On a Graph Calculus for Algebras of Relations*

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Abstract. We present a sound and complete logical system for deriving inclusions between graphs from inclusions between graphs, taken as hypotheses. Graphs provide a natural tool for expressing relations and reasoning about them. Here we extend this system to a sound and complete one to cope with proofs from hypotheses. This leads to a system dealing with complementation. Other approaches using pictures for relations use as bases the theory of allegories or rewriting systems. Our formalism is more widely applicable and provides a common denominator of these approaches.

Key words: Relational language, reasoning from hypotheses, graph calculus, completeness, complementation.

1 Introduction

This paper presents a sound and complete logical system for deriving inclusions between graphs from a set of inclusions between graphs, taken as hypotheses. Traditionally, formulas are written on a single line. S. Curtis and G. Lowe [4] suggest a more visually appealing alternative: using graphs for expressing relations and reasoning about them in a natural way. Although Curtis and Lowe give motivation, present the ideas of using graphs to prove inclusions between relations and illustrate how to apply such a calculus to justify the inference of an inclusion from a set of hypotheses, no proper treatment of these ideas as a logical system seems to have been presented. A proper formulation of the logical system +RG was presented [6]: a playful logical calculus to derive graphs from graphs, shown to be sound, complete and decidable, for the valid inclusions without the empty relation and complementation.

The main motivation of the present work is to obtain a graph calculus that can be applied to algebras of relations. As complementation can be defined in terms of intersection, union, the universal and the empty relations, here we extend the system +RG to cope with the empty relation and derivations from

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hypotheses. This is more than a simple and natural extension. The proof of completeness for this non-decidable system involves much more elaborated work.

Pictures have been proposed by some authors as a tool to help investigating and applying relational formalisms. Here, we mention two main lines.

The approach based on the *theory of allegories* [1–4, 11] views pictures as arrows in a (unitary pretabular) allegory [8] and uses laws directly associated to the valid allegorical identities for transforming pictures. Results of the theory of allegories are used to show that two pictures can be proved equal by using the laws on pictures iff they represent the same relation.

The approach based on the *rewriting systems* [12–14] endows pictures with a relational semantics, which allows them to be interpreted as terms of an algebraic language. A rewriting mechanism for pictures is built as a variant of the algebraic approach to graph rewriting. The way one can use rewriting sequences as proofs leads to a general and flexible tool for the proof of relational algebraic identities.

Our approach [3, 6, 7] may be called the *logic systematic* approach: pictures are considered as ordinary formulas of a (non-orthodox) logical system and a set of inference rules is provided for deriving pictures from pictures. This approach emphasizes notions of normal form and homomorphism for pictures, which are used to prove the inclusions and equalities.

Each one of these approaches has its own flavor, techniques of investigations and line of results. Nevertheless, they are not completely disjoint, sharing characteristics whose interactions deserve further investigation. The work reported here may also be viewed as a contribution in this direction. We thus provide a new formalism, which is more widely applicable and provides a common denominator of the above three lines of investigation.

The structure of this paper is as follows. In Section 2, we briefly review the relational framework. In Section 3, we introduce our non-negative graph relational framework. In Section 4, we present a derivation system for our graph relational framework and examine some of its aspects: strategy for derivations, soundness and completeness. In Section 5, we indicate how to extend it to handle complementation. In Section 6, we show an application of the graph calculus: proving the main result in [9] as a corollary of our completeness result. In the concluding section, we comment on some on-going work and perspectives. The Appendix complements the main text with figures and proofs of the results.

2 Relational Framework

We now briefly review the relational framework.

Abstractly, relation algebras can be defined by a set of identities specifying the behavior of the Boolean and Peircean operators as follows. The former operators behave as in Boolean algebras. The latter operators behave as in involuted monoid theory. One also adds an identity expressing a geometric aspect of the interaction of Boolean and Peircean operators [10, 16].

The non-negative relational language RL^- is the fragment of the relational algebra language with no occurrences of complementation. The RL^- *terms* or simply *terms*, typically denoted by R, S, T, are generated from the set of relational variables $\textsc{Rvar} = \{r_i : i \in \omega\}$ by applying the relational operators E, O, I, $^{\intercal}$, $\sqcap$, $\sqcup$, and $\circ$, as usual. We use $\mathcal{T}$ for the set of all (RL^-) terms. The *non-negative relational inclusions* and *equalities* are the expressions of the forms $R \sqsubseteq S$ and $R \equiv S$, respectively.

A *model* is a pair $\mathcal{M} = (M, r_i^{\mathcal{M}})_{i\in\omega}$, where M is a nonempty universe and $r_i^{\mathcal{M}} \subseteq M \times M$ for every $i \in \omega$. The *meaning* $[\![R]\!]_{\mathcal{M}}$ of a term R in a model $\mathcal{M}$ is defined as in the relational case (excluding all references to complementation). Formally, given a model $\mathcal{M}$ with universe M, we interpret the symbols as follows: symbols E, O and I as the relations $M^2 := M \times M$, the empty relation and $\{(a, a) : a \in M\}$, respectively, and symbols $\sqcap$, $\sqcup$, $^{\intercal}$ and $\circ$ as intersection, union, conversion and composition of relations, respectively. Satisfaction, consequence and validity are defined as usual: $\mathcal{M} \models R \sqsubseteq S$ iff $[\![R]\!]_{\mathcal{M}} \subseteq [\![S]\!]_{\mathcal{M}}$ and $\mathcal{M} \models R \equiv S$ iff $[\![R]\!]_{\mathcal{M}} = [\![S]\!]_{\mathcal{M}}$. Given a set of inclusions Δ, we use $\mathsf{Mod}\,\Delta$ for the class of models satisfying every inclusion in Δ and we define consequence by $\Delta \models R \sqsubseteq S$ iff $\mathsf{Mod}\,\Delta \subseteq \mathsf{Mod}\,\{R \sqsubseteq S\}$. The validities are the consequences of the empty set.

3 Non-negative Graph Relational Framework

We now introduce syntax and semantics of our graph relational framework.

In the non-negative graph relational framework RG^- relations are represented by (directed pseudo multi) graphs having two distinguished nodes and arcs labeled by non-negative relational terms. We consider a fixed set of *nodes* $\textsc{Inod} = \{x_n : n \in \omega\}$, typically denoted by x, y, z, u, v, w.

A *slice* is a structure $S = (N, A, x, y)$, where N is a finite nonempty set of nodes; $A \subseteq N \times \mathcal{T} \times N$ is a set of labeled *arcs* ($\mathcal{T}$ is the set of all terms), with x, y being, not necessarily distinct, distinguished nodes in N. A *non-negative relational graph*, or simply a *graph*, is a finite set of slices. We often identify a single-slice graph with its slice.

