Positive Fork Graph Calculus

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Outline

- Motivation
- Positive fork graph language
- Soundness and completeness
- Corollaries
- Ongoing work

Main Task

To design a graph calculus to decide
the positive identities and inclusions of fork algebras.

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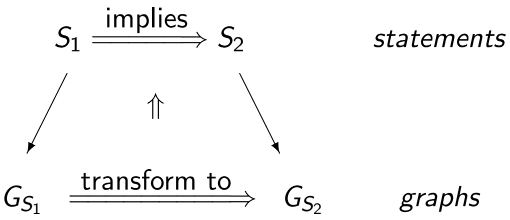
Reasoning with graphs



Write the statements as graphs and reason on the graphs.

Fork algebras

FA = RA + storing and retrieving data

- a, b : objects
- $a \star b$: structured object
- $R \nabla S = \{(a, b \star c) : aRb \wedge aSc\}$

To decide

- equalities such as

$$(R \triangledown S) \circ (T \triangledown U)^{-1} = (R \circ T^{-1}) \cap (S \circ U^{-1})$$

- inclusions such as

$$R \triangledown (S \cap T) \subseteq (R \triangledown S) \cap (R \triangledown T)$$

Positive fork language

Syntax

- Variables for relations $\{r, s, t, \ldots\}$
- Terms

$$R ::= \mathsf{E} \mid \mathsf{I} \mid r \mid R^{-1} \mid R \cap R \mid R \cup R \mid R \circ R \mid R \triangledown R$$

- Inclusions $R \subseteq S$
- Equalities $R = S$

Semantics

- Structured universe $(M, \star)$
  - $M \neq \emptyset$
  - $\star : M \times M \to M$ is injective
- Structured model $\mathfrak{M} = (M, \star, r^{\mathfrak{M}})$
  - $(M, \star)$ structured universe
  - $r^{\mathfrak{M}} \subseteq M \times M$ for every r

Meaning of terms

- $[\![R \cap S]\!]_{\mathfrak{M}} ::= [\![R]\!]_{\mathfrak{M}} \cap [\![S]\!]_{\mathfrak{M}}$
- $[\![R \cup S]\!]_{\mathfrak{M}} ::= [\![R]\!]_{\mathfrak{M}} \cup [\![S]\!]_{\mathfrak{M}}$
- $[\![R \circ S]\!]_{\mathfrak{M}} ::= [\![R]\!]_{\mathfrak{M}} \circ [\![S]\!]_{\mathfrak{M}}$
- $[\![R \nabla S]\!]_{\mathfrak{M}} ::= [\![R]\!]_{\mathfrak{M}} \nabla [\![S]\!]_{\mathfrak{M}}$
- $[\![R^{-1}]\!]_{\mathfrak{M}} ::= [\![R]\!]_{\mathfrak{M}}{}^{-1}$
- $[\![\mathrm{I}]\!]_{\mathfrak{M}} ::= \{(a, a) : a \in M\}$
- $[\![\mathrm{E}]\!]_{\mathfrak{M}} ::= M \times M$

- Inclusions and equalities

$$R = S \text{ iff } R \subseteq S \text{ and } S \subseteq R$$

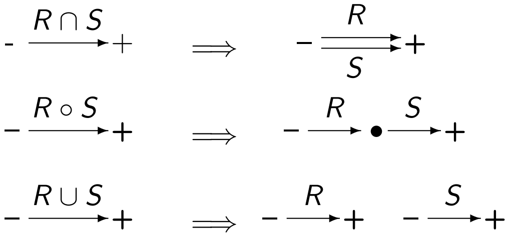
- Main idea

$$\begin{array}{ccc} G_R & & G_S \\ \Updownarrow & & \Updownarrow \\ \mathrm{NF}\, G_R & \xrightarrow[]{\text{compare}} & \mathrm{NF}\, G_S \end{array}$$

Graphs

- Idea of graph:
  - Arcs labeled by terms.
  - Two distinguished nodes: *input*, *output*.
  - Paths represent restrictions.

