LAGOS 2009

V Latin-American Algorithms, Graphs and Optimization Symposium

The P vs. NP-complete dichotomy of some challenging problems in graph theory

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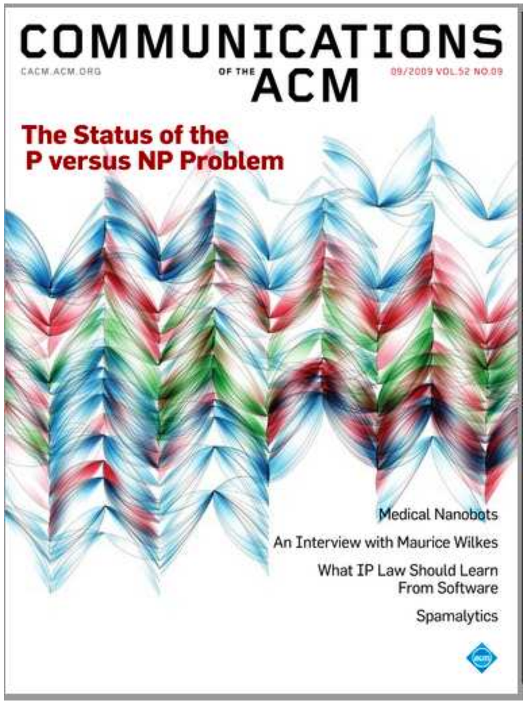
November 2009

Overview

Central problem in theoretical computer science: the P vs. NP problem

Are there questions whose answer can be quickly checked, but which require an impossibly long time to solve by any direct procedure?



september 2009

Overview

Central problem in theoretical computer science: the P vs. NP problem

Are there questions whose answer can be quickly checked, but which require an impossibly long time to solve by any direct procedure?

Classification into P or NP-complete of two challenging problems in graph theory

Complexity-separating problems and three full dichotomies

Two long-standing problems in graph theory

Perfect graphs: Chvátal's SKEW PARTITION is polynomial

Intersection graphs: Roberts–Spencer's CLIQUE GRAPH is NP-complete

V. Chvátal – *J. Combin. Theory Ser. B* 1985

F. Roberts, J. Spencer – *J. Combin. Theory Ser. B* 1971

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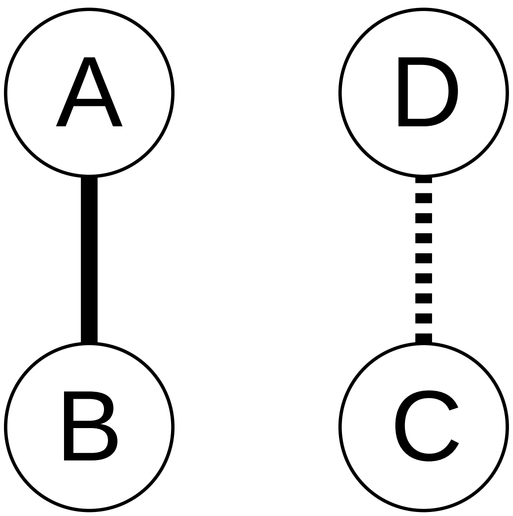
Skew partition

SKEW PARTITION

Instance: Graph $G = (V, E)$

Question: Does V admit a partition into 4 nonempty parts A, B, C, D such that each vertex in A is adjacent to each vertex in B and each vertex in C is nonadjacent to each vertex in D ?



$A \cup B$ is a skew cutset

SKEW PARTITION generalizes

- STAR CUTSET
- CLIQUE CUTSET
- HOMOGENEOUS SET

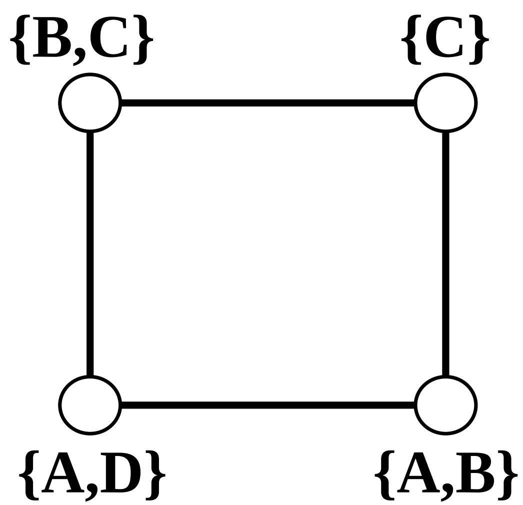
V. Chvátal – *J. Combin. Theory Ser. B* 1985

List skew partition

LIST SKEW PARTITION

Instance: Graph $G = (V, E)$ and, for each $v \in V$, a list $L(v) \subseteq \{A, B, C, D\}$

Question: Does V admit a skew partition into 4 parts A, B, C, D such that each vertex v is assigned to a part in $L(v)$?



Instance G, L admits a list skew partition

T. Feder, P. Hell, S. Klein, R. Motwani – *SIAM J. Discrete Math.* 2003

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Recursive algorithm:

Number of subproblems $T(n)$ encountered during recursive skew partitioning satisfies nested recurrences of the form:

$$T(n) \leq 4\,T(9n/10)$$

Running time $O(n^{100})$ challenges the notion:

polynomial-time solvable = efficiently solvable in practice

“Finding skew partitions efficiently”

J. Algorithms 2000 (with Sulamita Klein, Yoshiharu Kohayakawa, Bruce Reed)

Is LIST PARTITION harder than NONEMPTY PART PARTITION ?

