GRAFOS ALEATÓRIOS

DEF: O GRAFO ALEATÓRIO DE ERDŐS-RÉNYI É O GRAFO OBTIDO DE K_n AO MANTERMOS CADA ARESTA INDEPENDENTEMENTE COM PROBABILIDADE p.

→ É UMA DISTRIBUIÇÃO DE PROBABILIDADE.

→ Qual a PROBABILIDADE DE $G(n,p)$ CONTER UM TRIÂNGULO?

→ limiar

$A = \{K_3 \subseteq G(n,p)\}$ $\quad p_n \to 0 \quad$ $f \ll g$ SSE $\frac{f}{g} \to 0$

TEOREMA: SE $p \ll 1/n$, ENTÃO $\mathbb{P}(K_3 \subseteq G(n,p)) \to 0$ QUANDO $n \to 0$

PROVA: SEJA X A VARIÁVEL QUE CONTA O NÚMERO DE TRIÂNGULOS EM $G(n,p)$.

NOTE QUE $\mathbb{P}(K_3 \subseteq G(n,p)) = \mathbb{P}(X \geq 1)$

HÁ $\binom{n}{3}$ TRIÂNGULOS EM K_n, E CADA TRIÂNGULO ESTÁ CONTIDO EM $G(n,p)$ COM PROBABILIDADE p^3. TEMOS

$$\mathbb{E}(X) = p^3 \binom{n}{3} \leq p^3 n^3 = (pn)^3 \longrightarrow 0$$

PELA DES. DE MARKOV,

$$\mathbb{P}(X \geq 1) \leq \frac{\mathbb{E}(X)}{1} \longrightarrow 0$$

QUANDO $n \to \infty$

$p \ll 1/n$,

EX: $p(n) = \frac{1}{n^{1+\varepsilon}}$ PARA QUALQUER $\varepsilon > 0$, TEMOS

$$\frac{f}{g} = \frac{1/n^{1+\varepsilon}}{1/n} = \frac{1}{n^{\varepsilon}} \longrightarrow 0$$

<u>TEOREMA</u>: SE $p \gg 1/n$, ENTÃO $\mathbb{P}(K_3 \subseteq G(n,p)) \to 1$ QUANDO $n \to \infty$.

$\hookrightarrow \quad \dfrac{1/n}{p} \to 0 \iff pn = \dfrac{p}{1/n} \to \infty$

<u>PROVA</u>: SEJA X A VARIÁVEL QUE CONTA O NÚMERO DE TRIÂNGULOS EM $G(n,p)$.

HÁ $\binom{n}{3}$ TRIÂNGULOS EM K_n, E CADA TRIÂNGULO ESTÁ CONTIDO EM $G(n,p)$ COM PROBABILIDADE p^3. TEMOS

$$\mathbb{E}(X) = p^3 \binom{n}{3} \gtrsim \frac{n^3}{6} \cdot p^3 = \frac{(np)^3}{6} \to \infty$$

VAMOS CALCULAR A VARIÂNCIA DE X: $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

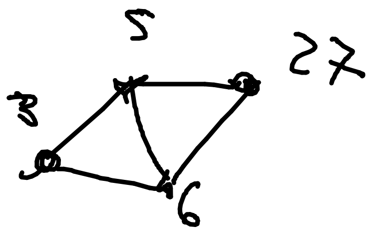
NOTE QUE $\quad X = \sum_{T \subseteq \binom{[n]}{3}} \mathbb{1}(T \text{ INDUZ TRIÂNGULO EM } G(n,p))$

LOGO

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6} = \frac{n^3}{6} + O(n^2)$$

NOTE QUE $X = \sum_{T \subseteq \binom{[n]}{3}} \mathbb{1}_{(T \text{ INDUZ TRIÂNGULO EM } G(n,p))}$

LOGO $X^2 = \left(\sum_{T \subseteq \binom{[n]}{3}} \mathbb{1}_{(T \text{ INDUZ TRIÂNGULO EM } G(n,p))} \right)^2$

