FUNÇÕES INTEIRAS

• PISOS E TESOS

$\lfloor x \rfloor$ = MAIOR INTEIRO MENOR OU IGUAL A X (Floor)

$\lceil x \rceil$ = MENOR INTEIRO MAIOR OU IGUAL A X (ceil)

EX: $X = \frac{1}{2}$

$\lfloor -\frac{1}{2} \rfloor = -1$

$\neq$

$-\lfloor \frac{1}{2} \rfloor = 0$

OBSERVAÇÕES

1) $\lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil$

2) $\lfloor -x \rfloor = -\lfloor x \rfloor$ SE E SOMENTE SE X É INTEIRO

SE X NÃO É INTEIRO, ENTÃO $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$

3) $\lceil x \rceil - \lfloor x \rfloor = \begin{cases} 0 & \text{SE } X \text{ INTEIRO} \\ 1 & \text{CASO CONTRÁRIO} \end{cases}$

4) $X - 1 < \lfloor x \rfloor$ E $\lceil x \rceil < X + 1$: $X - 1 < \lfloor x \rfloor \leqslant X \leqslant \lceil x \rceil < X + 1$

5) $\lfloor -x \rfloor = -\lceil x \rceil$ E $\lceil -x \rceil = -\lfloor x \rfloor$

RESUMINDO

$$\lfloor x \rfloor = n \iff x - 1 < n \leq x < n + 1$$

$$\lceil x \rceil = n \iff n - 1 < x \leq n < x + 1$$

- É POSSÍVEL MOVER UM INTEIRO PARA DENTRO DO PISO/TETO

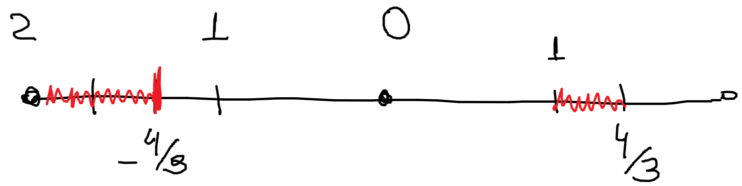
$$\lfloor x + n \rfloor = \lfloor x \rfloor + n, \qquad n \in \mathbb{Z}$$
$$\lceil x + n \rceil = \lceil x \rceil + n$$

- OUTRAS PROPRIEDADES, $x \in \mathbb{R}$, $n \in \mathbb{Z}$

$$\begin{array}{lcl}
x < n & & \lfloor x \rfloor < n \\
n < x & \iff & n < \lceil x \rceil \\
x \leq n & & \lceil x \rceil \leq n \\
n \leq x & & n \leq \lfloor x \rfloor
\end{array}$$

Parte Fracionária.

$$\{x\} = x - \lfloor x \rfloor$$



Ex: $\left\{\frac{3}{2}\right\} = \frac{1}{2}$

$$\left\{\frac{4}{3}\right\} = \frac{4}{3} - 1 = \frac{1}{3}$$

$$\left\{-\frac{4}{3}\right\} = -\frac{4}{3} - \left\lfloor -\frac{4}{3} \right\rfloor$$

$$= -\frac{4}{3} + 2 = \frac{2}{3}$$

Obs: $\{x\} + \{-x\} = \begin{cases} 0 & \text{se } x \in \mathbb{Z} \\ 1 & \text{se } x \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$

Note que se $x = n + \theta$, em que $n \in \mathbb{Z}$ e $0 \leq \theta < 1$,

então $\lfloor x \rfloor = n$ e $\{x\} = \theta$

SE $x, y \in \mathbb{R}$.

$$\lfloor x+y \rfloor = \lfloor \lfloor x \rfloor + \{x\} + \lfloor y \rfloor + \{y\} \rfloor$$

$$= \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor$$

SABEMOS QUE

$$0 \leqslant \{x\}, \{y\} < 1$$

$$0 \leqslant \{x\} + \{y\} < 2$$

$$\Rightarrow 0 \leqslant \lfloor \{x\} + \{y\} \rfloor \leqslant 1$$

EX: $\log_2 35$ $\quad \lceil \log_2 35 \rceil$

COMO $\underset{32}{2^5} < 35 < \underset{64}{2^6}$, TEMOS $5 < \log_2 35 < 6$

$$\Rightarrow \lceil \log_2 35 \rceil = 6$$

• QUAL O NÚMERO DE BITS NECESSÁRIO P/ ESCREVER n EM BINÁRIO $\lceil \log_2 n \rceil$?