	list	nonempty part
SKEW	n^{100}	n^6

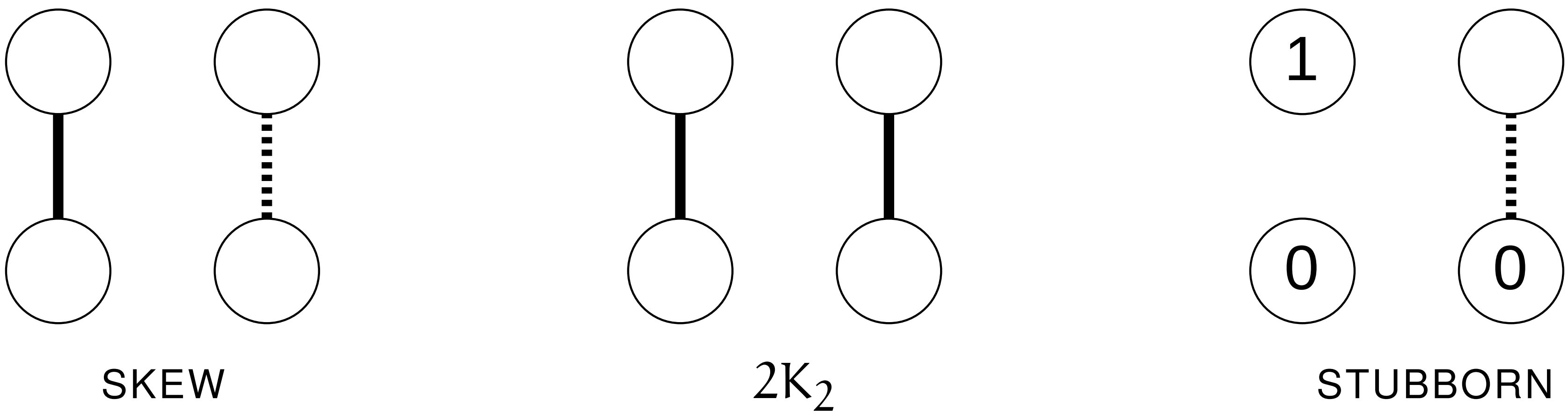
Lists capture additional constraints: nonempty part, cardinality of parts, specify for each vertex a list of parts in which the vertex is allowed

W. Kennedy, B. Reed – *KyotoCGGT Lecture Notes in Comput. Sci.* 2007

Is LIST PARTITION harder than NONEMPTY PART PARTITION ?

	list	nonempty part
SKEW	n^{100}	n^6
$2K_2$	N	O
STUBBORN	Q	P

N: NP-complete, P: polynomial, Q: quasi-polynomial, O: open



T. Feder, P. Hell, S. Klein, R. Motwani – *SIAM J. Discrete Math.* 2003

W. Kennedy, B. Reed – *KyotoCGGT Lecture Notes in Comput. Sci.* 2007

K. Cameron, E. Eschen, C. Hoàng, R. Sritharan – *SIAM J. Discrete Math.* 2007

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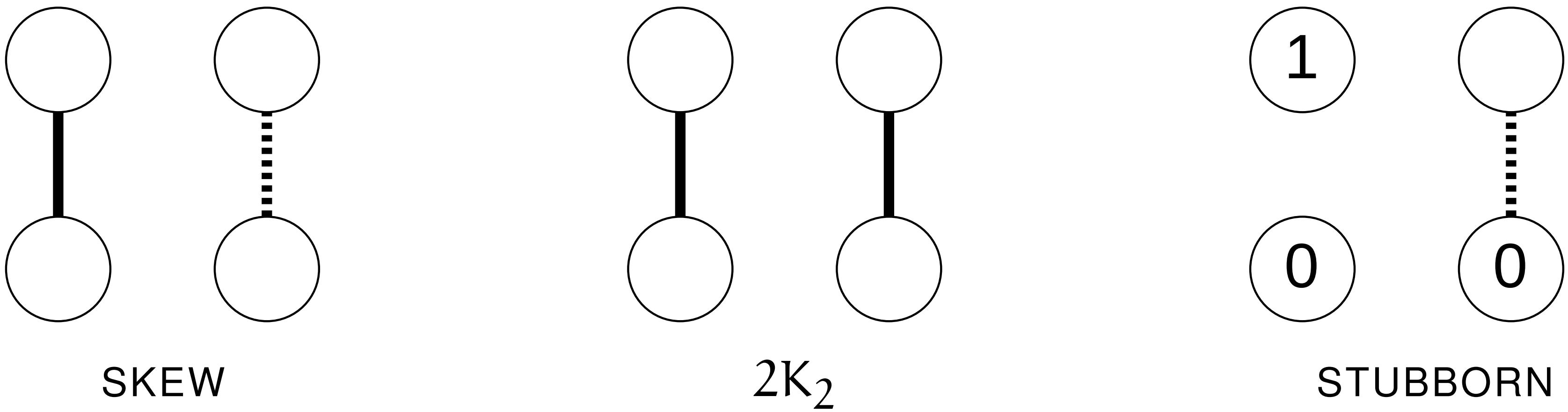

Is $2K_2$-PARTITION complexity-separating?

Two long-standing problems in graph theory

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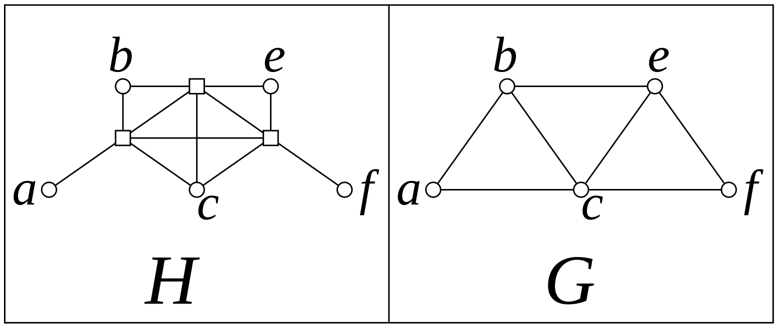
Clique graph

CLIQUE GRAPH

Instance: Graph G

Question: Is there a graph H such that graph G is the intersection graph of the cliques of graph H ?



