Staggered Quantum Walk Algorithm for Element Distinctness

Alexandre S. Abreu[1]
Franklin L. Marquezino[1]
Luis A.B. Kowada[2]

santiago@cos.ufrj.br

Paper 66

1 PESC/COPPE - Universidade Federal Rio de Janeiro-UFRJ
2 IC - Universidade Federal Fluminense - UFF

November 2016

PESC Programa de Engenharia de Sistemas e Computação

Outline

Section 1

Staggered Quantum Walk Model

Staggered Quantum Walk Model

- We say that a set of disjoint cliques (**polygons or clusters**) $C = \{c_1, \cdots, c_T\}$ is a tessellation for a graph G, if the union of these cliques covers all vertices of G.
- A graph is T-tessellable if T is the smallest number of tessellations such that the union of these tessellations covers all edges of the graph.
- Given a graph G with N vertices, we define two or more **tessellations** in G.

PESC Programa de Engenharia de Sistemas e Computação

Staggered Quantum Walk Model

- Polygons of different overlapping tessellations necessarily share **at least** 1 **vertex**.

Proposition 1 ((PORTUGAL, 2016))

A graph is 2-tessellable if and only if its clique graph is 2-colorable.

Evolution Operators

- Each polygon defines a **unitary vector** in the Hilbert space $\mathcal{H}^N$.

$$|\varphi_i\rangle = \sum_{j \in \varphi_i} \psi_{i,j} |j\rangle \tag{1}$$

- Each **tessellation** defines one **unitary operator** as

$$U_i = 2\left(\sum_{k=0}^{m_i - 1} |\varphi_i\rangle\langle\varphi_i| \right) - I \tag{2}$$

where m_i is the number of polygons of the tessellation and I is the identity matrix.

Relation between Staggered model and Szegedy's model

- Quantum walks in 2-**tessellable** graphs, where **just a unique vertex** is shared by polygons of different tessellations can be cast into a quantum walk in Szegedy's model (SZEGEDY, 2004).
- **Every quantum walk** in Szegedy's model can be cast into a quantum walk in Staggered model with overlapping polygons sharing just one vertex. Thus Szegedy's model is a particular case of Staggered model.

Section 2

Algorithm for Element Distinctness

Element Distinctness problem

- Given a list $\langle x_1, \ldots, x_n \rangle$, such that the elements $x_1, ..., x_n \in \{1, \ldots, M\}$, we want to find if there are indices $i \neq j$, such that $x_i = x_j$;
- Generalization: $\langle x_1, \ldots, x_n \rangle$, such that the elements $x_1, ..., x_n \in \{1, \ldots, M\}$, we want to find if there are k indices $i_1 \neq \cdots \neq i_k$ such that $x_{i_1} = \cdots = x_{i_k}$.

Definitions of Variables

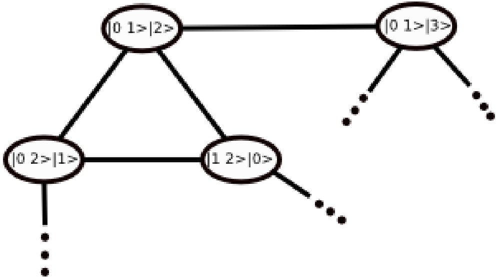
- Let $r = \lfloor N^{k/(k+1)} \rfloor$, where k is the number of collisions;
- S, such that $S \subseteq [N]$ and $|S| = r$, where $[N]$ is the set of indices, and;
- y, such that $y \in [N], y \notin S$.
- For each set S we have $(N - r)$ values y associated.
- Let each pair (S, y) be a vertex for a graph G.
- $|S\rangle|y\rangle$, in the space $\mathcal{H} = \binom{N}{r}(N - r)$, represents the vertices.
- We need $O(r \log N)$ qubits of memory to save these vertices.

PESC Programa de Engenharia de Sistemas e Computação

Graph Construction

Definition 1

We define a graph G with $\binom{N}{r}(N-r)$ vertices. A vertex v corresponds to a pair (S, y). Will exist an edge between two vertices v and v', for $v = (S, y)$ and $v' = (S', y')$, if and only if, either (i) $S' = S$ and $y \neq y'$, or (ii) $S' = S \cup \{y\} \backslash \{y'\}$.