EX: $0, 1 \rightsquigarrow 1$ BIT

$2 - 3 \rightsquigarrow 2$ BITS

$\vdots$

$2^k - 2^{k+1} - 1 \rightsquigarrow k+1$ BITS

SE n PRECISA DE m BITS, ENTÃO

$$2^{m-1} \leq n < 2^m$$

LOGO, $m - 1 \leq \log_2 n < m$. PORTANTO $\lfloor \log_2 n \rfloor = m - 1$

$$\Rightarrow m = \lfloor \log_2 n \rfloor + 1$$

O QUE ACONTECE COM $\lceil \lfloor x \rfloor \rceil$?

$x \in [m, m+1)$ $[m+1, m+2)$

ENTÃO $\lfloor x \rfloor = m$

FÁCIL: $\lfloor x \rfloor$, POIS $\lfloor x \rfloor$ JÁ É INTEIRO

EX: $\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \left\lfloor \sqrt{x} \right\rfloor$ É VERDADEIRO OU FALSO?

VERDADE QUANDO x É INTEIRO

$\lfloor x \rfloor = x - \{x\}$

TOME $m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor$. TEMOS $m \leq \sqrt{\lfloor x \rfloor} < m+1$

$$m^2 \leq \lfloor x \rfloor < (m+1)^2$$

$$x \geq (m+1)^2$$
$$\lfloor x \rfloor \geq (m+1)^2$$

$$m^2 \leq x - \{x\} < (m+1)^2$$

$$\underline{m^2 + \{x\}} \leq x < (m+1)^2 + \{x\}$$

ENTÃO $m^2 \leq x < (m+1)^2$

$$m \leq \sqrt{x} < m+1$$

$$m \leq \left\lfloor \sqrt{x} \right\rfloor < m+1 \Rightarrow \left\lfloor \sqrt{x} \right\rfloor = m$$

EXERCÍCIO: PROVE QUE $\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil$ PARA TODO $x \in \mathbb{R}$.

$g(x) = \frac{x+r}{m}$

$\lfloor \frac{x+r}{m} \rfloor = \lfloor g(x) \rfloor = \lfloor g(\lfloor x \rfloor) \rfloor = \lfloor \frac{\lfloor x \rfloor + r}{m} \rfloor$

PROP: SEJA f UMA FUNÇÃO CONTÍNUA, ESTRITAMENTE CRESCENTE

E TAL QUE

(*) SE $f(x) \in \mathbb{Z}$, ENTÃO $x \in \mathbb{Z}$

EX: $\sqrt[k]{\ }$, $\log_a$, $e \in \mathbb{Z}$

ENTÃO TEMOS $\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor$ E $\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$

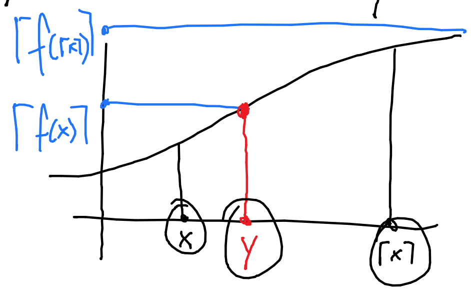
$f(x)$, $f(\lceil x \rceil)$, $(m-2, m-1]$, $(m-1, m]$

PROVA (TETO): SE $x = \lceil x \rceil$, NÃO HÁ O QUE FAZER: $\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$

ENTÃO PODEMOS SUPOR QUE $x < \lceil x \rceil$. COMO f É EST. CRESC., ENTÃO $f(x) < f(\lceil x \rceil)$

LOGO, TEMOS $\lceil f(x) \rceil \leq \lceil f(\lceil x \rceil) \rceil$

SE $\lceil f(x) \rceil < \lceil f(\lceil x \rceil) \rceil$, COMO f É CONTÍNUA ENTÃO EXISTE y COM $x \leq y < \lceil x \rceil$ T.Q. $f(y) = \lceil f(x) \rceil$



POR (*) COMO $f(y) \in \mathbb{Z}$, TEMOS $y \in \mathbb{Z}$

MAS ENTÃO y É UM INTEIRO EM $[x, \lceil x \rceil)$, UM ABSURDO

INTERVALOS

- INT. FECHADO : $[\alpha .. \beta] = \{x \in \mathbb{R} : \alpha \leq x \leq \beta\}$
- INT. ABERTO : $(\alpha .. \beta) = \{x \in \mathbb{R} : \alpha < x < \beta\}$
- INT SEMI ABERTOS : $(\alpha .. \beta]$ E $[\alpha , \beta)$

PERGUNTA: QUANTOS INTEIROS ESTÃO CONTIDOS EM TAIS INTERVALOS?