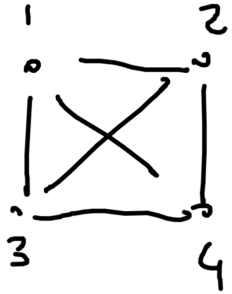


$$\mathbb{E}[X^2] = \sum_{T_1 \in \binom{[n]}{3}} \sum_{T_2 \in \binom{[n]}{3}} \mathbb{P}(T_1 \cup T_2 \subseteq G(n,p))$$

$$= \sum_{|T_1 \cap T_2| = 3} \mathbb{P}(T_1 \cup T_2 \subseteq G) + \sum_{|T_1 \cap T_2| \leq 1} \mathbb{P}(T_1 \cup T_2 \subseteq G) + \underline{\sum_{|T_1 \cap T_2| = 2} \mathbb{P}(T_1 \cup T_2 \subseteq G)}$$

$$\leq \underbrace{\sum_{T \in \binom{[n]}{3}} \mathbb{1}_T}_{\shortparallel \atop \mathbb{E}[X]} + \mathbb{E}[X]^2 + p^5 n^4$$

K_4



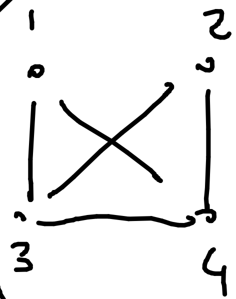
$T_i = \{1, 2, 3, 4\} - i$

$T_1 = \{2, 3, 4\}$

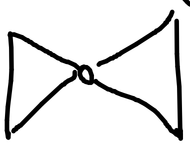
$T_3 = \{1, 2, 4\}$

$\Pi_{T_1} \cdot \Pi_{T_1} = \Pi_{T_1}$

$T_1 \cup T_2$



p^5



$$X = \Pi_{T_1} + \Pi_{T_2} + \Pi_{T_3} + \Pi_{T_4}$$

$$X^2 = \left(\Pi_{T_1} + \Pi_{T_2} + \Pi_{T_3} + \Pi_{T_4}\right)^2$$

$$= \Pi_{T_1} \cdot \Pi_{T_1} + \Pi_{T_2} \cdot \Pi_{T_1} + \Pi_{T_3} \cdot \Pi_{T_1} + \Pi_{T_4} \cdot \Pi_{T_1}$$

$$+ \Pi_{T_1} \cdot \Pi_{T_2} + \Pi_{T_2} \cdot \Pi_{T_2} + \Pi_{T_3} \cdot \Pi_{T_2} + \Pi_{T_4} \cdot \Pi_{T_2}$$

$$+ \Pi_{T_1} \cdot \Pi_{T_3} + \Pi_{T_2} \cdot \Pi_{T_3} + \Pi_{T_3} \cdot \Pi_{T_3} + \Pi_{T_4} \cdot \Pi_{T_3}$$

$$+ \Pi_{T_1} \cdot \Pi_{T_4} + \Pi_{T_2} \cdot \Pi_{T_4} + \Pi_{T_3} \cdot \Pi_{T_4} + \Pi_{T_4} \cdot \Pi_{T_4}$$

$$= \Pi_{T_1}$$

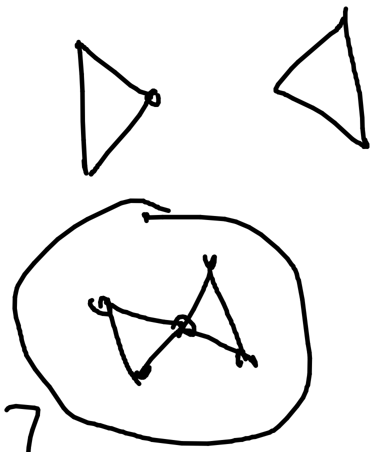
$$\mathbb{E}[X] = \sum_{T} \mathbb{P}(T \subseteq G(n,p))$$

$$\mathbb{E}[X]^2 = \sum_{T_1} \sum_{T_2} \mathbb{P}(T_1 \subseteq G(n,p)) \cdot \mathbb{P}(T_2 \subseteq G(n,p))$$

$$\geq \sum_{|T_1 \cap T_2| \leq 1} \mathbb{P}(T_1 \subseteq G(n,p)) \cdot \mathbb{P}(T_2 \subseteq G(n,p))$$