Constructions of the Tessellations

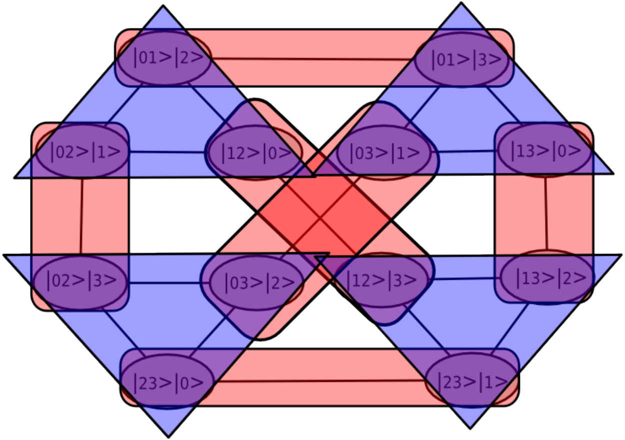
Definition 2

We define two tessellations on graph G. The first one is defined by polygons that cover cliques, where for each pair of vertices (v, v') in those cliques, such that $v = (S, y)$ and $v' = (S', y')$, we have $S = S'$ and $y \neq y'$. The second one is defined for polygons that cover cliques, where for each pair of vertices (v, v') in those cliques, such that $v = (S, y)$ and $v' = (S', y')$, we have $S' = S \cup \{y\} \backslash \{y'\}$.

PESC Programa de Engenharia de Sistemas e Computação

Example

Let $[N] = [0, 1, 2, 3]$. For $k = 2$, we have $r = 2$.



Some Propositions

Proposition 2

Every graph constructed by Definitions 1 and 2 can be cast into a bipartite graph.

Proposition 3

A graph constructed by Definitions 1 and 2 always has at most 1 *vertex in the intersection of two polygons of distinct tessellations.*

PESC
Programa de Engenharia de Sistemas e Computação

Algorithm's Evolution Operators

Each tessellation generates a unity operator as

$$U_0 = 2\left[\sum_{i=0}^{\binom{N}{r}-1}\left(\frac{1}{N-r}|S_i,y\rangle\langle S_i,y| + \sum_{\substack{y'\neq y\\ y'\notin S}}\frac{1}{N-r}|S_i,y'\rangle\langle S_i,y'|\right)\right] - I$$

and

$$U_1 = 2\left[\sum_{i=0}^{\binom{N}{r}-1}\left(\frac{1}{r+1}|S_i,y\rangle\langle S_i,y| + \sum_{\substack{y'\neq y\\ y'\in S}}\frac{1}{r+1}|S_i',y'\rangle\langle S_i',y'|\right)\right] - I,$$

where $|S_i'\rangle = |S_i \cup \{y\} \backslash \{y'\}\rangle$.

The Staggered Quantum Algorithm

Algorithm 1: *Element k-distinctness Algorithm (Single-solution)*

1. Generate the uniform superposition

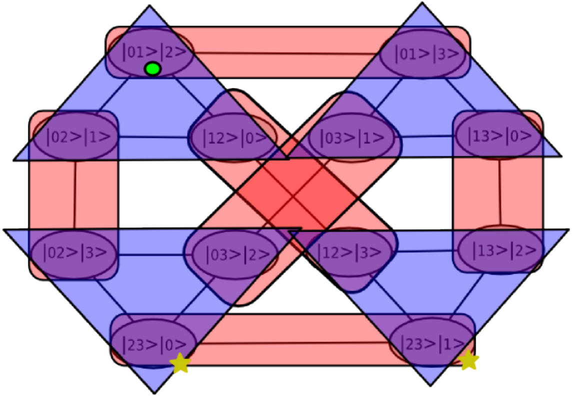
$$\frac{1}{\sqrt{\binom{N}{r}(N-r)}} \sum_{|S|=r, y \notin S} |S, y\rangle.$$

2. $t_1 = O((N/r)^{k/2})$ times repeat:
   (i) Apply the conditional phase flip ($|S, y\rangle \rightarrow -|S, y\rangle$), such that $x_{i_1} = ... = x_{i_k}$ for $i_1 \neq ... \neq i_k$, and $i_1, ..., i_k \in S$.
   (ii) Apply $U_1 U_0$ $t_2 = O(\sqrt{r})$ times.
3. Measure the final state. Check if S contains a k-collision and answer "There is a k-collision" or "There is no k-collision" according to the result.

PESC Programa de Engenharia de Sistemas e Computação

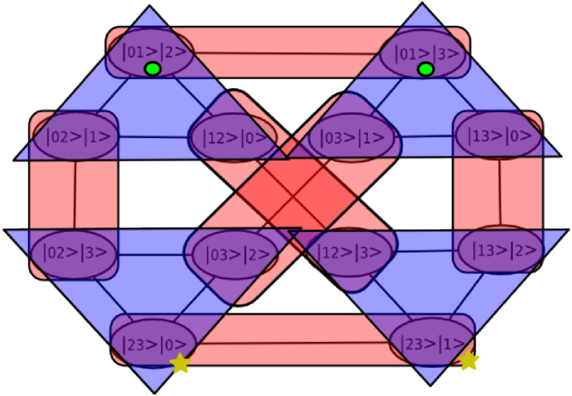
Execution example

For simplicity, let's ignore the superposition supposing that the walker starts at the node $|01\rangle|2\rangle$.