We call a slice *basic* iff the labels of its arcs are relational variables or I and a *basic* graph is one whose slices are basic. The RG^- *inclusions* and *equalities* are expressions of the forms $G \sqsubseteq H$ and $G \equiv H$, respectively.

We now present semantics: slices and graphs denote binary relations.

Given a slice $S = (N, A, x, y)$ and a model $\mathcal{M}$ with universe M, an $\mathcal{M}$-*assignment* for S, denoted by $g : S \to \mathcal{M}$, is a function $g : N \to M$ such that $(gu, gv) \in [\![R]\!]_{\mathcal{M}}$ for every arc uRv in A. Now, the *meaning* of a slice S in a model $\mathcal{M}$ is the binary relation $[\![S]\!]_{\mathcal{M}}$ on M defined by $(a, b) \in [\![S]\!]_{\mathcal{M}}$ iff $gx = a$ and $gy = b$, for some assignment $g : S \to \mathcal{M}$. The *meaning* of a graph G in a model $\mathcal{M}$, $[\![G]\!]_{\mathcal{M}}$, is the union of the meanings of its slices.

We now extend some notions for terms to graphs. *Satisfaction* is as expected: $\mathcal{M} \models G \sqsubseteq H$ iff $[\![G]\!]_{\mathcal{M}} \subseteq [\![H]\!]_{\mathcal{M}}$ and $\mathcal{M} \models G \equiv H$ iff $[\![G]\!]_{\mathcal{M}} = [\![H]\!]_{\mathcal{M}}$. Given a set of inclusions Γ, we use $\mathsf{Mod}\,\Gamma$ for the class of models satisfying every inclusion in Γ, and we define *consequence* by $\Gamma \models G \sqsubseteq H$ iff $\mathsf{Mod}\,\Gamma \subseteq \mathsf{Mod}\,\{G \sqsubseteq H\}$. The *validities* are the consequences of the empty set. Also, graphs G and H are *equivalent* iff $G \equiv H$ is valid.

We connect the relational and graphical frameworks by associating single-slice graphs to terms. Given a term R, its graph is $G_R := \{(\{x,y\},\{xRy\},x,y)\}$. Note that R and G_R have the same meaning in every model.

4 Non-negative Graph Relational Calculus

We now introduce a derivation system for our graph relational framework.

The deductive apparatus of RG^- is given by a set of graph transforming rules: some rules transform a graph into an equivalent one, and a rule to compare graphs. To state the transformation rules, we use the *node substitution* notation $\frac{u}{v}$ for replacing u by v, which we extend naturally to pairs and triples as well as sets, e.g., for a set A of arcs, we put $A\frac{u}{v} := \{w\frac{u}{v}Rz\frac{u}{v} : wRz \in A\}$.

The *transformation rules* are given in Tables 1, 2 and 3. Table 1 covers the labels of the graphs. Table 2 gives the crucial rule for comparing graphs. Table 3 introduces the new rule for hypotheses. The rules in Tables 1 and 3 can be applied in both directions; each one of these rules is an abbreviation for two rules: downward and upward. The rules in Table 1 allow the *elimination* (downwards) and the *introduction* (upwards) of the operators.

Table 1. Elimination/Introduction rules for transforming graphs.

$$\mathsf{Unv}\quad \frac{G \cup \{(N, A \cup \{u\mathsf{E}v\}, x, y)\}}{G \cup \{(N, A, x, y)\}}$$

$$\mathsf{Idn}\quad \frac{G \cup \{(N, A \cup \{u\mathsf{I}v\}, x, y)\}}{G \cup \{(N\frac{u}{v}, A\frac{u}{v}, x\frac{u}{v}, y\frac{u}{v})\}}$$

$$\mathsf{Cnv}\quad \frac{G \cup \{(N, A \cup \{uR^{\mathsf{T}}v\}, x, y)\}}{G \cup \{(N, A \cup \{vRu\}, x, y)\}}$$

$$\mathsf{Int}\quad \frac{G \cup \{(N, A \cup \{uR \sqcap Sv\}, x, y)\}}{G \cup \{(N, A \cup \{uRv, uSv\}, x, y)\}}$$

$$\mathsf{Cmp}\quad \frac{G \cup \{(N, A \cup \{uR \circ Sv\}, x, y)\}}{G \cup \{(N \cup \{w\}, A \cup \{uRw, wSv\}, x, y)\}} \quad \text{if } w \notin N$$

$$\mathsf{Uni}\quad \frac{G \cup \{(N, A \cup \{uR \sqcup Sv\}, x, y)\}}{G \cup \{(N, A \cup \{uRv\}, x, y), (N, A \cup \{uSv\}, x, y)\}}$$

$$\mathsf{Vd}\quad \frac{G \cup \{(N, A \cup \{u\mathsf{O}v\}, x, y)\}}{G}$$

We will explain each two-way rule in the downward direction. Each rule in Table 1 involves the application of the local transformation specified in the rule, leaving the rest of the graph untouched. The meaning of the graph is to remain unchanged. Soundness of the rules follows from these explanations.

Rules in Table 1 are similar to those of +RG [7], with one new rule to deal with the O operator.

Rule Unv allows erasing an arc labeled by E from a slice. Rule Idn allows erasing an arc $u\mathsf{I}v$ and a node u, renaming nodes and redirecting arcs accordingly. Rule Cnv allows replacing arcs: $uR^{\intercal}v$ by vRu. Rule Int allows replacing an arc $uR \sqcap Sv$ by arcs uRv and uSv. Rule Cmp allows replacing an arc $uR \circ Sv$ by arcs uRw and wSv, with a new node w. Rule Uni allows replacing a slice T having an arc $uR \sqcup Sv$ by two other slices T_R and T_S, obtained from T by replacing the arc $uR \sqcup Sv$ by new arcs uRv and uSv, respectively. The new rule Vd allows erasing a slice having an arc $u\mathsf{O}v$.

We can reduce each graph G to a basic graph βG equivalent to G by applying the elimination rules in Table 1 (except Idn).

The next example will illustrate the idea of arc preservation and motivate the Graph Cover rule (GrCvr, in Table 2).

Example 1. Consider the terms $R := r \circ (s \sqcap t)$ and $S := (r \circ s) \sqcap (r \circ t)$. To establish the term inclusion $R \sqsubseteq S$, we form the corresponding term graphs G_R and G_S and reduce them to basic forms βG_R and βG_S, shown in Figure 1. Now, consider the node mapping $\theta : x_S \mapsto x_R$, $y_S \mapsto y_R$, $u \mapsto w$, $v \mapsto w$. We see that it preserves arcs, mapping arcs in βG_S to arcs in βG_R. So, we will be able to finish the derivation by applying the Graph Cover rule.