- Transformation on graphs:



- Compare the transformed graphs:
  - Homomorphism.

- OK for the relational language
(ENTCS 2006, IGPL 2006, LNAI 2008, IC to appear).

- What about the positive relational language with fork?

Example

- Fork axioms:

$$(\mathsf{I} \triangledown \mathsf{E})^{-1} \triangledown (\mathsf{E} \triangledown \mathsf{I})^{-1} \subseteq \mathsf{I}$$

$$(r \triangledown s) \circ (t \triangledown q)^{-1} = (r \circ t^{-1}) \cap (s \circ q^{-1})$$

$$(r \circ (\mathsf{I} \triangledown \mathsf{E})) \cap (s \circ (\mathsf{E} \triangledown \mathsf{I})) = r \triangledown s$$

- Fork distributes over intersection:

$$r \triangledown (s \cap t) = (r \triangledown s) \cap (r \triangledown t)$$

Equational proof

Part I

- $\mathsf{E} \triangledown \mathsf{I}$ is injective:

$$(\mathsf{E} \triangledown \mathsf{I}) \circ (\mathsf{E} \triangledown \mathsf{I})^{-1} =$$

$$(\mathsf{E} \circ \mathsf{E}^{-1}) \cap (\mathsf{I} \circ \mathsf{I}^{-1}) =$$

$$(\mathsf{E} \circ \mathsf{E}) \cap (\mathsf{I} \circ \mathsf{I}) =$$

$$\mathsf{E} \cap \mathsf{I} \quad \subseteq \quad \mathsf{I}$$

Equational proof

Part II

- R is injective $\Longrightarrow (S \cap T) \circ R = (S \circ R) \cap (T \circ R)$:

Graph proof from hypotheses.

Equational proof

Part III

- Fork distributes over intersection:

$R \triangledown (S \cap T) =$

$(R \circ (\mathsf{I} \triangledown \mathsf{E})) \cap ((S \cap T) \circ (\mathsf{E} \triangledown \mathsf{I})) =$ (by Parts I and II)

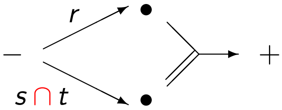
$(R \circ (\mathsf{I} \triangledown \mathsf{E})) \cap (S \circ (\mathsf{E} \triangledown \mathsf{I})) \cap (T \circ (\mathsf{E} \triangledown \mathsf{I})) =$

$(R \circ (\mathsf{I} \triangledown \mathsf{E})) \cap (S \circ (\mathsf{E} \triangledown \mathsf{I})) \cap (R \circ (\mathsf{I} \triangledown \mathsf{E})) \cap (T \circ (\mathsf{E} \triangledown \mathsf{I})) =$

$(R \triangledown S) \cap (R \triangledown T)$

Graph proof

$$- \xrightarrow{\; r \;\nabla\; (s \cap t) \;} +$$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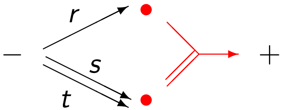
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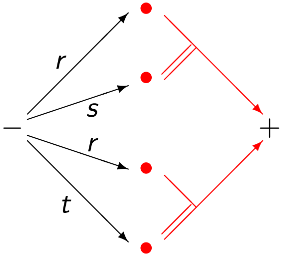
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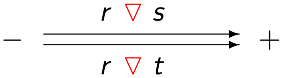
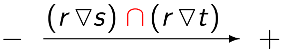



$(r \nabla s) \cap (r \nabla t)$
$-$
$+$

Positive fork graph calculus

- Labels

$$R ::= \mathsf{E} \mid \mathsf{I} \mid r \mid R^{-1} \mid R \sqcap R \mid R \sqcup R \mid R \circ R \mid R \triangledown R$$