SE $\alpha, \beta \in \mathbb{Z}$, ENTÃO $[\alpha .. \beta)$ CONTÉM EXATAMENTE $\beta - \alpha$ INTEIROS

SE $\alpha, \beta \in \mathbb{R}$, E $n \in \mathbb{Z}$, TEMOS

$$\underline{n \in [\alpha .. \beta)} \iff \alpha \leq n < \beta \iff \lceil \alpha \rceil \leq n < \lceil \beta \rceil$$

$$\iff n \in \underline{[\lceil \alpha \rceil .. \lceil \beta \rceil)}$$ POSSUI $\lceil \beta \rceil - \lceil \alpha \rceil$ INTEIROS

$$n \in (\alpha .. \beta], \iff \alpha < n \leq \beta \iff \lfloor \alpha \rfloor < n \leq \lfloor \beta \rfloor$$

$$\iff n \in (\lfloor \alpha \rfloor .. \lfloor \beta \rfloor]$$

LOGO, $[\alpha .. \beta)$ E $(\alpha .. \beta]$ POSSUEM $\lceil \beta \rceil - \lceil \alpha \rceil$ E $\lfloor \beta \rfloor - \lfloor \alpha \rfloor$ INTEIROS, RESP.

$[\alpha..\beta)$ e $(\alpha..\beta]$ POSSUEM $\lceil\beta\rceil - \lceil\alpha\rceil$ e $\lfloor\beta\rfloor - \lfloor\alpha\rfloor$ INTEIROS, RESP.

ANALOGAMENTE

$[\alpha..\beta]$ POSSUI $\lfloor\beta\rfloor - \lceil\alpha\rceil + 1$ INTEIROS

$(\alpha..\beta)$ POSSUI $\lceil\beta\rceil - \lfloor\alpha\rfloor - 1$ INTEIROS

PROBLEMA: QUANTOS INTEIROS n ENTRE 1 E $\underbrace{1000000}_{10^6}$ SÃO DIVISÍVEIS POR $\lfloor \sqrt[3]{n} \rfloor$?

$$\left[k^2 \leq m < \frac{(k+1)^3}{k}\right]$$

$$\left[m \in \left[k^2 .. \frac{(k+1)^3}{k}\right)\right]$$

EX: SE $1 \leq n \leq 7$, $\lfloor \sqrt[3]{n} \rfloor = 1$

SE $8 \leq n \leq 26$, TEMOS $\sqrt[3]{n} < 3$. LOGO $\lfloor \sqrt[3]{n} \rfloor = 2$

ENTÃO CONTAMOS OS PARES.

$$\underset{\overset{\shortparallel}{10^4}}{k}, \underset{=10^2}{m}, \underset{=10^6}{n}$$

SEJA V O NÚMERO DE INTEIROS n ENTRE 1 E 1000000 DIVISÍVEIS POR $\lfloor \sqrt[3]{n} \rfloor$

$$V = \sum_{n=1}^{10^6} [n \text{ É DIVISÍVEL POR } \lfloor \sqrt[3]{n} \rfloor] = \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor]\,[k \backslash n]\,[1 \leq n \leq 10^6]$$

$[1;8)\ [8,27)\ [27,64)\ [64,125)\ \ldots\ [\ \ 10^6)\ 10^6$

$$= \sum_{k,m,n} [k^3 \leq n < (k+1)^3]\,[n = m \cdot k]\,[1 \leq n \leq 10^6]$$

$$\overset{10^6}{=} 1 + \sum_{k,m} [k^3 \leq m \cdot k < (k+1)^3]\,[1 \leq k < 10^2] = 1 + \sum_{k,m} \left[m \in \left[k^2 .. \frac{(k+1)^3}{k}\right)\right]\,[1 \leq k \leq 10^2]$$