G is the clique graph of H

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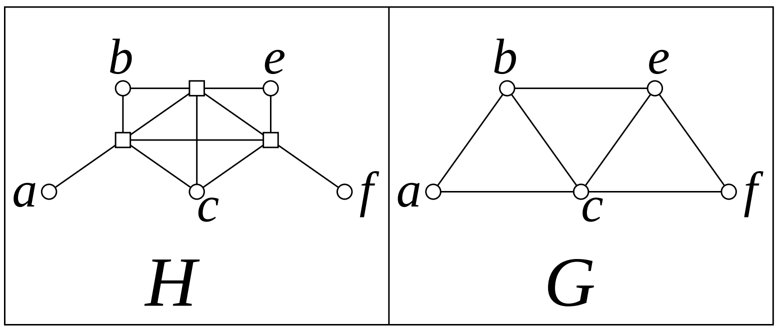


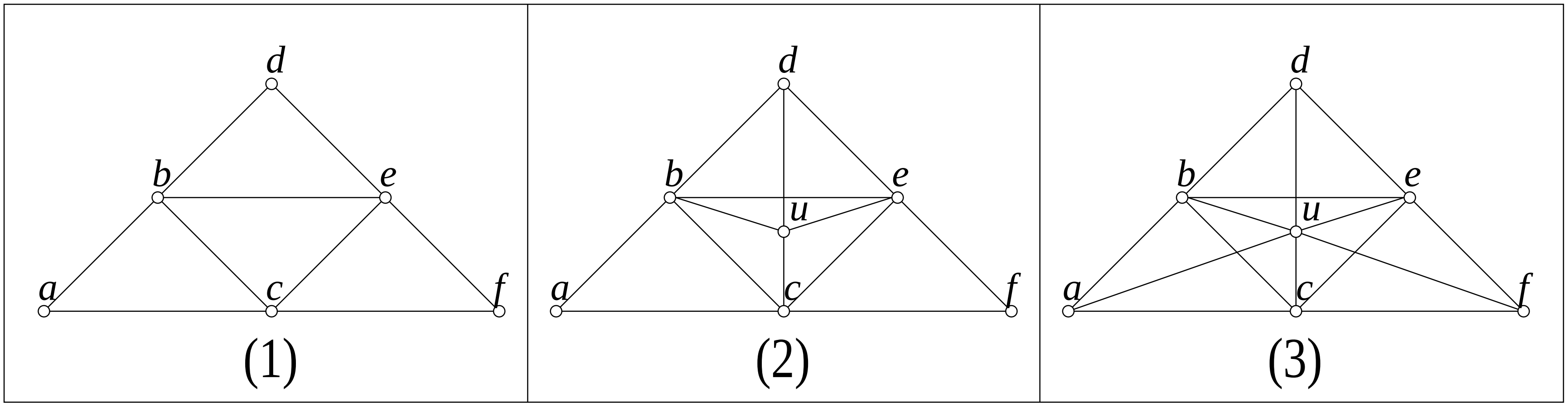
G is the clique graph of H

RS-family: G is a clique graph if and only if G admits an edge-cover by complete sets satisfying the Helly property (mutually intersecting members have nonempty total intersection)

CLIQUE GRAPH is NP: RS-family of size $\leq |E(G)|$ gives H such that
$$|V(H)| \leq |V(G)| + |E(G)|$$

F. Roberts, J. Spencer – *J. Combin. Theory Ser. B* 1971

Clique graphs and clique-Helly graphs



(1) clique-complete

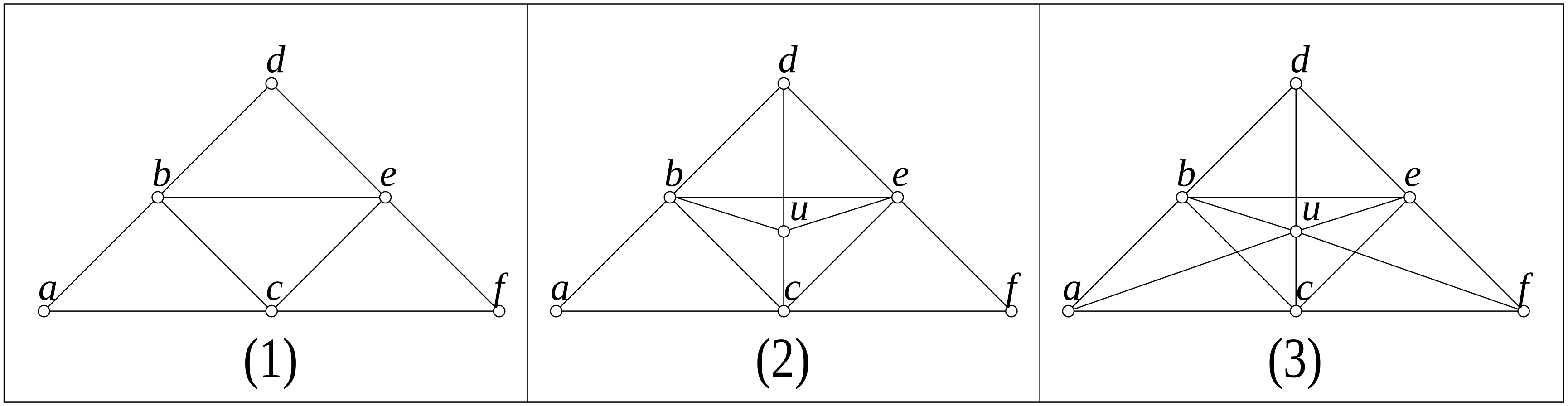
(2) clique-complete

(3) clique-complete

C. Lucchesi, C. Mello, J. Szwarcfiter – *Discrete Math.* 1998

Clique graphs and clique-Helly graphs

Clique-Helly graph: clique family satisfies the Helly property



(1) clique-complete, but non clique-Helly

(2) clique-complete, non clique-Helly

(3) clique-complete, clique-Helly

R. Hamelink – *J. Combin. Theory Ser. B* 1968

J. Szwarcfiter – *Ars Combin.* 1997

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Clique graphs and clique-Helly graphs