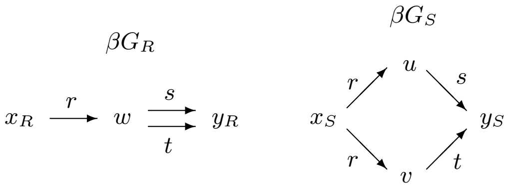

Fig. 1.

Given slices $S = (N, A, x, y)$ and $S' = (N', A', x', y')$, a *draft homomorphism* $\theta : S' \xrightarrow{d} S$ is a function $\theta : N' \to N$ such that if $u'Rv' \in A'$, then $\theta u'R\theta v' \in A$. A *homomorphism* $\theta : S' \to S$ is a draft homomorphism $\theta : S' \xrightarrow{d} S$ such that $\theta x' = x$ and $\theta y' = y$. Given graphs G and H, we say that H *covers* G, denoted by $G \leftarrow H$, iff for each slice $S \in G$ there exist a slice $T \in H$ and a homomorphism $\theta : T \to S$. The downward rule GrCvr (Table 2), allows one to replace a graph by another one covering it.

Soundness of GrCvr follows since draft homomorphisms transfer assignments by composition: if $\theta : T \xrightarrow{d} S$ and $g : S \to \mathcal{M}$, then $g.\theta : T \to \mathcal{M}$.

Table 2. Graph Cover rule.

$$\mathsf{GrCvr}\ \frac{G}{H} \quad \text{if } G \leftarrow H$$

The next example will illustrate the idea of "gluing" slices and motivate the hypothesis rule Hyp_Γ (Table 3).

Example 2. Consider the inclusion $(r \circ s) \sqcap (r \circ t) \sqsubseteq r \circ (s \sqcap t)$. It is well-known that this inclusion is not valid, but it does hold if r is functional, which can be expressed by the inclusion $r^\top \circ r \sqsubseteq \mathsf{I}$. With the notation of Example 1, we wish to derive the inclusion $S \sqsubseteq R$ from the hypothesis $T_1 \sqsubseteq T_2$, where $T_1 := r^\top \circ r$ and $T_2 := \mathsf{I}$. As before, we reduce the question to graph inclusions: deriving $G_S \sqsubseteq G_R$ from $G_{T_1} \sqsubseteq G_{T_2}$. The corresponding basic forms appear in Figures 1 and 2. Now, there is no homomorphism from βG_R to βG_S: there is no node in βG_S to map w to. But, the lefthand side βG_{T_1} of the hypothesis occurs in βG_S. If we add to this occurrence the righthand side βG_{T_2}, we can then reduce the result to a slice into which we can find a homomorphism from βG_R. Indeed, we have a draft homomorphism $\theta : \beta G_{T_1} \xrightarrow{d} \beta G_S$ given by $\theta x_{T_1} := u$ $\theta y_{T_1} := v$ and $\theta z := y_S$. We now "glue" slice βG_{T_2} onto βG_S using the nodes $\theta x_{T_1} = u$ and $\theta y_{T_1} = v$, obtaining slice T in Figure 2. Now, an application of the downward Idn rule to T yields a slice isomorphic to βG_R.

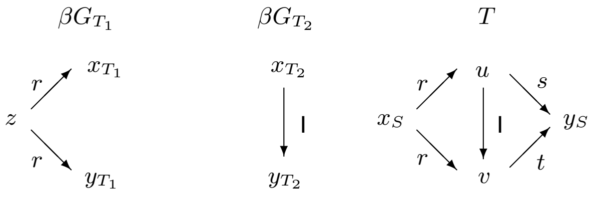

Fig. 2. "Glued" slice: βG_{T_2} into βG_S.

The rule for hypotheses uses a few concepts which we now introduce. Consider slices $S = (N_S, A_S, x, y)$ and $T = (N_T, A_T, w, z)$, as well as designated nodes $u, v \in N_S$. The result of *gluing* T onto S via (u, v) is the slice defined by $\mathsf{glue}_{(u,v)}(T, S) := (N_S \uplus N_T{}^{\frac{w}{u}\frac{z}{v}}, A_S \uplus A_T{}^{\frac{w}{u}\frac{z}{v}}, x, y)$. One glues a graph H by gluing its slices: $\mathsf{glue}_{(u,v)}(H, S) := \{\mathsf{glue}_{(u,v)}(T, S) : T \in H\}$. Now, given a slice $S' = (N', A', x', y')$ and a draft homomorphism $\theta : S' \xrightarrow{d} S$, we set $\mathsf{glue}_{\theta}(H, S) := \mathsf{glue}_{(\theta x', \theta y')}(H, S)$.

Rule Hyp_{Γ} allows one to glue a graph in a slice of a graph.

Table 3. Hypothesis rule.

$$\mathsf{Hyp}_{\Gamma}\ \frac{G \cup \{S\}}{G \cup \mathsf{glue}_{\theta}(H, S)} \quad \text{if } G' \cup \{S'\} \sqsubseteq H \in \Gamma \text{ and } \theta : S' \xrightarrow{d} S$$

The notion of derivation is standard. By a *Γ-derivation* we mean a sequence $G_0, \ldots, G_n$ of graphs such that for each $i \in \{1, \ldots, n\}$, graph G_i is obtained from graph G_{i-1} by application of one of the rules in Tables 1, 2 and 3. We say that a graph inclusion $G \sqsubseteq H$ is *derivable from a set Γ of graph inclusions*, denoted by $\Gamma \vdash G \sqsubseteq H$, iff there is a Γ-derivation $G_0, \ldots, G_n$ such that $G_0 = G$ and $G_n = H$. Examples of derivations (without hypotheses) appear in [6] and [7].

Soundness of the rules in Tables 1 and 2 is clear from the explanations given above. Soundness of rule Hyp_{Γ} in Table 3 follows from Lemma 1, whose proof is given in the Appendix.

Lemma 1. *If $\mathcal{M} \models G' \cup \{S'\} \sqsubseteq H$, then $[\![S]\!]_{\mathcal{M}} \subseteq [\![\mathsf{glue}_{\theta}(H, S)]\!]_{\mathcal{M}}$, for any slice S and draft homomorphism $\theta : S' \xrightarrow{d} S$.*

Now, to establish a graph inclusion $G \sqsubseteq H$ from a set Γ of graph inclusions, all in basic form, we can use the strategy of iterating both removal from G of every slice covered by H and application of Hyp_{Γ} to a slice in G. If and when graph G has no more slices, we can conclude that $\Gamma \models G \sqsubseteq H$. The reason is as follows. Starting with the graphs $G_0^- := G$ and $G_0^+ := \emptyset$, we have graphs such that $\Gamma \vdash G \sqsubseteq G_n^- \cup G_n^+$ with G_n^+ covered by H. Thus, if $G_n^- = \emptyset$, we have $\Gamma \vdash G \sqsubseteq G_n^+$, and, by the graph cover rule GrCvr, $\Gamma \vdash G_n^+ \sqsubseteq H$. Hence, when this construction does terminate, we have $\Gamma \vdash G \sqsubseteq H$. Figure 3 in the Appendix gives a gist of this construction.