SEJA V O NÚMERO DE INTEIROS n ENTRE 1 E 1000000 DIVISÍVEIS POR $\left\lfloor \sqrt[3]{n} \right\rfloor$

$$V = \sum_{n=1}^{10^6} \left[n \text{ é divisível por } \left\lfloor \sqrt[3]{n} \right\rfloor \right] = \sum_{k,n} \left[k = \left\lfloor \sqrt[3]{n} \right\rfloor \right] \left[k \backslash n \right] \left[1 \leq n \leq 10^6 \right]$$

$$= \sum_{k,m,n} \left[k^3 \leq n < (k+1)^3 \right] \left[n = m \cdot k \right] \left[1 \leq n \leq 10^6 \right]$$

$$\left\lceil \frac{k^3 + 3k^2 + 3k + 1}{k} \right\rceil$$

$$= 1 + \sum_{k,m} \left[k^3 \leq m \cdot k < (k+1)^3 \right] \left[1 \leq k \leq 10^2 \right]$$

$$= \left\lceil k^2 + 3k + 3 + \frac{1}{k} \right\rceil$$

$$= 1 + \sum_{k,m} \underline{\left[m \in \left[k^2 .. \frac{(k+1)^3}{k} \right) \right]} \left[1 \leq k \leq 10^2 \right]$$

$10^6 \rightarrow$?

$$= \cancel{k^2} + 3k + 3 + 1$$

$$- \cancel{k^2}$$

?

$$= 1 + \sum_{k=1}^{10^2} \left(\left\lceil \frac{(k+1)^3}{k} \right\rceil - \left\lceil k^2 \right\rceil \right) = 1 + \underline{\sum_{k=1}^{100} (3k + 4)} = 1 + 4 \cdot 100 + 3 \, \frac{101 \cdot 100}{2}$$

$$\zeta(K) = \sum_{J\geq 1} \frac{1}{J^K} = \frac{1}{1^K} + \frac{1}{2^K} + \frac{1}{3^K} + \frac{1}{4^K} + \cdots$$

$$S = x^2 + x^4 + x^6 + \cdots$$
$$x^2 S = x^4 + x^6 + \cdots$$
$$S = \frac{x^2}{1-x^2}$$

1) MOSTRE QUE $\displaystyle\sum_{K\geq 2} (\zeta(K) - 1) = 1$

$$\sum_{K\geq 2} (\zeta(K) - 1) = \sum_{K\geq 2} \sum_{J\geq 2} \frac{1}{J^K} = \sum_{J\geq 2} \sum_{K\geq 2} \frac{1}{J^K} = \sum_{J\geq 2} \frac{(1/J)^2}{(1-1/J)} = \sum \frac{1}{J^{\cancel{2}}} \cdot \frac{\cancel{J}}{J-1} = \sum_{J\geq 2} \frac{1}{J(J-1)}$$

$$= \sum_{J\geq 2} \frac{1}{J-1} - \frac{1}{J} = \sum_{J\geq 2} \frac{1}{J-1} - \sum_{J\geq 2} \frac{1}{J} = 1 + \sum_{J\geq 2} \frac{1}{J} - \sum_{J\geq 2} \frac{1}{J} = 1$$

$$\sum_{J\geq 2} \frac{1}{J-1} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \qquad \sum_{J\geq 2} \frac{1}{J} = \frac{1}{2} + \frac{1}{3} + \cdots$$

$$\frac{1}{J^2-1} = \frac{1}{(J-1)(J+1)} = \frac{1}{2}\left(\frac{1}{J-1} - \frac{1}{J+1}\right)$$

2) Qual o valor de $\displaystyle\sum_{K\geq 1} (\zeta(2K) - 1)$

$$\sum_{K\geq 1} (\zeta(2K) - 1) = \sum_{K\geq 1} \sum_{J\geq 2} \frac{1}{J^{2K}} = \sum_{J\geq 2} \frac{(1/J)^2}{1-(1/J)^2} = \sum_{J\geq 2} \frac{1}{J^2-1} = \frac{1}{2}\left(\sum_{J\geq 2} \frac{1}{J-1} - \sum_{J\geq 2} \frac{1}{J+1}\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$$

$$\zeta(K) = \sum_{J \geq 1} \frac{1}{J^K} = \frac{1}{1^K} + \frac{1}{2^K} + \frac{1}{3^K} + \frac{1}{4^K} + \cdots$$

