consequently, if there is a path $P = (p_1, \ldots, p_m)$ in $\overline{O}_k$, then $O_k$ has $n$ disjoint paths $(q_i + (i - 1), \ldots, q_m + (i - 1))$, for $1 \leq i \leq n$, such that $q_j \in p_j$, for $1 \leq j \leq m$. Finally, the subgraph of $O_k$ induced by $\sigma(r_1)$ is the cycle $r_1, r_1 + k, r_1 + 2k, \ldots, r_1 + (n - 1)k$ and the subgraph of $O_k$ induced by $\sigma(r_2)$ is the cycle $r_2, r_2 + 1, r_2 + 2, \ldots, r_2 + (n - 1)$. Basically, we traverse all the $n$ disjoint paths in $O_k$, and carefully pick edges from the cycles induced by $\sigma(r_1)$ and $\sigma(r_2)$ in order to connect $n$ paths into a single hamiltonian path.

Given a path $Q$, we denote by $\overline{Q}$ the path $Q$ traversed from the last to the first vertex. Given two paths $Q_1, Q_2$ with no vertices in common and such that the last vertex of $Q_1$ is adjacent to the first vertex of $Q_2$, we denote by $Q_1 \circ Q_2$ the path obtained by the vertices of $Q_1$, followed by the vertices of $Q_2$.

By the definition of a useful path, $P = (p_1, \ldots, p_m)$ is hamiltonian in $\overline{O}_k$, $m = |V(O_k)|/n$, $p_1 = \sigma(r_1)$, and $p_m = \sigma(r_2)$. For $1 \leq i \leq n$, there are $n$ disjoint paths $P_i$ of the following form: $P_i = (q_i + (i - 1), \ldots, q_m + (i - 1))$ with $q_1 + (i - 1) \in \sigma(r_1)$, $q_m + (i - 1) \in \sigma(r_2)$ and $q_j + (i - 1) \in p_j$.

Because of the cycle induced by $\sigma(r_1)$, and because $n = 2k + 1$, we have that $q_1 + i$ is adjacent to $q_1 + i + k$. It follows that $\overline{P}_{i+1} \circ P_{i+k+1}$ is a valid path. Considering the cycle induced by $\sigma(r_2)$, $q_m + i$ is adjacent to $q_m + i + 1$. Therefore, $P_i \circ \overline{P}_{i+1}$ is a valid path as well. Consequently, the following is a valid path: $Q_i = P_i \circ \overline{P}_{i+1} \circ P_{i+k+1} \circ \overline{P}_{i+k+2}$, where $P_i = (q_1 + (i - 1), \ldots, q_m + (i - 1)), \overline{P}_{i+1} = (q_m + i, \ldots, q_1 + i), P_{i+k+1} = (q_1 + i + k, \ldots, q_m + i + k),$ and $\overline{P}_{i+k+2} = (q_m + i + k + 1, \ldots, q_1 + i + k + 1)$. \

The idea is to build a hamiltonian path $Q_1 \circ Q_3 \circ Q_5 \circ \ldots$. If $k$ is odd, then we show that $R_{odd} = Q_1 \circ Q_3 \circ \ldots \circ Q_{k-2}$ is a valid path. Because of the cycle induced by $\sigma(r_1)$, we know that the last vertex of $Q_i$, $q_i + i + k + 1$, is adjacent to $q_i + i + 1$, the first vertex of $Q_{i+2}$, since $P_{i+2} = (q_i + i + 1, \ldots, q_{m+i} + i + 1)$. Also, $R_{odd}$ contains either $P_i$ or $\overline{P_i}$, for $i \in \{1, \ldots, 2k+1\} \setminus \{k, k+1, 2k+1\}$. To include the missing paths, we define the hamiltonian path in $O_k$ as $H_{odd} = R_{odd} \circ P_k \circ \overline{P_{k+1}} \circ P_{2k+1}$. The full construction of a hamiltonian path in $O_3$ is illustrated in Figure 3. We omit the construction of the hamiltonian path for $k$ even, since it is similar to the case $k$ odd.

Since there is a useful path in $\overline{B}_k$ for $1 \leq k \leq 17$ [Shields and Savage 1999, Shields et al. 2009], $O_k$ has a hamiltonian path for $1 \leq k \leq 17$.

4. Conclusion and Open Problems

In our thesis, we showed a relationship between the reduced graphs $\overline{B}_k$ and $\overline{O}_k$, and determined a hamiltonian path in the odd graph $O_k$ by using a useful path in the reduced graph $\overline{O}_k = \overline{B}_k$. In this way, we determine hamiltonian paths in $O_k$ for $k$ up to 17. Further improved results for the middle-levels problem can be used to find hamiltonian paths in $O_k$ for larger values of $k$ [Bueno et al. 2009, Bueno 2009]. It is natural to ask whether a hamiltonian cycle in $O_k$ can be constructed in a similar manner.

All hamiltonian paths known for the reduced graph $\overline{B}_k$ were determined by computational methods, using heuristics. Finding an useful path in the reduced graph $\overline{B}_{17}$ [Shields et al. 2009] took more than 20 days of processing on an AMD Athlon 3500+. Further studies in the structure of the reduced graph may help finding useful paths faster, and possibly determine whether all reduced graphs have a useful path. It is important to note that even if the reduced graph does not have a useful path, the corresponding odd graph may still have a hamiltonian path.
Two different kinds of approximations for hamiltonian cycles in the middle-levels graphs are known. [Savage and Winkler 1995] showed that $B_k$ has a cycle containing at least 86.7% of the graph vertices, for $k \geq 18$. [Horák et al. 2005] showed the middle-levels graph has a closed spanning 2-walk. We proved that, for every $k$ even, $O_k$ has a closed spanning 2-walk. Moreover, for every $k$ odd, $O_k$ has a closed spanning trail in which every vertex appears at most twice [Bueno and Horák 2009, Bueno 2009].

Since vertex-transitive graphs defined by a single parameter, such as the odd graphs and the middle-levels graphs, are not known to have hamiltonian paths, Lovász’s conjecture [Lovász 1970] remains an open challenge to this day.

References