The above construction may fail to terminate. A simple example, based on the density-like term inclusion $r \sqsubseteq r \circ r$, will illustrate why. Imagine that the graph inclusion $G_r \sqsubseteq G_{r \circ r}$ is in Γ. The application of $G_r \sqsubseteq \beta G_{r \circ r}$ to a slice will produce a larger slice still having G_r within it, so this may lead to an infinite sequence of increasing slices. See also Figure 4 in the Appendix.

We show (in the Appendix) that, whenever the above construction fails to terminate, we have such an infinite sequence of increasing basic slices, from which we can obtain a canonical counter-model.

To define a canonical model, we need an auxiliary concept. Given a slice $S = (N, A, x, y)$, we define the relation $\sim_S$ on N by $u \sim_S v$ iff there is an arc

$u\mathsf{I}v \in A$. We use $\sim_S^*$ for the equivalence closure of $\sim_S$. The *canonical model* $\mathcal{S}$ based on a sequence $(S_n)_{n\in\omega}$ of increasing basic slices is defined as follows. Form the set N of all nodes occurring in the sequence and define the (equivalence) relation on N by $u \sim v$ iff $u {\sim}_{S_n}^* v$, for some n. The universe is the quotient $\tilde{N}$ of N by $\sim$, with natural map $q : N \to \tilde{N}$. Now define the relations by $(\tilde{u}, \tilde{v}) \in r^{\mathcal{S}}$ iff there exist n such that the arc $u'rv'$ is in S_n, with $u' \sim u$ and $v' \sim v$.

The canonical model has the factorization property: given a basic slice T, a function $g : N_T \to \tilde{N}$ is an assignment $g : T \to \mathcal{S}$ iff there exists a draft homomorphism $\theta : T \xrightarrow{d} S_n$, such that $g = q.\theta$. This leads to the crucial property of the canonical model: $\mathcal{S} \models \Gamma$ and $(\tilde{x}_{S_0}, \tilde{y}_{S_0}) \in [\![H]\!]_{\mathcal{S}}$ iff H covers some S_n, for any basic graph H. We then have completeness of our graph calculus.

Theorem 1. *If $\Gamma \models G \sqsubseteq H$, then $\Gamma \vdash G \sqsubseteq H$.*

The following corollary shows that the graph calculus achieves the aim it was originally designed for.

Corollary 1. *Let $\{R_i \sqsubseteq S_i : i \in I\} \cup \{R \sqsubseteq S\}$ be a set of relational inclusions. Then $\{R_i \sqsubseteq S_i : i \in I\} \models R \sqsubseteq S$ iff $\{G_{R_i} \sqsubseteq G_{S_i} : i \in I\} \vdash G_R \sqsubseteq G_S$.*

5 Full-relational Graph Calculus

We now indicate how to handle complementation, so as to obtain a full-relational graph calculus. We know that the complement of a relation $X \subseteq M \times M$ can be characterized as the unique relation $Y \subseteq M \times M$ such that $X \cap Y \subseteq \emptyset$ and $M \times M \subseteq X \cup Y$. So, whenever we have a complemented term $\overline{R}$ within the label of an arc, we select a new relational variable r_R, replace $\overline{R}$ by r_R and add the two hypotheses $R \sqcap r_R \sqsubseteq \mathsf{O}$ and $\mathsf{E} \sqsubseteq R \sqcup r_R$.

Example 3. Consider the inclusion $(r^{\top} \circ \overline{r \circ s}) \sqcap s \sqsubseteq \mathsf{O}$. We will indicate how one can establish it, using $(r \circ s) \sqcap \overline{r \circ s} \sqsubseteq \mathsf{O}$ and $\mathsf{E} \sqsubseteq (r \circ s) \sqcup \overline{r \circ s}$ as hypotheses. The term graphs $G_{r\circ s}$ and $G_{(r^{\top}\circ\overline{r\circ s})\sqcap s}$ are respectively equivalent (by Table 1) to $\{T\}$, with $T := (\{w, v, z\}, \{wrv, vsz\}, w, z)$, and $\{S\}$, with $S := (\{x, u, y\}, \{urx, xsy, u\overline{r \circ s}y\}, x, y)$. We have a draft homomorphism $\theta : T \xrightarrow{d} S$, yielding $\mathsf{glue}_{\theta}(\{T\}, \{S\}) = \{T'\}$, where

$$T' := (N_S \cup N_T, \{urx, xsy, u\overline{r \circ s}y, wrv, vsz, w\mathsf{I}u, z\mathsf{I}y\}, x, y).$$

Now, by Table 1, graph $\{T'\}$ is equivalent to

$$H := \{(N_S \cup N_T, \{urx, xsy, u\overline{r \circ s}y, ur \circ sy\}, x, y)\},$$

which is equivalent (by $(r \circ s) \sqcap \overline{r \circ s} \sqsubseteq \mathsf{O}$) to the empty graph βG_{O}.

The preceding example shows a rather simple case of how one can handle complementation. More generally, we can use an overall strategy of employing the hypotheses as follows: use $\mathsf{E} \sqsubseteq \overline{R} \sqcup R$ to expand a slice into two, and use $\overline{R} \sqcap R \sqsubseteq \mathsf{O}$ to erase a slice with parallel arcs $\overline{R}$ and R.

Example 4. We indicate how to show that complementation inverts inclusion: $r \sqsubseteq s$ yields $\overline{s} \sqsubseteq \overline{r}$. We start with the single-slice term graph $G_{\overline{s}}$. We expand it (by $\mathsf{E} \sqsubseteq \overline{r} \sqcup r$) to a two-slice graph G_1, which is equivalent (by $\overline{s} \sqcap s \sqsubseteq \mathsf{O}$) to a single-slice graph G_2. Now, we have a draft homomorphism $\theta : G_{\overline{r}} \overset{d}{\to} G_2$, so, the downward graph cover rule GrCvr (in Table 2) yields the desired graph $G_{\overline{r}}$.