(a) $S = x^2 + x^3 + x^4 \ldots$

$xS = x^3 + x^4 + \ldots$

$(1-x)S = x^2 \rightsquigarrow S = \frac{x^2}{1-x}$

(b) $S = x^2 + x^4 + x^6 + \cdots$

$x^2 S = x^4 + x^6 + \ldots$

$S = \frac{x^2}{1-x^2}$

1) MOSTRE QUE $\sum_{K \geq 2} (\zeta(K) - 1) = 1$

$x = \frac{1}{J}$

$$\sum_{K \geq 2} \frac{1}{J^K} = \sum_{K \geq 2} x^K \overset{(a)}{=} \frac{x^2}{1-x} = \frac{1/J^2}{1 - 1/J} = \frac{1}{J} \cdot \frac{1}{J-1}$$

$$= \frac{1}{J-1} - \frac{1}{J}$$

PROVA:

$$\sum_{K \geq 2} \sum_{J \geq 2} \frac{1}{J^K} = \sum_{J \geq 2} \underline{\sum_{K \geq 2} \frac{1}{J^K}} = \sum_{J \geq 2} \frac{1}{J-1} - \sum_{J \geq 2} \frac{1}{J}$$

$$= \sum_{J \geq 1} \frac{1}{J} - \sum_{J \geq 2} \frac{1}{J} = \frac{1}{1} = 1$$

$$\zeta(2K) = \sum_{J \geq 1} \frac{1}{J^{2K}} = \frac{1}{1} + \frac{1}{2^{2K}} + \frac{1}{3^{2K}} \cdots$$

2) Qual o valor de $\sum_{K \geq 1} (\zeta(2K) - 1)$

$x = \frac{1}{J}$

$$\sum_{K \geq 1} \sum_{J \geq 2} \frac{1}{J^{2K}} = \sum_{J \geq 2} \sum_{K \geq 1} \frac{1}{J^{2K}}$$

$$\sum_{K \geq 1} \frac{1}{J^{2K}} = \sum_{K \geq 1} x^{2K} = x^2 + x^4 + x^6 + \ldots$$

$$\overset{(b)}{=} \frac{x^2}{1-x^2} = \frac{1}{J^2 - 1} = \frac{1}{(J-1)(J+1)}$$

2) Qual o valor de $\sum_{k\geq 1} \left(\zeta(2k) - 1\right)$

$$\zeta(2k) = \sum_{j\geq 1} \frac{1}{j^{2k}} = \frac{1}{1} + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} \dots$$

$$\sum_{k\geq 1}\sum_{j\geq 2} \frac{1}{j^{2k}} = \sum_{j\geq 2}\sum_{k\geq 1} \frac{1}{j^{2k}}$$

$$x = \frac{1}{j}$$

$$\sum_{k\geq 1} \frac{1}{j^{2k}} = \sum_{k\geq 1} x^{2k} = x^2 + x^4 + x^6 + \dots$$

$$= \frac{x^2}{1-x^2} = \frac{1}{j^2-1} = \frac{1}{(j-1)(j+1)}$$

$$= \frac{-1/2}{j+1} + \frac{+1/2}{j-1}$$

$$= \frac{1}{2}\left(\frac{-(j-1) + (j+1)}{(j+1)(j-1)}\right)$$

$$\frac{1}{(j+1)(j-1)} = \frac{a}{j+1} + \frac{b}{j-1} = \frac{a(j-1) + b(j+1)}{(j+1)(j-1)}$$

$$a(j-1) + b(j+1) = (a+b)j + (b-a) = 1$$

$$\begin{cases} a+b = 0 \\ b-a = 1 \end{cases}$$

$$\begin{cases} 2b = 1 \\ 2a = -1 \end{cases}$$

$$= \frac{1}{2}\left(-\sum_{j\geq 2} \frac{1}{j+1} + \sum_{j\geq 2} \frac{1}{j-1}\right)$$

$$\sum^{\infty}_{1}$$

$$\frac{1}{3} + \cancel{\frac{1}{4}} + \dots \qquad \frac{1}{1} + \frac{1}{2} + \cancel{\frac{1}{3} + \frac{1}{4}}$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$$