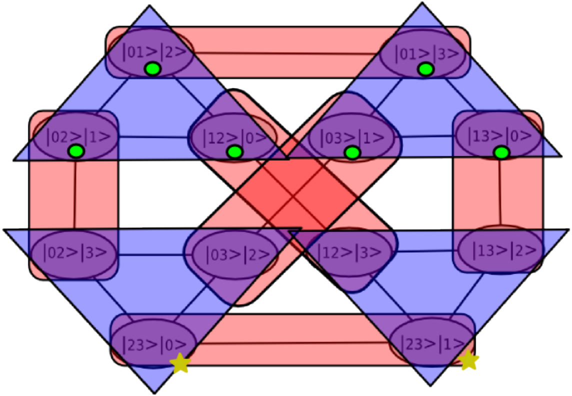
Execution example

We apply the operator U_0.



Execution example

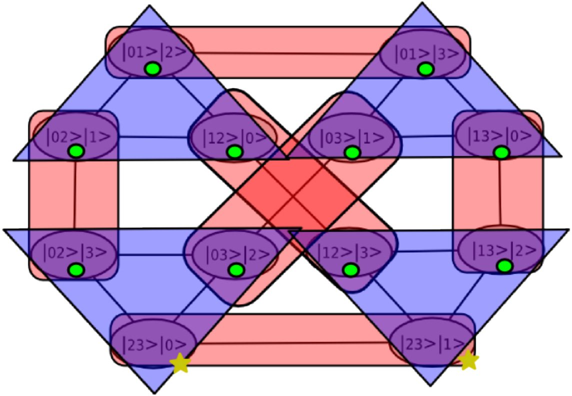
We apply the operator U_1.



PESC Programa de Engenharia de Sistemas e Computação

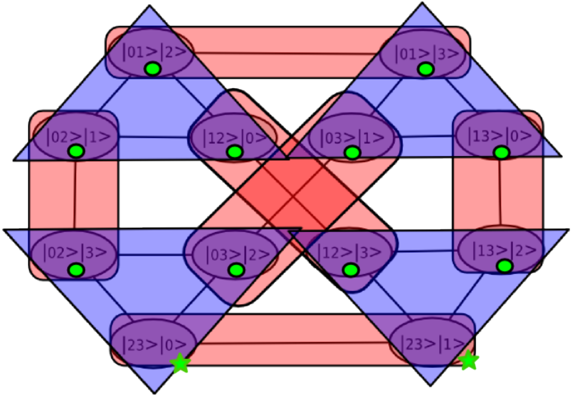
Execution example

We apply the operator U_0, again.



Execution example

44 We apply the operator U_1, again.



Correctness and Complexity of Algorithm

- We proved that we can cast the algorithm presented into Ambainis' algorithm for Element Distinctness (AMBAINIS, 2007);
- Ambainis' algorithm uses a quantum walk in a bipartite graph, needing $O(N^{k/(k+1)})$ queries and $O(r(\log N + \log M))$ qubits of memory.

PESC Programa de Engenharia de Sistemas e Computação

Relation with Ambainis' algorithm

Proposition 4

The initial states of Ambainis' algorithm and the algorithm presented are equivalents.

Proposition 5

The evolution operators of Ambainis' algorithm and the algorithm presented are equivalents.

Proposition 6 ((PORTUGAL et al., 2016))

Every instance of a staggered quantum walk in a 2-tessellable graph with at most one vertex in common in the intersection of two polygons of different tessellations can be cast into a Szegedy's quantum walk.

Final Considerations

- By the previously propositions we can cast the graph presented into a bipartite graph.
- As the initial states and the evolution operators of the both algorithms are equivalents, after T applications of the evolution operators on the initial state, we will reach the same final state.
- Thus, the algorithm presented needs just $O(N^{k/k+1})$ steps of quantum walks to find a answer, however, needing $O(r \log N)$ qubits of memory.

PESC
Programa de Engenharia de Sistemas e Computação

References

AMBAINIS, A. Quantum algorithm for element distinctness. *SIAM Journal on Computing*, v. 37, n. 1, p. 210—-239, 2007.

PORTUGAL, R. Staggered quantum walks on graphs. *arXiv preprint arXiv:1603.02210*, 2016.

PORTUGAL, R. et al. The staggered quantum walk model. *Quantum Information Processing*, Springer, v. 15, n. 1, p. 85–101, 2016.

SZEGEDY, M. Quantum speed-up of markov chain based algorithms. In: IEEE. *Foundations of Computer Science, 2004. Proceedings. 45th Annual IEEE Symposium on*. [S.l.], 2004. p. 32–41.

Thank you!

santiago@cos.ufrj.br

This presentation is available in
www.cos.ufrj.br/∼santiago/ccisPresentation.pdf