Example 5. As part of De Morgan's Theorem K [19] one has the implication *if* $r \circ s \sqsubseteq t$, *then* $r^{\top} \circ \overline{t} \sqsubseteq \overline{s}$. To establish this, it is enough to show

$$\{r \circ s \sqsubseteq t, s \sqcap \overline{s} \sqsubseteq \mathsf{O}, \mathsf{E} \sqsubseteq s \sqcup \overline{s}, t \sqcap \overline{t} \sqsubseteq \mathsf{O}, \mathsf{E} \sqsubseteq t \sqcup \overline{t}\} \models r^{\top} \circ \overline{t} \sqsubseteq \overline{s}, \qquad (1)$$

where $\overline{s}$ and $\overline{t}$ are taken themselves as the new relational variables $r_{\overline{s}}$ and $r_{\overline{t}}$. Figure 5, in the Appendix, presents a graph-calculus derivation establishing (1).

6 Proof Theory for Linear Lattices

M. Haiman [9] presented a set of rules based on graphs to prove the (infinitary) Horn sentences of the form

$$\&_{i \in I}(P_i \leq Q_i) \Rightarrow P \leq Q, \qquad (2)$$

where P_i, Q_i, $i \in I$, and P, Q, are lattices terms, which are valid in all linear lattices. We show how to translate (2) in the relational language and prove that the main result of [9] follows from the strong soundness and completeness of the graph calculus.

Let $\textsc{Lvar} = \{p_i : i \in I\}$ be a set of lattices variables. The *lattice terms*, typically denoted by P, Q, are generated from the p_is by applications of the lattice constructing terms operators $\wedge$ and $\vee$. A *lattice inclusion* is an expression of the form $P \leq Q$, where P and Q are lattice terms.

A *lattice* is a structure $\mathcal{L} = (L, \wedge^{\mathcal{L}}, \vee^{\mathcal{L}}, \leq^{\mathcal{L}})$, where $(L, \leq^{\mathcal{L}})$ is a partially ordered set and for every $a, b \in L$, the elements $a \wedge^{\mathcal{L}} b$ and $a \vee^{\mathcal{L}} b$ are, respectively, the inf and sup of a and b according to $\leq^{\mathcal{L}}$. An *assignment* of the lattice variables in a lattice $\mathcal{L}$ is a mapping $v : \textsc{Lvar} \to L$. The *value* of a term under an assignment is the element $[\![P]\!]^v_{\mathcal{L}}$ of L defined by the following rules: $[\![p_i]\!]^v_{\mathcal{L}} := vp_i$, $[\![P \wedge Q]\!]^v_{\mathcal{L}} := [\![P]\!]^v_{\mathcal{L}} \wedge^{\mathcal{L}} [\![Q]\!]^v_{\mathcal{L}}$ and $[\![P \vee Q]\!]^v_{\mathcal{L}} := [\![R_P]\!]^v_{\mathcal{L}} \vee^{\mathcal{L}} [\![R_Q]\!]^v_{\mathcal{L}}$. Let $\{P_j \leq Q_j : j \in J\} \cup \{P \leq Q\}$ be a set of lattice inclusions and K be a class of lattices. We say that $P \leq Q$ is a *consequence of* $\{P_j \leq Q_j : j \in J\}$ *in* K when for every lattice $\mathcal{L} \in \mathsf{K}$ and assignment v of the lattice variables in $\mathcal{L}$, the assumption that $[\![P_j]\!]^v_{\mathcal{L}} \leq^{\mathcal{L}} [\![Q_j]\!]^v_{\mathcal{L}}$, for every $j \in J$, implies $[\![P]\!]^v_{\mathcal{L}} \leq^{\mathcal{L}} [\![Q]\!]^v_{\mathcal{L}}$.

It is well known that given the set M, the structure $\mathcal{E} = (\mathsf{Eq}M, \wedge^{\mathcal{E}}, \vee^{\mathcal{E}}, \leq^{\mathcal{E}})$ is a lattice when $\mathsf{Eq}M$ is the set of all equivalence relations on M, $\leq^{\mathcal{E}}$ and $\wedge^{\mathcal{E}}$ are, respectively, the set inclusion and set intersection, and $X \vee^{\mathcal{E}} Y$ is defined as

$$X \vee^{\mathcal{E}} Y := (X \mid Y) \cup (X \mid Y \mid X) \cup (X \mid Y \mid X \mid Y) \cup \ldots$$

An important class of lattices of equivalence relations is defined by requiring that $X \vee^{\mathcal{E}} Y := X \mid Y$. Since this condition is equivalent to requiring commutativity of the composition $\mid$, we say that $\mathcal{L}$ is a *lattice of commuting equivalence* *relations*. Also, call a lattice *linear* when it is isomorphic to a lattice of commuting equivalence relations.

In [9] M. Haiman presented a set of rules based on graphs to prove the (infinitary) Horn sentences of the form $\&_{j\in J}(P_j \leq Q_j) \Rightarrow P \leq Q$, where P_j, Q_j, $j \in J$, and P, Q, are lattices terms, which are valid in all linear lattices. In the terminology above this means that the system proposed in [9] proves the sentence $\&_{j\in J}(P_j \leq Q_j) \Rightarrow P \leq Q$ iff $P \leq Q$ is a consequence of the set $\{P_j \leq Q_j : j \in J\}$ in the class of all linear lattices. Now, we prove that the main result of [9] follows from the strong completeness of the graph calculus.

We assume that the set of lattice variables $\textsc{Lvar} := \{p_i : i \in I\}$ is in one-to-one correspondence with the set of relation symbols $\textsc{Rsym} := \{r_i : i \in I\}$ and that these sets are disjoint and fixed throughout.

First, we define the translation mapping from lattice terms P to relational terms R_P, by $R_{p_i} := r_i$ $(i \in I)$, $R_{P\wedge Q} := R_P \sqcap R_Q$, $R_{P\vee Q} := R_P \circ R_Q$. According to the above, we say that a $+\mathsf{RG}$ inclusion $R \sqsubseteq S$ is *linear lattice like* iff the only operators occurring in $R \sqsubseteq S$ are $\sqcap$ and $\circ$. Second, we denote by LLat the set of linear lattices like inclusions containing as elements all the inclusions of the form $\mathsf{I} \sqsubseteq r$, $r^{\mathsf{T}} \sqsubseteq r$, $r \circ r \sqsubseteq r$, and $r \circ s \sqsubseteq s \circ r$, where r, s are relation symbols.

Theorem 2. *Let* $\{P_j \leq Q_j : j \in J\} \cup \{P \leq Q\}$ *be a set of lattice inclusions. Then the following are equivalent:*

(a) $\&_{j\in J}\, P_j \leq Q_j \Rightarrow P \leq Q$ *is valid in the class of all linear lattices;*

(b) $\{R_{P_j} \sqsubseteq R_{Q_j} : j \in J\} \cup \mathsf{LLat} \vdash R_P \sqsubseteq R_Q$ *in* $+\mathsf{RG}$.