ESPECTRO

- O ESPECTRO DE REAL α É MULTICONJUNTO → CONJUNTO QUE PODE CONTER MAIS DE UMA CÓPIA DO MESMO ELTO

$$SPEC(\alpha) = \{\lfloor\alpha\rfloor, \lfloor 2\alpha\rfloor, \lfloor 3\alpha\rfloor, \dots\}$$
$$= \{\lfloor k\alpha\rfloor : k \in \mathbb{N}\}$$

EX: $SPEC(1/3)$

```
for k ∈ range(15):
    print(floor(k * 1/3))
```

$$SPEC(1/3) = \{0, 0, 1, 1, 1, 2, 2, $$

k	1	2	3	4	5	6	7	8
	0	0	1	1	1	2	2	

EX: $SPEC(3.14) = \{$

NOTE QUE SE $\alpha \neq \beta$ ENTÃO $\text{SPEC}(\alpha) \neq \text{SPEC}(\beta)$

$\beta > \alpha \rightsquigarrow m(\beta - \alpha) \geq 1$ (TOME $m \geq \left\lceil \frac{1}{\beta - \alpha} \right\rceil$)

$m\beta \geq m\alpha + 1$

TEMOS $m\beta \geq m\alpha + 1$ PORTANTO $\lfloor m\beta \rfloor \geq \lfloor m\alpha \rfloor + 1$

COMPAREMOS OS ESPECTROS DE $\sqrt{2}$ E $2+\sqrt{2}$

$\text{SPEC}(\sqrt{2}) = \{1, 2, 4, 5, 7, 8, 9, \ldots\}$

$\text{SPEC}(2+\sqrt{2}) = \{3, 6, 10, 13, 17, 20, 23$ $\quad 23 = 9 + 2\cdot 7$

$7 + 2\cdot 5 \qquad 8 + 2\cdot 6$

n-ÉSIMO TERMO DE $\text{SPEC}(\sqrt{2})$

OBS: O n-ÉSIMO TERMO EM $\text{SPEC}(2+\sqrt{2})$ É $\lfloor n(2+\sqrt{2}) \rfloor = 2n + \underline{\lfloor n\sqrt{2} \rfloor}$

$\text{SPEC}(\sqrt{2})$ E $\text{SPEC}(2+\sqrt{2})$ PARTICIONAM OS INTEIROS

PROP: $\operatorname{SPEC}(\sqrt{2})$ E $\operatorname{SPEC}(2+\sqrt{2})$ PARTICIONAM OS INTEIROS

$$\operatorname{SPEC}(\sqrt{2}) \cap \operatorname{SPEC}(2+\sqrt{2}) = \emptyset$$

$$\operatorname{SPEC}(\sqrt{2}) \cup \operatorname{SPEC}(2+\sqrt{2}) = \mathbb{N}$$

PROVA: DENOTE $N(\alpha, n)$ O NÚMERO DE ELTS $\leq n$ EM $\operatorname{SPEC}(\alpha)$

$$N(\alpha,n) = \sum_{k>0} \left[\lfloor k\alpha \rfloor \leq n\right] = \sum_{k>0}\left[\lfloor k\alpha\rfloor < n+1\right] = \sum_{k>0} \underbrace{\left[k\alpha < n+1\right]}_{\left[k < \frac{n+1}{\alpha}\right]}$$

$$= \sum_{k}\left[1 \leq k < \frac{n+1}{\alpha}\right] = \left\lceil \frac{n+1}{\alpha} \right\rceil - 1$$

$$(n+1)\left(\frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}}\right)$$

$$(n+1)\left(\frac{2+\sqrt{2}+\sqrt{2}}{\sqrt{2}\,(2+\sqrt{2})}\right)$$

ASSIM, TEMOS

$$\frac{N(\sqrt{2}, n) + N(2+\sqrt{2}, n)}{\text{INTEIRO!}} = \left\lceil \frac{n+1}{\sqrt{2}} \right\rceil + \left\lceil \frac{n+1}{2+\sqrt{2}} \right\rceil - 2$$

$$= \left\lfloor \frac{n+1}{\sqrt{2}} \right\rfloor + \left\lfloor \frac{n+1}{2+\sqrt{2}} \right\rfloor = \frac{n+1}{\sqrt{2}} - \left\{\frac{n+1}{\sqrt{2}}\right\} + \frac{n+1}{2+\sqrt{2}} - \left\{\frac{n+1}{2+\sqrt{2}}\right\}$$

$$= n+1 \underbrace{- \left\{\frac{n+1}{\sqrt{2}}\right\} - \left\{\frac{n+1}{2+\sqrt{2}}\right\}}_{=1}$$