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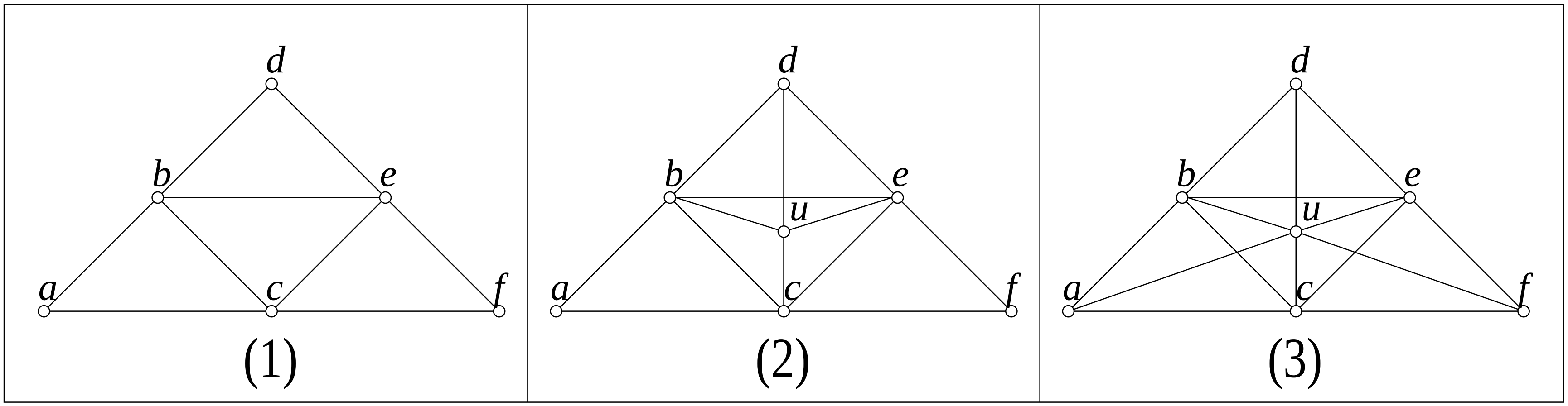

(1) clique-complete, but non clique-Helly, non clique graph

(2) clique-complete, non clique-Helly, but clique graph

(3) clique-complete, clique-Helly, clique graph

R. Hamelink – *J. Combin. Theory Ser. B* 1968

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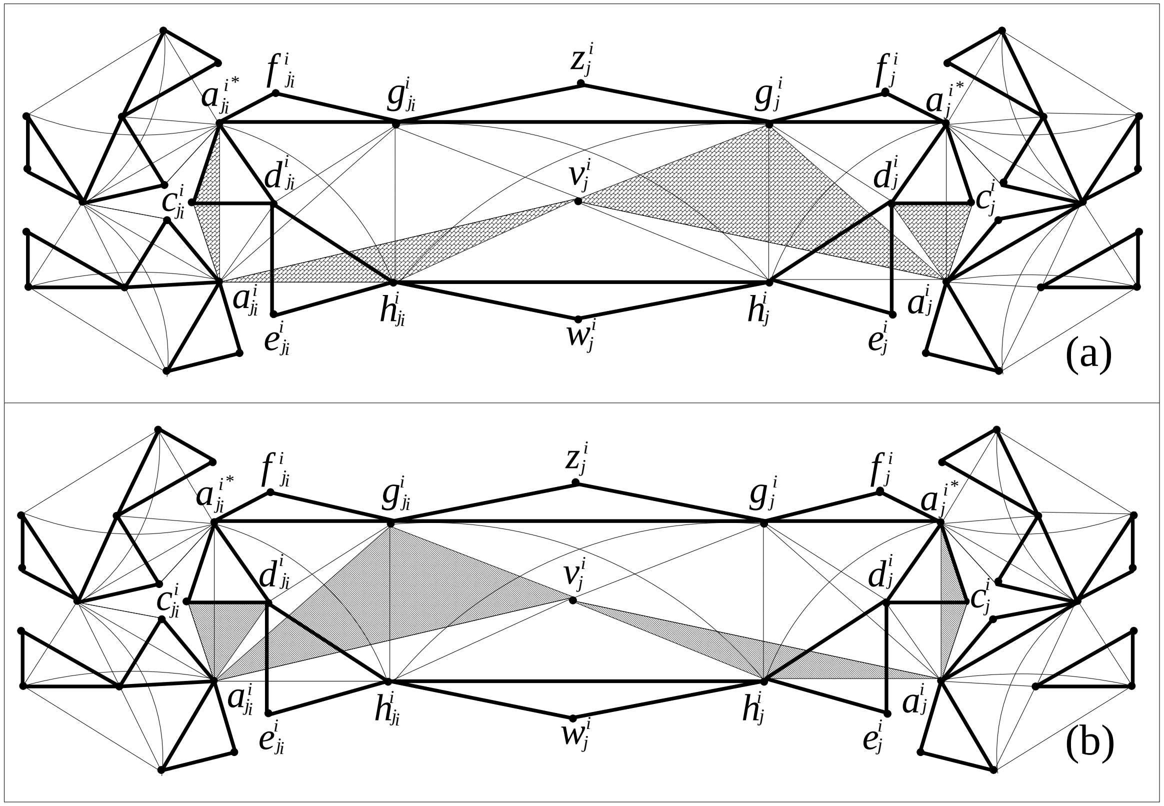
Clique graph gadget: a catwalk for variable u_i



RS-family of G_I must contain either the false triangles in (a) or the true triangles in (b). All bold triangles must belong to the RS-family.