7 Conclusion

We have presented a sound and complete logical system for deriving inclusions between graphs from inclusions between graphs. We can extend our system to a full-relational graph calculus with complementation, as indicated in Section 4. Our system uses linear derivations, in contrast with other systems for relations [15].

The monotonicity rule was suggested to handle simple hypotheses [4]. Our Rule Hyp_Γ was inspired by it, but it involves more elaborated formulation. The proof of our main result, the completeness theorem, gives a strategy for deriving a graph inclusion from a set of graph inclusions.

The importance of the the main result of [9] and, consequently, of our Theorem 2, is discussed in [5, 17, 20, 18], where the proof theory for linear lattices is extended and applied in the study of expressions involving joins and meets of subspaces of vectors spaces. The main differences between the graph calculi presented in these works and ours is that they do not use neither the notion of normal form for graphs nor that of homomorphism between graphs. As future work we intend to investigate the exact relationship between their calculi and ours.

Appendix

Proof of Lemma 1 Given $(a, b) \in [\![S]\!]_{\mathcal{M}}$, there is an assignment $g : S \to \mathcal{M}$ such that $gx = a$ and $gy = b$. This induces the assignment $g.\theta : S' \to \mathcal{M}$. So, with $c := g.\theta x_{S'}$ and $d := g.\theta y_{S'}$, we have $(c, d) \in [\![S']\!]_{\mathcal{M}} \subseteq [\![G' \cup S']\!]_{\mathcal{M}}$, whence $(c, d) \in [\![H]\!]_{\mathcal{M}}$ (as $\mathcal{M} \models G' \cup \{S'\} \sqsubseteq H$). Thus, $(c, d) \in [\![T]\!]_{\mathcal{M}}$, for some slice T of H, so, there is an assignment $g' : T \to \mathcal{M}$ such that $(c, d) = (g'x_T, g'y_{S'})$. Now, define $g'' : N_S \uplus N_T \to M$ naturally to agree with g on N_S and with g' on N_T. This gives an assignment $g'' : \mathsf{glue}_{\theta}(T, S) \to \mathcal{M}$, with $(g''x_S, g''y_S) = (a, b)$. Therefore, $(a, b) \in [\![\mathsf{glue}_{\theta}(T, S)]\!]_{\mathcal{M}} \subseteq [\![\mathsf{glue}_{\theta}(H, S)]\!]_{\mathcal{M}}$. □

Proof of Theorem 1 The proof is based on the construction of a tree with nodes labeled by basic graphs, based on applications of derived rules iHyp_{Γ} and GrRCvr (Tables 4 and 5). These derived rules uses a few concepts which we now introduce.

Given a slice $S' = (N', A', x', y')$, an *i-draft homomorphism* $\theta : S' \xrightarrow{i} S$ is a function $\theta : N' \to N$ such that (1) if $u' \mathsf{I} v' \in A'$, then $\theta u' {\sim}_S^* \theta v'$, and (2) if $u'Rv' \in A'$, then there exist an arc $uRv \in A$ such that $\theta u' {\sim}_S^* u$ and $\theta v' {\sim}_S^* v$. A *relaxed homomorphism* $\theta : S' \xrightarrow{r} S$ is an i-draft homomorphism $\theta : S' \xrightarrow{i} S$ such that $\theta x' {\sim}_S^* x$ and $\theta y' {\sim}_S^* y$.

Given graphs G and H, we say that H *r-covers* G, denoted by $G \xleftarrow{r} H$, iff for each slice $S \in G$ there exist a slice $T \in H$ and a relaxed homomorphism $\theta : T \xrightarrow{r} S$.

Table 4. Graph Relaxed Cover rule.

$$\mathsf{GrRCvr}\ \frac{G}{H} \quad \text{if } G \xleftarrow{r} H$$

Consider slices $S = (N_S, A_S, x, y)$ and $T = (N_T, A_T, w, z)$, as well as designated nodes $u, v \in N_S$. The result of *i-gluing* T onto S via (u, v) is the slice defined by $\mathsf{iglue}_{(u,v)}(T, S) := (N_S \uplus N_T, A_S \uplus A_T \cup \{u \mathsf{I} x_T, v \mathsf{I} y_T\}, x_S, y_S)$. One i-glues a graph H by i-gluing its slices: $\mathsf{iglue}_{(u,v)}(H, S) := \{\mathsf{iglue}_{(u,v)}(T, S) : T \in H\}$. Given a slice $S' = (N', A', x', y')$ and a i-draft homomorphism $\theta : S' \xrightarrow{i} S$, we set $\mathsf{iglue}_{\theta}(H, S) := \mathsf{iglue}_{(\theta x', \theta y')}(H, S)$.

A set of graph inclusions Γ is *basic* iff every inclusion $G \sqsubseteq H \in \Gamma$ is such that G and H are basic. Let us consider, wlog, that G, H, and Γ are basic and that each inclusion in Γ has a single-slice graph at the left-hand side.

Given slice S, define

Table 5. i-Hypothesis rule.

$$\mathsf{iHyp}_{\Gamma}\ \frac{G \cup \{S\}}{G \cup \mathsf{iglue}_{\theta}(H,S)} \quad \text{if } G' \cup \{S'\} \sqsubseteq H \in \Gamma \text{ and } \theta : S' \xrightarrow{d} S$$

- $H(S) := \{(\theta, T) : T \in H \text{ and } \theta : T \xrightarrow{r} S\}$ and
- $\Gamma(S) := \{(\theta, G' \cup \{S'\} \sqsubseteq H') : G' \cup \{S'\} \sqsubseteq H' \in \Gamma \text{ and } \theta : S' \xrightarrow{d} S\}$.

We say that a slice S is *red* iff $H(S) \neq \emptyset$. We say that a slice S is *yellow* iff $H(S) = \emptyset$ and $\Gamma(S) = \emptyset$. We say that a slice S is *green* iff $H(S) = \emptyset$ and $\Gamma(S) \neq \emptyset$. We say that a graph G is *red* iff every slice in G is red.

Consider an enumeration $S_1 \sqsubseteq H_1, S_2 \sqsubseteq H_2, \ldots, S_n \sqsubseteq H_n, \ldots$ of graph inclusions in Γ such that each graph inclusion appears infinitely many times in the enumeration.

We shall build a tree whose nodes are labeled by graphs beginning with its root, labeled by G, and expanding the tree considering the slices of each leaf, based on the following idea: we will have no gain in expanding the tree from red slices, from yellow slices it is impossible to expand the tree (since expansion is made based on the hypotheses in Γ), green slices are the ones used to expand the tree.