$0 < \quad < 1 \qquad 0 < \quad < 1$

$$= n$$

$$(n+1)\left(\frac{1}{\sqrt{2}}+\frac{1}{2+\sqrt{2}}\right)$$

$$(n+1)\left(\frac{2+\sqrt{2}+\sqrt{2}}{\sqrt{2}(2+\sqrt{2})}\right)=(n+1)\left(\frac{2+2\sqrt{2}}{2\sqrt{2}+2}\right)=n+1$$

MOD

- DIVISÃO INTEIRA

- DIVIDIR n POR m SIGNIFICA ENCONTRAR $q, r \in \mathbb{N}$ T.Q.

$$n = q \cdot m + r \quad \text{E} \quad \underline{0 \leq r < m}$$

- TEMOS $q = \left\lfloor \frac{n}{m} \right\rfloor$, QUEREMOS UMA NOTAÇÃO PARA r

$\leadsto$ DENOTAMOS POR $r = n \text{ MOD } m$

$\left\lfloor \frac{5}{-3} \right\rfloor = \left\lfloor -\frac{5}{3} \right\rfloor - 2$

- ASSIM, TEMOS $n = m \left\lfloor \frac{n}{m} \right\rfloor + n \text{ MOD } m$

- EM PARTICULAR, $n \text{ MOD } m = n - m \left\lfloor \frac{n}{m} \right\rfloor$

EX:

$5 \text{ MOD } 3 = 5 - 3 \left\lfloor \frac{5}{3} \right\rfloor = 2$

$5 \text{ MOD } -3 = 5 + 3 \left\lfloor \frac{5}{-3} \right\rfloor = 5 - 6 = -1$

$-5 \text{ MOD } 3 = -5 - 3 \left\lfloor \frac{-5}{3} \right\rfloor = 1$

$-5 \text{ MOD } -3 = -5 + 3 \left\lfloor \frac{-5}{-3} \right\rfloor = -2$

ASSIM, TEMOS

$0 \leq x \bmod y < y$ PARA $y > 0$

$0 \geq x \bmod y > y$ PARA $y < 0$

$$n \bmod m = n - m\left\lfloor \frac{n}{m} \right\rfloor$$

$$x = \lfloor x \rfloor + \{x\}$$

$$\{x\} = x - 1\lfloor x/1 \rfloor$$

- DEFINIMOS $x \bmod 0 = x$
- A PARTE FRACIONÁRIA PODE SER ESCRITA COMO $x \bmod 1$
- DISTRIBUTIVIDADE: $c(x \bmod y) = (cx) \bmod (cy)$

PROVA: $c(x \bmod y) = c\left(x - y\left\lfloor \frac{x}{y} \right\rfloor\right) = cx - cy\left\lfloor \frac{x}{y} \right\rfloor = cx - cy\left\lfloor \frac{cx}{cy} \right\rfloor$

$$= (cx) \bmod (cy) \quad \blacksquare$$

$$= \frac{cx}{cy}$$

n ELTS EM m CAIXAS

"BALANCEADO" = A DIF. ENTRE DUAS CAIXAS É NO MÁX 1.

TEMOS $\left\lfloor \frac{n}{m} \right\rfloor$ OU $\left\lceil \frac{n}{m} \right\rceil$ ELTS

↳ EXATAMENTE EM $n \bmod m$ CAIXAS

- NÃO É DIFÍCIL VERIFICAR QUE A K-ÉSIMA CAIXA POSSUI $\left\lceil \frac{n-k+1}{m} \right\rceil$ ELTS

$$\underbrace{\left\lceil \frac{n}{m} \right\rceil \left\lceil \frac{n}{m} \right\rceil \left\lceil \frac{n}{m} \right\rceil}_{n \bmod m} \left\lfloor \frac{n}{m} \right\rfloor$$