“The complexity of clique graph recognition”

Theoret. Comput. Sci. 2009 (with Liliana Alcon, Luerbio Faria, Marisa Gutierrez)

NP-completeness ongoing guide

Identification of an interesting problem, of an interesting graph class

Categorization of the problem according to its complexity status

Problems and complexity-separating graph classes

Graph classes and complexity-separating problems

Johnson's NP-completeness column 1985

Golumbic–Kaplan–Shamir's sandwich problems 1995

Spinrad's book 2003

Complexity-separating graph classes

	VERTEXCOL	EDGECOL	MAXCUT
perfect	P	N	N
chordal	P	O	N
split	P	O	N
strongly chordal	P	O	O
comparability	P	N	O
bipartite	P	P	P
permutation	P	O	O
cographs	P	O	P
proper interval	P	O	O
split-proper interval	P	P	P

N: NP-complete P: polynomial O: open

Johnson's NP-completeness column 1985

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C. Simone, C. Mello – *Theoret. Comput. Sci.* 2006

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Split vs. chordal

Split = chordal $\cap$ complement chordal = partition into stable and clique

Johnson's NP-Completeness Column:

> "Every known hardness result for chordal graphs also applies to split graphs!"

Open problems: EDGE COLORING, CLIQUE GRAPH

Split vs. chordal

Split = chordal ∩ complement chordal = partition into stable and clique

Johnson's NP-Completeness Column:

> "Every known hardness result for chordal graphs also applies to split graphs!"

Spinrad's book:

> "Split graphs often are at the core of algorithms and hardness results for chordal graphs."

Same complexity:

- CLIQUE, VERTEX COLORING are linear time
- DOMINATING SET, MAXCUT, HAMILTON CYCLE are NP-complete

Separated in complexity:

- TRIANGLE PACKING, PATHWIDTH

Open problems: EDGE COLORING, CLIQUE GRAPH

Complexity-separating problems

	VERTEXCOL	edgecol	MAXCUT	SANDWICH
perfect	P	N	N	O
chordal	P	O	N	N
split	P	O	N	P
strongly chordal	P	O	O	N
comparability	P	N	O	N
bipartite	P	P	P	P
permutation	P	O	O	N
cographs	P	O	P	P
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split-proper interval	P	P	P	P

N: NP-complete   P: polynomial   O: open

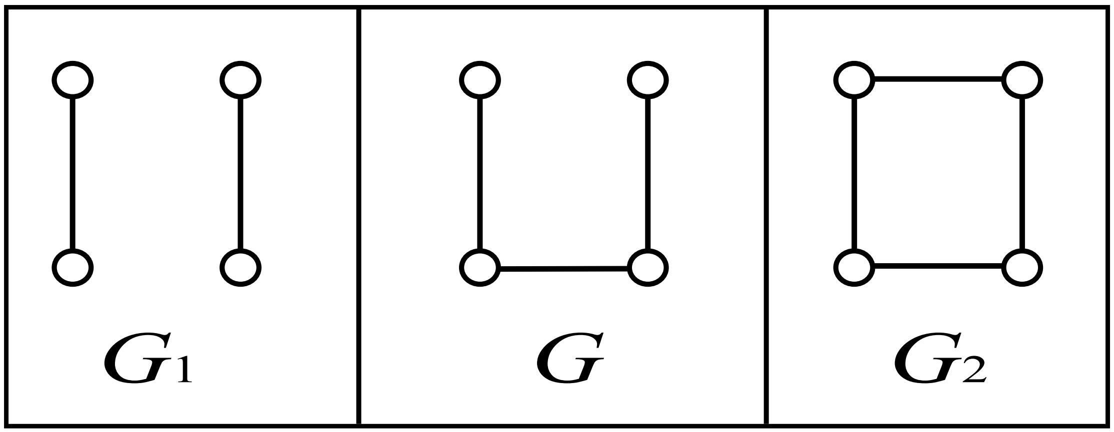
M. C. Golumbic, H. Kaplan, R. Shamir – *J. Algorithms* 1995

Graph sandwich problem

GRAPH SANDWICH PROBLEM FOR PROPERTY Π

Instance: Two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$

Question: Is there a graph $G = (V, E)$ with $E_1 \subseteq E \subseteq E_2$ that satisfies property Π ?