We say that a slice $S' = (N', A', x', y')$ is an *extension* of a slice $S = (N, A, x, y)$ iff $N \subseteq N'$, $A \subseteq A'$, $x = x'$, and $y = y'$. Then every slice in $\mathsf{glue}_{\theta}(H,S)$ is an extension of S and, if $\theta : S' \xrightarrow{d} S$, then θ can be considered as a draft homomorphism from S' to every extension of S. If G is a graph whose slices are extensions of S and $\theta : S' \xrightarrow{d} S$, define $\mathsf{glue}_{\theta}(H,G) := \{\mathsf{glue}_{\theta}(H,S'') : S'' \in G\}$. Define also $\mathsf{glue}_{\theta_1 \cdots \theta_n}(H,G) := \mathsf{glue}_{\theta_1}(H, \mathsf{glue}_{\theta_2 \cdots \theta_n}(H,S))$.

Define $\mathcal{T}_0$ to be the tree whose only node is $G_0 = G$. Given $\mathcal{T}_k$, define $\mathcal{T}_{k+1}$ as follows. For each leaf G' of $\mathcal{T}_k$, for each $S \in G'$, if S is green, then add $\mathsf{glue}_{\theta_1 \cdots \theta_n}(H_{k+1}, S)$ as a son of G' and label its arc with S, where $\{\theta_1, \ldots, \theta_n\}$ is the set of all draft homomorphisms from S_{k+1} to S. If there is no draft homomorphism from S_{k+1} to S, add S itself as a son of G'. When a leaf in $\mathcal{T}_k$ does not have green slices, it gains no sons and remains a leaf in $\mathcal{T}_{k+1}$. Define tree $\mathcal{T} = \bigcup_{i \in \mathbb{N}} \mathcal{T}_i$.

Given a node G in $\mathcal{T}$, we say that G is red* iff G is red or for all $S \in G$, slice S is red or G has a son G' labeled S such that G' is red*. If G_0 is red*, then one can obtain a derivation of $G \sqsubseteq H$ from Γ, as follows. We start with a sequence of graphs $(G'_1, \ldots, G'_n)$, obtained as follows. Graph G'_1 is G_0 and G'_{k+1} is the result of a series of applications of Hyp_{Γ} on G'_k, one for each draft homomorphism from S_{k+1} to each $S \in G'_k$.

It is immediate to transform $(G'_1, \ldots, G'_n)$ into a derivation of $G \sqsubseteq H$ from Γ. Include H after G'_n, since H can be obtained from G'_n by an application of

GrCvr (because every slice in G'_n is red). If G, H, or Γ are not basic, include steps of applications of the rules in Table 1 eliminating operators, except I, and introducing operators, to obtain H from the result of application rule of GrCvr on graph G'_n.

If G_0 is not red*, then one can construct a model $\mathcal{M}$ of Γ that is not a model of $G \sqsubseteq H$. If tree $\mathcal{T}$ has a graph with a yellow slice $S = (N, A, x, y)$, then define $\mathcal{M} = (M, r_i^{\mathcal{M}})_{i \in \mathbb{N}}$ with $M = \tilde{N}$ and $r_i^{\mathcal{M}} = \{(\tilde{u}, \tilde{v}) : uRv \in A\}$, where $\sim$ is $\sim_S^*$. Otherwise, there is an infinite path of graphs having green slices in tree $\mathcal{T}$. From this path we can obtain a chain $(S_n)_{n\in\omega}$ of green slices. In this case, take $\mathcal{M}$ to be the canonical model on $(S_n)_{n\in\omega}$.

In any case (tree $\mathcal{T}$ having a yellow slice or a chain of green slices), the constructed model $\mathcal{M}$ is such that $\mathcal{M} \models \Gamma$ and $\mathcal{M} \not\models G \sqsubseteq H$. In fact, $\mathcal{M} \models \Gamma$ since in the construction of $\mathcal{T}$ all possible applications of Hyp_Γ, related to each inclusion of Γ, are effectively applied on every slice in chain $(S_n)_{n\in\omega}$. Also, $(\tilde{x}_{S_0}, \tilde{y}_{S_0}) \in [\![S_0]\!]_{\mathcal{M}} \subseteq [\![G]\!]_{\mathcal{M}}$ but there is no slice $T \in H$ such that $(\tilde{x}_{S_0}, \tilde{y}_{S_0}) \in [\![T]\!]_{\mathcal{M}}$. Otherwise, witness assignment g would be a relaxed homomorphism from T to S_0 and S_0 would be red. □

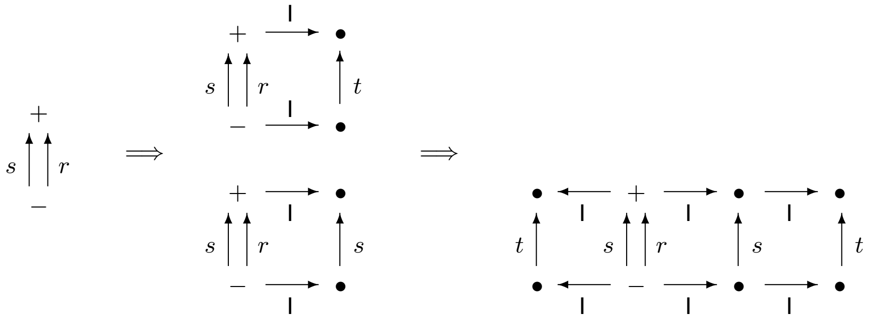

Fig. 3. Construction for $\{r \sqsubseteq t \sqcup s, s \sqsubseteq t\} \vdash r \sqcap s \sqsubseteq t$.

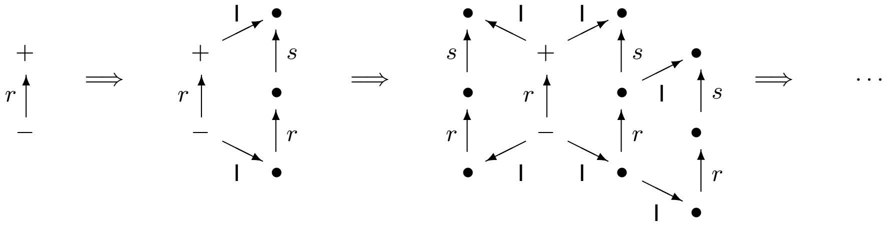

Fig. 4. Construction for $\{r \sqsubseteq r \circ s\} \not\models r \sqsubseteq s$.