$n - m\left\lfloor \frac{n}{m} \right\rfloor$ $\quad k = n \bmod m + 1$

$$\frac{n - (n \bmod m) \cancel{+1}}{m}$$

- ISSO DÁ A SEGUINTE FÓRMULA:

$$n = \left\lceil \frac{n}{m} \right\rceil + \left\lceil \frac{n-1}{m} \right\rceil + \left\lceil \frac{n-2}{m} \right\rceil + \cdots + \left\lceil \frac{n-m+1}{m} \right\rceil = \sum_{k=1}^{m} \left\lceil \frac{n-k+1}{m} \right\rceil$$

$$= \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n+1}{m} \right\rfloor + \cdots + \left\lfloor \frac{n+m-1}{m} \right\rfloor = \sum_{k=1}^{m} \left\lfloor \frac{n+k-1}{m} \right\rfloor$$

EX: $\sum_{0 \leq k < m} \left\lfloor \frac{nk+x}{m} \right\rfloor$ com $n, m \in \mathbb{Z}$ e $x \in \mathbb{R}$ e $m > 0$

$$\left\lfloor \frac{\lfloor x \rfloor}{m} \right\rfloor^{+r} = \left\lfloor \frac{x}{m} \right\rfloor^{+r}$$

CASOS PEQUENOS

$n=0$, temos $\left\lfloor \frac{x}{m} \right\rfloor + \left\lfloor \frac{x}{m} \right\rfloor + \dots + \left\lfloor \frac{x}{m} \right\rfloor = m \left\lfloor \frac{x}{m} \right\rfloor$

$n=1$, temos $\left\lfloor \frac{x}{m} \right\rfloor + \left\lfloor \frac{x+1}{m} \right\rfloor + \dots + \left\lfloor \frac{x+m-1}{m} \right\rfloor = \lfloor x \rfloor$

$$\left\lfloor \frac{\lfloor x \rfloor}{m} \right\rfloor + \left\lfloor \frac{\lfloor x \rfloor + 1}{m} \right\rfloor + \dots + \left\lfloor \frac{\lfloor x \rfloor + m - 1}{m} \right\rfloor = \lfloor x \rfloor$$

Usando $n = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n+1}{m} \right\rfloor + \dots + \left\lfloor \frac{n+m-1}{m} \right\rfloor$ com $n = \lfloor x \rfloor$

$m=1$ temos $\lfloor x \rfloor$

$m=2$ temos dois casos:

n par: $\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{n+x}{2} \right\rfloor = 2 \left\lfloor \frac{x}{2} \right\rfloor + \frac{n}{2}$

n ímpar: $\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+n}{2} \right\rfloor = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} + \frac{n-1}{2} \right\rfloor = \overbrace{\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor}^{\lfloor x \rfloor} + \frac{n-1}{2} = \lfloor x \rfloor + \frac{n-1}{2}$

$$\sum_{0 \le k < m} \left\lfloor \frac{mk + x}{m} \right\rfloor$$

$$\left\lfloor \frac{x}{m} \right\rfloor + \left\lfloor \frac{x+1}{m} \right\rfloor + \cdots + \left\lfloor \frac{x+m-1}{m} \right\rfloor = \lfloor x \rfloor$$

QUANDO $m = 3$, TEMOS $\left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x+m}{3} \right\rfloor + \left\lfloor \frac{x+2m}{3} \right\rfloor$

n MOD $3 = 0$, $n/3$ E $2n/3$ SÃO INTEIROS, TEMOS $3\left\lfloor \frac{x}{3} \right\rfloor + n$

n MOD $3 = 1$: $\frac{n-1}{3}$ E $\frac{2n-2}{3}$ SÃO INTEIROS,

$$\left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x+1}{3} \right\rfloor + \frac{n-1}{3} + \left\lfloor \frac{x+2}{3} \right\rfloor + \frac{2n-2}{3} = \lfloor x \rfloor + n - 1$$

$\lfloor x \rfloor + n - 1$

n MOD $3 = 2$: $\frac{n-2}{3}$ $\frac{2n-4}{3}$

ESTAMOS OBTENDO ALGO $a\left\lfloor \frac{x}{a} \right\rfloor + bm + c$ COM $b = \frac{m-1}{2}$ E $a = MDC(m, n)$

MÁXIMO
DIVISOR
COMUM