Pois T_1 e T_2 são ind.

$$= \sum_{|T_1 \cap T_2| \leq 1} \mathbb{P}(\underline{T_1 \cup T_2 \subseteq G(n,p)})$$



$$\{T_1 \subseteq G(n,p)\} \cap \{T_2 \subseteq G(n,p)\} = \{T_1 \cup T_2 \subseteq G(n,p)\}$$

$$\mathbb{E}[X^2] = \mathbb{E}[X] + \mathbb{E}[X]^2 + p^5 n^4$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \mathbb{E}[X] + \cancel{\mathbb{E}[X]^2} + p^5 n^4 \cancel{- \mathbb{E}[X]^2}$$

$$= \mathbb{E}[X] + p^5 n^4 \ll \mathbb{E}[X]^2$$

$$\frac{\mathbb{E}[X] + p^5 n^4}{\mathbb{E}[X]^2} = \frac{1}{\mathbb{E}[X]} + \frac{p^5 n^4}{\mathbb{E}[X]^2}$$

$$\mathbb{E}[X] = p^3 \binom{n}{3} \geq \frac{n^3}{6} \cdot p^3 = \frac{(np)^3}{6} \to \infty$$

$$\mathbb{E}[X]^2 \geq \frac{(np)^6}{36}$$

$$n^2 p \to \infty$$

$$\leq \underbrace{\frac{1}{\mathbb{E}[X]}}_{\to 0} + \overbrace{\frac{p^5 n^4}{p^6 n^6 / 36}}^{\to 0} \to 0$$

$$\frac{p^5 n^4}{p^6 n^6 / 36} = \frac{36}{p n^2}$$

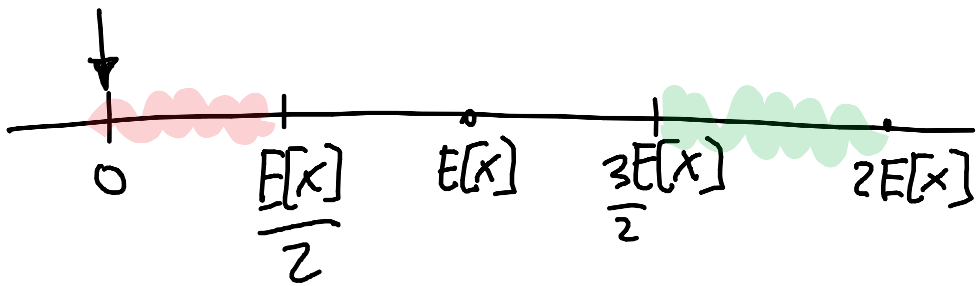
$$\mathrm{Var}(X) \ll \mathbb{E}[X]^2$$

$$\mathbb{P}(|X-\mathbb{E}[X]| \geq \lambda) \leq \frac{\mathrm{Var}(X)}{\lambda^2}$$

$$\{K_3 \subseteq G(n,p)\} = \{X \geq 1\}$$

$$\mathbb{P}(X=0) \leq \mathbb{P}\left(X \leq \frac{\mathbb{E}[X]}{2}\right) \leq \mathbb{P}\left(|X-\mathbb{E}[X]| \geq \frac{\mathbb{E}[X]}{2}\right) \leq \frac{4\cdot \mathrm{Var}(X)}{\mathbb{E}[X]^2} \longrightarrow 0$$

∎



$$\mathbb{P}(|X-\mathbb{E}[X]| \geq \lambda) \leq \frac{\mathrm{Var}(X)}{\lambda^2}$$

FUNÇÕES LIMIARES

→ DIZEMOS QUE $1/n$ É UM LIMIAR PARA O EVENTO $\{K_3 \subseteq G(n,p)\}$

→ É POSSÍVEL GENERALIZAR $1/n$ PARA OUTROS CLIQUES E PARA OUTROS GRAFOS "BALANCEADOS"

→ O QUE ACONTECE QUANDO $p = \frac{c}{n}$

PODEMOS ENCONTRAR UMA FUNÇÃO EXPLÍCITA $f(c)$ QUE DIZ A PROBABILIDADE DO EVENTO $\{K_3 \subseteq G(n,p)\}$