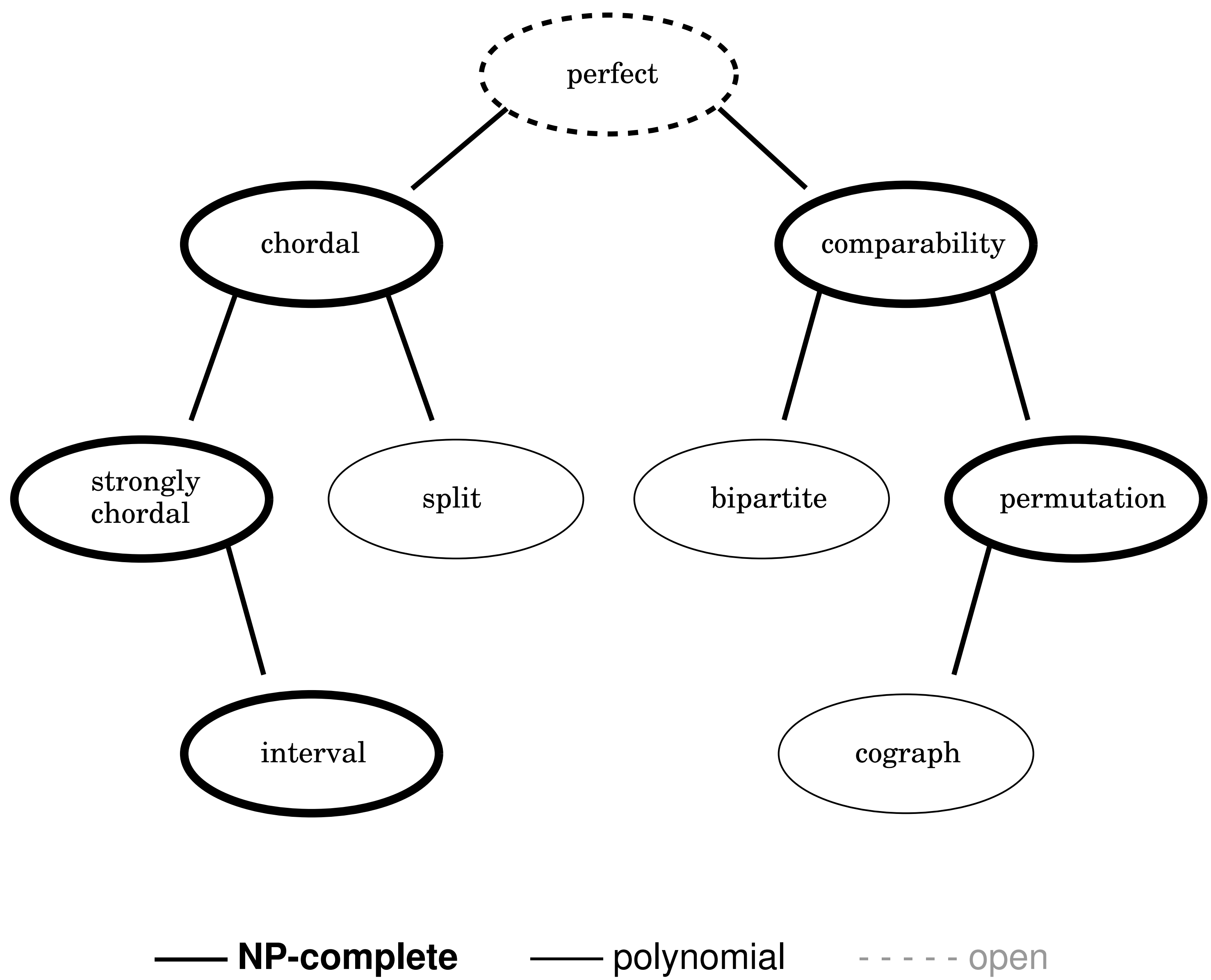


G is a sandwich split graph

The sandwich problem generalizes the recognition problem

M. C. Golumbic, H. Kaplan, R. Shamir – *J. Algorithms* 1995

Sandwich problems for perfect graph classes



Three full dichotomies

Classes for which every problem is classified into P or NP-complete:

SANDWICH PROBLEM

EDGE COLORING

GRID EMBEDDING

Three full dichotomies

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The three nonempty part problem

Full dichotomy for the RECOGNITION PROBLEM:

STABLE CUTSET, 3-COLORING are the only NP-complete

T. Feder, P. Hell, S. Klein, R. Motwani – *SIAM J. Discrete Math.* 2003

The three nonempty part problem

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Full dichotomy for the SANDWICH PROBLEM:

61 interesting problems: 19 NP-complete, 42 polynomial

HOMOGENEOUS SET SANDWICH PROBLEM is polynomial

CLIQUE CUTSET SANDWICH PROBLEM is NP-complete

Full dichotomy for the GENERALIZED SPLIT GRAPH SANDWICH PROBLEM:

(2,1)-GRAPH SANDWICH PROBLEM is NP-complete

“The polynomial dichotomy for three nonempty part sandwich problems”

Discrete Appl. Math. 2009 (with Rafael Teixeira, Simone Dantas)

Three full dichotomies

Classes for which every problem is classified into P or NP-complete:

SANDWICH PROBLEM

EDGE COLORING

GRID EMBEDDING

Graphs with no cycle with a unique chord

χ-bounded graph class: $\chi \leq f(\omega)$

Perfect graph: $\chi = \omega$

Line graph: $\chi \leq \omega + 1$, the Vizing bound

N. Trotignon, K. Vušković – *J. Graph Theory* 2009

Graphs with no cycle with a unique chord

χ-bounded graph class: $\chi \leq f(\omega)$

Perfect graph: $\chi = \omega$

Line graph: $\chi \leq \omega + 1$, the Vizing bound

Which choices of forbidden induced subgraphs give χ-bounded class?

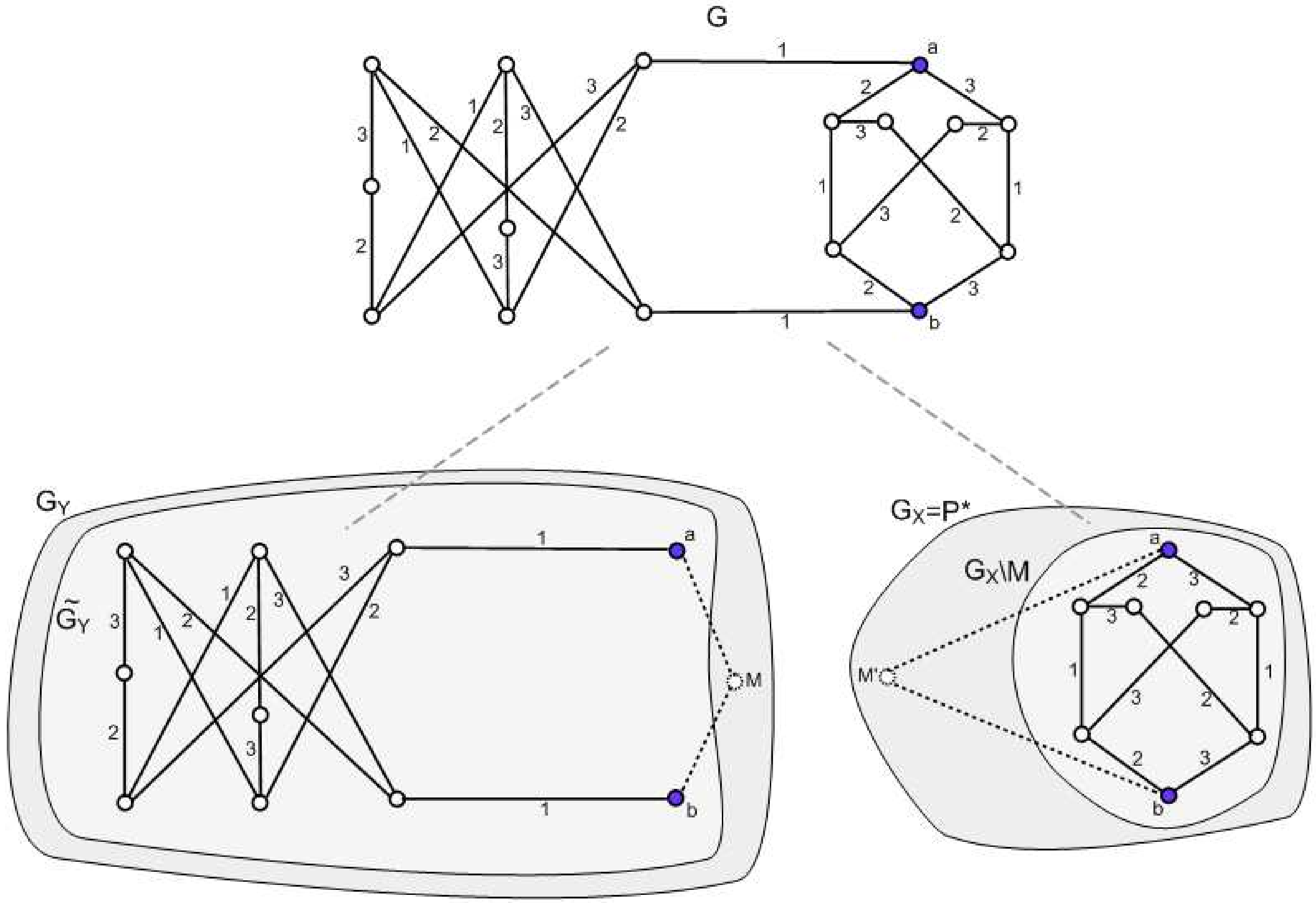
Graphs with no cycle with a unique chord: $\chi \leq \omega + 1$