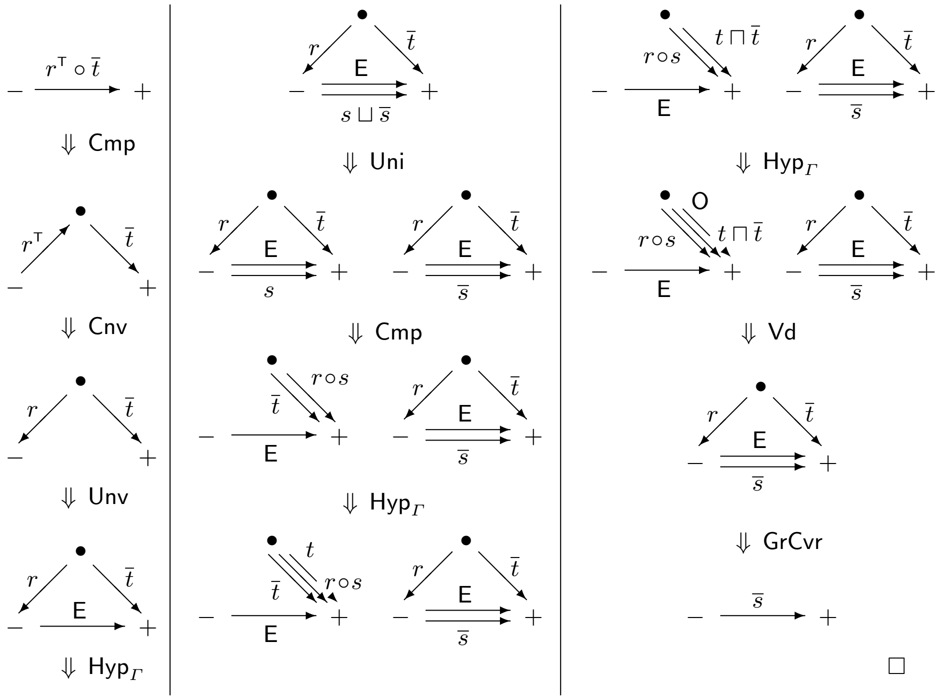

Fig. 5. Graph derivation for De Morgan's Theorem K.

Proof of Theorem 2 (a)⇒(b) Suppose, for a contradiction, that $\&_{j \in J} P_j \leq Q_j \Rightarrow P \leq Q$ is valid in all linear lattices but $\{R_{P_j} \sqsubseteq S_{Q_j} : j \in J\} \cup \mathsf{LLat} \nvdash R_P \sqsubseteq S_Q$ in $+\mathsf{RG}$. Then, by the strong completeness of the graph calculus $\{R_{P_j} \sqsubseteq S_{Q_j} : j \in J\} \cup \mathsf{LLat} \not\models R_P \sqsubseteq S_Q$. So, there is a model $\mathcal{M}$ such that $\mathcal{M} \models \{R_{P_j} \sqsubseteq S_{Q_j} : j \in J\}$, $\mathcal{M} \models \mathsf{LLat}$, but $\mathcal{M} \not\models R_P \sqsubseteq S_Q$. Now, take $\mathcal{L} = (\{r_i^{\mathcal{M}} : i \in I\}, \cap, \mid, \subseteq)$ and $v : \textsc{Lvar} \rightarrow \{r_i^{\mathcal{M}} : i \in I\}$ such that $vp_i := r_i^{\mathcal{M}}$. We have that $\mathcal{L}$ is a lattice of commuting equivalence relations and that $[\![P]\!]^v_{\mathcal{L}} = [\![R_P]\!]_{\mathcal{M}}$ for every lattice term P. So, for $\mathcal{L}$ and v we have $[\![P_j]\!]^v_{\mathcal{L}} \subseteq [\![Q_j]\!]^v_{\mathcal{L}}$, for every $j \in J$, but $[\![P]\!]^v_{\mathcal{L}} \not\subseteq [\![Q]\!]^v_{\mathcal{L}}$, a contradiction with $P \leq Q$ is a consequence of $\{P_j \leq Q_j : j \in J\}$ in the class of all linear lattices.

(b)⇒(a) Suppose, for a contradiction, that $\{R_{P_j} \sqsubseteq S_{Q_j} : j \in J\} \cup \mathsf{LLat} \vdash R_P \sqsubseteq S_Q$ in $+\mathsf{RG}$ but $\&_{j \in J} P_j \subseteq Q_j \Rightarrow P \subseteq Q$ is not valid in the class of all linear lattices. So, there is a linear lattice $\mathcal{L} = (L, \wedge^{\mathcal{L}}, \vee^{\mathcal{L}}, \leq^{\mathcal{L}})$ and an assignment $v : \textsc{Lvar} \rightarrow L$ such that $[\![P_j]\!]^v_{\mathcal{L}} \leq^{\mathcal{L}} [\![Q_j]\!]^v_{\mathcal{L}}$, for every $j \in J$, but $[\![P]\!]^v_{\mathcal{L}} \not\leq^{\mathcal{L}} [\![Q]\!]^v_{\mathcal{L}}$. Now, since $\mathcal{L}$ is a linear lattice, there are sets M and E, where E is a set of equivalence relations on M, $M = \bigcup_{X \in E} \mathsf{Fld}X$, being $\mathsf{Fld}X$ the field of the relation X, and a function $f : L \rightarrow E$ such that $\mathcal{E} = (E, \subseteq, \cap, \mid)$ is a lattice of commuting equivalence relations and f is an isomorphism from $\mathcal{L}$ onto $\mathcal{E}$. Taking $\mathcal{E}$ and observing that fv is an assignment for $\textsc{Lvar}$ in $\mathcal{E}$, since f is an isomorphism, we have also that $[\![P_j]\!]^{fv}_{\mathcal{E}} \subseteq [\![Q_j]\!]^{fv}_{\mathcal{E}}$, for every $j \in J$, but

$[\![P]\!]^{fv}_{\mathcal{E}} \not\subseteq [\![Q]\!]^{fv}_{\mathcal{E}}$. Now, take $\mathcal{M} = (M, \{r_j^{\mathcal{M}} : j \in J\})$, where $r_j^{\mathcal{M}} := f(v(p_j))$, for every $j \in J$. We have that $\mathcal{M}$ is a model, $\mathcal{M} \models \mathsf{LLat}$, and that $[\![R_P]\!]_{\mathcal{M}} = [\![P]\!]^{fv}_{\mathcal{E}}$ for every lattice term P. So, for $\mathcal{M}$ we have $[\![R_{P_j}]\!]_{\mathcal{M}} \subseteq [\![R_{Q_j}]\!]_{\mathcal{M}}$, for every $j \in J$, but $[\![R_P]\!]_{\mathcal{M}} \not\subseteq [\![R_Q]\!]_{\mathcal{M}}$, a contradiction with the strong soundness theorem for $+\mathsf{RG}$. □

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