- Slice $S = (N, T, A, x, y)$, where
  - $N \neq \emptyset$ is a set of nodes
  - $T \subseteq N \times N \times N$ is a set of node equations
  - $A \subseteq N \times \text{Labels} \times N$ is a set of arcs
  - $x, y \in N$

- Graph $G = (N_j, T_j, A_j, x_j, y_j)_{j \in J}$ is a finite set of slices

Meaning of a graph

- Assignment for a slice S_j in $\mathfrak{M}$
$$g : N_j \to M$$

- Meaning of S in $\mathfrak{M}$
$$(a, b) \in [\![S]\!]_{\mathfrak{M}}$$
$$\text{iff}$$
$$\exists g \text{ such that}$$
$$gx = a$$
$$gy = b$$
$$\forall u{\star}v \rightarrowtail w \in T : gu \star gv = gw$$
$$\forall uRv \in A : (gu, gv) \in [\![R]\!]_{\mathfrak{M}}$$

- Meaning of G in $\mathfrak{M}$
$$[\![G]\!]_{\mathfrak{M}} = \bigcup_{j \in J} [\![S_j]\!]_{\mathfrak{M}}$$

Introduction/Elimination rules

Cnv $$\frac{N, T, A \cup \{uR^{-1}v\}, x, y}{N, T, A \cup \{vRu\}, x, y}$$

Int $$\frac{N, T, A \cup \{uR \cap Sv\}, x, y}{N, T, A \cup \{uRv, uSv\}, x, y}$$

Uni $$\frac{N, T, A \cup \{uR \cup Sv\}, x, y}{N, T, A \cup \{uRv\}, x, y \qquad N, T, A \cup \{uSv\}, x, y}$$

Cmp $$\frac{N, T, A \cup \{uR \circ Sv\}, x, y}{N \cup \{w\}, A \cup \{uRw, wSv\}, x, y} \text{ if } w \notin N$$

Introduction/Elimination rules

Constants

$$\text{Unv} \quad \frac{N, A \cup \{u\mathsf{E}v\}, x, y}{N, A, x, y}$$

$$\text{Idn} \quad \frac{N, A \cup \{u\mathsf{I}v\}, x, y}{N\frac{v}{u}, A\frac{v}{u}, x\frac{v}{u}, y\frac{v}{u}}$$

Introduction/Elimination rules

Fork

Frk

$$\frac{N, T, A \cup \{uR \triangledown Sv\}, x, y}{N \cup \{v_1, v_2\}, T \cup \{v_1 \star v_2 \rightarrowtail v\}, A \cup \{uRv_1, uSv_2\}, x, y}$$

if $v_1, v_2 \notin N$

Star rules

Tot $$\frac{N, T, A, x, y}{N \cup \{w\}, T \cup \{u{\star}v \rightarrowtail w\}, A, x, y} \quad \text{if } w \notin N$$

Fnc $$\frac{N, T \cup \{u{\star}v \rightarrowtail w_1, u{\star}v \rightarrowtail w_2\}, A, x, y}{N\frac{w_1}{w_2}, (T \cup \{u{\star}v \rightarrowtail w_1)\frac{w_1}{w_2}, A\frac{w_1}{w_2}, x\frac{w_1}{w_2}, y\frac{w_1}{w_2}}$$

Inj $$\frac{N, T \cup \{u_1{\star}v_1 \rightarrowtail w, u_2{\star}v_2 \rightarrowtail w\}, A, x, y}{N\frac{u_1}{u_2}\frac{v_1}{v_2}, (T \cup \{u_1{\star}v_1 \rightarrowtail w\})\frac{u_1}{u_2}\frac{v_1}{v_2}, A\frac{u_1}{u_2}\frac{v_1}{v_2}, x\frac{u_1}{u_2}\frac{v_1}{v_2}, y\frac{u_1}{u_2}\frac{v_1}{v_2}}$$