Structure theorem:

every graph in the class can be built from basic graphs

N. Trotignon, K. Vušković – *J. Graph Theory* 2009

Combining edge-colorings with respect to 2-cutset



Decomposition with respect to a proper 2-cutset $\{a, b\}$
G is Class 1: Δ colors suffice, but $G_X = P^*$ is Class 2: $\Delta + 1$ colors needed

Edge-coloring graphs with no cycle with a unique chord

Class $\mathcal{C}$ = graphs with no cycle with a unique chord

	$\Delta = 3$	$\Delta \geq 4$	regular
graphs of $\mathcal{C}$	N	N	N
4-hole-free graphs of $\mathcal{C}$	N	P	P
6-hole-free graphs of $\mathcal{C}$	N	N	N
$\{$4-hole, 6-hole$\}$-free graphs of $\mathcal{C}$	P	P	P

“Chromatic index of graphs with no cycle with a unique chord”

submitted to *Theoret. Comput. Sci.* (with Raphael Machado, Kristina Vušković)

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EDGECOL is N for k-partite r-regular, for each $k \geq 3$, $r \geq 3$

	$k \leq 2$	$k \geq 3$
k-partite graphs	P	N

“Chromatic index of graphs with no cycle with a unique chord”

submitted to *Theoret. Comput. Sci.* (with Raphael Machado, Kristina Vušković)

Class 2 = overfull implies EDGECOL is P

Overfull graph: $|E| > \Delta \left\lfloor \frac{|V|}{2} \right\rfloor$

Complete multipartite: Class 2 = overfull

Graphs with a universal vertex: Class 2 = overfull

Split-proper interval graphs: Class 2 = subgraph overfull

4-hole-free graphs of $\mathcal{C}$, with $\Delta \neq 3$: Class 2 = subgraph overfull

D. Hoffman, C. Rodger – *J. Graph Theory* 1992

M. Plantholt – *J. Graph Theory* 1981

C. Ortiz Z., N. Maculan, J. Szwarcfiter – *Discrete Appl. Math.* 1998

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Conjecture for edge-coloring chordal graphs:

Class 2 = subgraph overfull

“On edge-colouring indifference graphs”

Theoret. Comput. Sci. 1997 (with João Meidanis, Célia Mello)

Three full dichotomies

Classes for which every problem is classified into P or NP-complete:

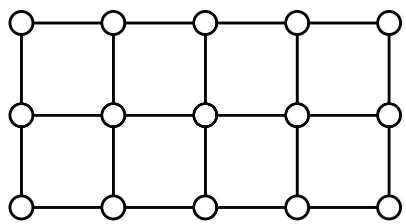
SANDWICH PROBLEM

EDGE COLORING

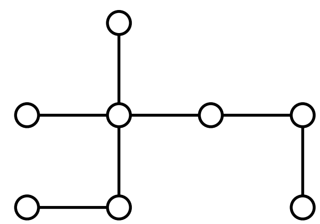
GRID EMBEDDING

Grid embedding

Graph theory: The recognition of partial grids is often stated as an open problem.



the grid $G_{3,5}$



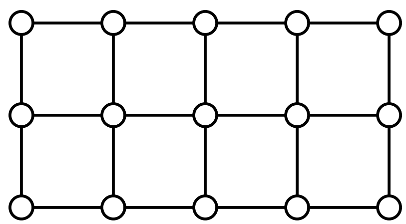
embedding for $\{1, 2, 4\}$-tree

A. Brandstädt, V.B. Le, et al. – Information system on graph class inclusions.
http://wwwteo.informatik.uni-rostock.de/isgci/, 2002

Grid embedding

Graph theory: The recognition of partial grids is often stated as an open problem.

Graph drawing: Deciding whether a graph admits a VLSI layout with unit-length edges is NP-complete.