Homomorphism

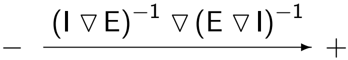
- $\phi : S \to S'$ is a *homomorphism* if
  - $\phi : N \to N'$
  - ϕ preserves node equations
  - ϕ preserves arc labels
  - $\phi x = x'$
  - $\phi y = y'$

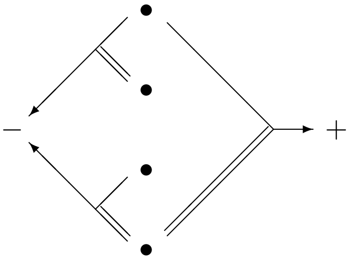
- H covers G if
  $\forall S$ in G, $\exists S'$ in H and $\exists \phi$ such that
  $\phi : S \to S'$ is a homomorphism

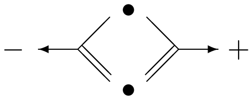
- Graph Cover rule

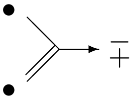
$$\text{GCv}\ \frac{G}{H} \quad \text{if } H \text{ covers } G$$

Graph proof of $(\mathsf{I} \triangledown \mathsf{E})^{-1} \triangledown (\mathsf{E} \triangledown \mathsf{I})^{-1} \sqsubseteq \mathsf{I}$



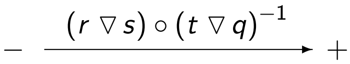


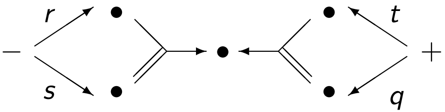




$$\mp$$

Graph proof of $(r \triangledown s) \circ (t \triangledown q)^{-1} = (r \circ t^{-1}) \sqcap (s \circ q^{-1})$

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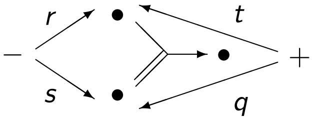
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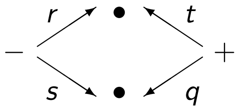
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$$- \xrightarrow{(r \circ t^{-1}) \cap (s \circ q^{-1})} +$$

Soundness and Completeness

Theorem (Soundness)

If $\vdash G_R \sqsubseteq G_S$, *then* $\models R \sqsubseteq S$.

Theorem (Completeness)

If $\models R \sqsubseteq S$ *then* $\vdash G_R \sqsubseteq G_S$.

Proof outline

Want to prove $\forall$ slice S of G $\exists$ slice S' of H and $\theta : S' \to S$.

1. Let S be a slice of G.
2. Close S by $\star$, preserving totality, functionality and injectivity.
3. From closed S construct a model $\mathfrak{M}_S$.
4. Prove that $(x_S, y_S) \in [\![G]\!]_{\mathfrak{M}_S}$.

5. $\models G \subseteq H$ gives $(x_S, y_S) \in [\![H]\!]_{\mathfrak{M}_S}$.
So, $\exists$ slice S' of H s.t. $(x_S, y_S) \in [\![S']\!]_{\mathfrak{M}_S}$ and $\exists g : N' \to N^\star$ s.t. $gx_{S'} = x_S$ and $gy_{S'} = y_S$.

6. Prove that $g(N') \subseteq N$.

7. Prove that g preserves arcs and tables.
So, $g : N' \to N$ us a homomorphism. ■

Corollaries

- Finitely given model property
- Decidability
- Normal form of proofs:

$$
\begin{array}{l}
G_R \\
\Downarrow \textit{elimination} \\
NF\ G_R \\
\uparrow \textit{cover} \\
NF\ G_S \\
\Uparrow \textit{elimination} \\
G_S
\end{array}
$$

Perspectives

- To provide sound and complete graph calculi for structured universes with weaker restrictions imposed on $\star$.
- To characterize the relational calculi for which one can provide a graph calculus.

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