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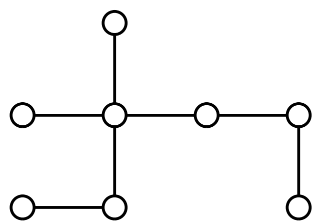

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S. N. Bhatt, S. S. Cosmadakis – *Inform. Process. Lett.* 1987

P vs. N dichotomy for degree-constrained partial grids

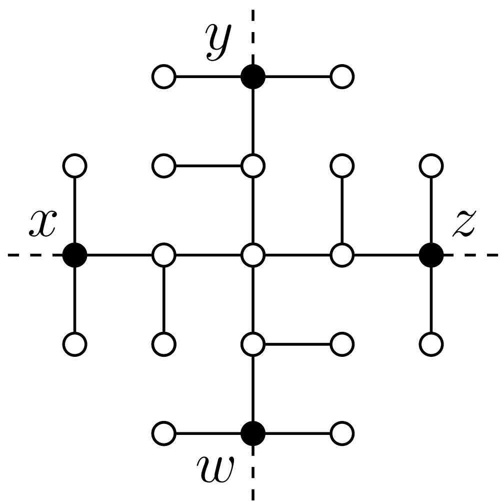
D	D-graphs	D-trees
{1}	P	P
{2}	P	—
{3}	P	—
{4}	P	—
{1,2}	P	P
{1,3}	N	O
{1,4}	P	P
{2,3}	N	—

D	D-graphs	D-trees
{2,4}	N	—
{3,4}	P	—
{1,2,3}	N [G89]	N [G89]
{1,2,4}	N [BC87]	N [BC87]
{1,3,4}	N	N
{2,3,4}	N	—
{1,2,3,4}	N [BC87]	N [BC87]

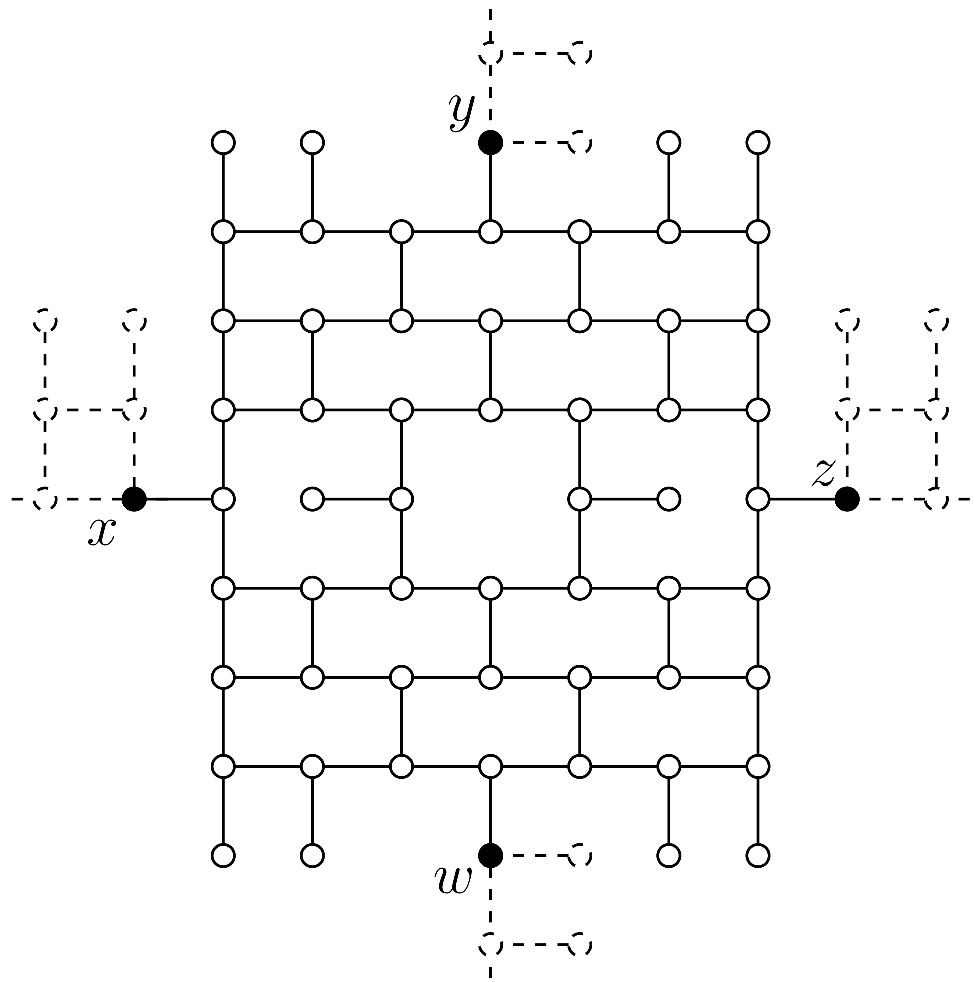
S. N. Bhatt, S. S. Cosmadakis – *Inform. Process. Lett.* 1987
A. Gregori – *Inform. Process. Lett.* 1989

"Complexity dichotomy on degree-constrained VLSI layouts with unit-length edges"
submitted to *LATIN 2010* (with Vinícius Sá, Guilherme Fonseca, Raphael Machado)

Partial grid gadgets



the windmill: $\{1, 3, 4\}$-gadget



the brick wall: $\{1, 3\}$-gadget

P vs. N dichotomy for degree-constrained partial grids

D	D-graphs	D-trees
{1}	P	P
{2}	P	—
{3}	P	—
{4}	P	—
{1,2}	P	P
{1,3}	N	O
{1,4}	P	P
{2,3}	N	—

D	D-graphs	D-trees
{2,4}	N	—
{3,4}	P	—
{1,2,3}	N [G89]	N [G89]
{1,2,4}	N [BC87]	N [BC87]
{1,3,4}	N	N
{2,3,4}	N	—
{1,2,3,4}	N [BC87]	N [BC87]

Is $\{1,3\}$-partial-grid recognition a complexity-separating problem?

S. N. Bhatt, S. S. Cosmadakis – *Inform. Process. Lett.* 1987
A. Gregori – *Inform. Process. Lett.* 1989

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Proposed complexity-separating questions

Is LIST PARTITION harder than NONEMPTY PART PARTITION?

Is CLIQUE GRAPH polynomial for split graph instances?

Is Class 2 = subgraph overfull for chordal graphs?

Is PARTIAL GRID polynomial for $\{1, 3\}$-tree instances?

References

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Thanks!