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Even Pairs in Bull-reducible Graphs

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Abstract. A bull is a graph with five vertices a, b, c, d, e and five edges ab, bc, cd, be, ce. A graph G is bull-reducible if no vertex of G lies in two bulls. An even pair is a pair of vertices such that every chordless path joining them has even length. We prove that for every bull-reducible Berge graph G with at least two vertices, either G or its complementary graph $\overline{G}$ has an even pair.

1. Introduction

A graph is *perfect* if for every induced subgraph H of G the chromatic number of H is equal to its clique number. Perfect graphs were defined by Claude Berge [1]. The study of perfect graphs led to several interesting and difficult problems. The first one is their characterization. Berge conjectured that a graph is perfect if and only if it contains no odd hole and no odd antihole, where a hole is a chordless cycle of length at least 4, and an antihole is the complementary graph of a hole. It has become customary to call *Berge graph* any graph that contains no odd hole and no antihole, and to call the above conjecture the "Strong Perfect Graph Conjecture". This conjecture was proved by Chudnovsky, Robertson, Seymour, and Thomas [4] in 2002. A second problem is the existence of a polynomial-time algorithm to color optimally the vertices of a perfect graph. This problem was solved in 1984 by Grötschel, Lovász and Schrijver [10] with an algorithm based on the ellipsoid method for linear programming. A third problem is the existence of a polynomial-time algorithm to decide if a graph is Berge. This was solved by Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [3] in 2002. There remains a number of interesting open problems in the context of perfect graphs. Some of them are related to the concept of even pair.

Even pairs: An *even pair* [18] in a graph G is a pair of vertices such that every chordless path between them has even length. A graph G is called a *quasi-parity*

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graph [18] if for every induced subgraph H of G on at least two vertices, either H has an even pair or $\overline{H}$ has an even pair. A graph G is called a *strict quasi-parity graph* [18] if every induced subgraph of G on at least two vertices has an even pair. Clearly, strict quasi-parity graphs are quasi-parity graphs. Meyniel [18] proved that every quasi-parity graph is perfect. The concept of even pair turned out to be very useful for proving that certain classes of Berge graphs are perfect and for designing optimization algorithms on special classes of perfect graphs. See [8] for a survey on this matter. Some questions of particular interest are the characterization of quasi-parity graphs and of strict-quasi-parity graphs. Hougardy [14, 15] (see also [8]) made two conjectures: (1) there is a family F of line-graphs of bipartite graphs such that a graph is a strict quasi-parity graph if and only if it does not contain an odd hole, an antihole, or a graph in F; (2) there is a family F' of line-graphs of bipartite graphs such that a graph is a quasi-parity graph if and only if it does not contain an odd hole, an odd antihole, or a graph in F'. These two conjectures are still unsolved.

Bull-free graphs: A *bull* is a graph with five vertices r, y, x, z, s and five edges ry, yx, yz, xz, zs; see Figure 1. We will frequently use the notation $r - yxz - s$ for such a graph. Chvátal and Sbihi [6] proved in 1987 that every bull-free Berge graph is perfect. Subsequently Reed and Sbihi [20] gave a polynomial-time algorithm for recognizing bull-free Berge graphs. De Figueiredo, Maffray and Porto [9] proved that every bull-free Berge graph is a quasi-parity graph, and that every bull-free Berge graph with no antihole is a strict quasi-parity graph. Hayward [11] proved that every bull-free graph with no antihole if perfectly orderable (see [5, 13] for this definition), as conjectured by Chvátal. These results also settled Hougardy's above two conjectures for bull-free graphs.

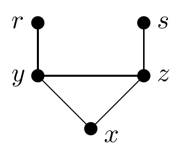

FIGURE 1. The bull $r - yxz - s$.

Bull-reducible graphs: A graph G is called *bull-reducible* if every vertex of G lies in at most one bull of G. Clearly, bull-free graphs are bull-reducible. Everett, de Figueiredo, Klein and Reed [7] proved that every bull-reducible Berge graph is perfect. Although this result now follows directly from the Strong Perfect Graph Theorem [4], the proof given in [7] is much simpler and leads moreover to a polynomial-time recognition algorithm for bull-reducible Berge graphs whose complexity is lower than that given for all Berge graphs in [3]. Here we will prove:

Theorem 1. *Let G be a bull-reducible Berge graph with at least two vertices. Then either G or $\overline{G}$ has an even pair.*

We note that this theorem settles Hougardy's above two conjectures in the case of bull-reducible graphs. The proof of this theorem is given in Section 3, while Section 2 presents some technical lemmas. We tend to follow the standard terminology of graph theory [2], but we will use the verb "sees" instead of "is adjacent to" and "misses" instead of "is not adjacent to".

2. Some technical lemmas

As in [20], call *wheel* a graph made of an even hole of length at least 6 plus a vertex that sees all vertices of this hole. Say that a proper subset H of vertices of a graph G is *homogeneous* if every vertex of $V(G) \setminus H$ either sees all vertices of H or misses all vertices of H and $2 \leq |H| \leq |V(G)| - 1$. We recall two lemmas from [7].

Lemma 2 ([7]**).** *Let G be a bull-reducible odd hole-free graph, and let C be a shortest even hole of length at least 6 in G, with its vertices colored alternately red and blue. Let v be any vertex in $V(G) \setminus V(C)$. Then v satisfies exactly one of the following conditions:*

- $N(v) \cap V(C) = \emptyset$;
- $N(v) \cap V(C) = V(C)$, *so C and v form a wheel;*
- $N(v) \cap V(C)$ *consists in either all red vertices and no blue vertex or all blue vertices and no red vertex;*
- $N(v) \cap V(C)$ *consists in either one, or two consecutive or three consecutive vertices of C;*
- $N(v) \cap V(C)$ *consists in two vertices at distance 2 along C;*
- *C has length 6 and $N(v) \cap V(C)$ consists in four vertices such that exactly three of them are consecutive.* □

Lemma 3 (Wheel Lemma [7]**).** *Let G be a bull-reducible odd hole-free graph. If G contains a wheel, then G contains a homogeneous set.* □

Now we give a few more lemmas that will be useful in the proof of the main result.

Lemma 4. *Let G be a bull-reducible odd hole-free graph. Let $P = u_0\text{-}\cdots\text{-}u_r$ be a chordless path of G of odd length $r \geq 5$, and let c be a vertex of $V(G) \setminus V(P)$ that sees u_0 and u_r. Then up to symmetry we have either:*

1. $N(c) \cap V(P) = V(P)$;
2. $N(c) \cap V(P) = \{u_0, u_1, u_r\}$ *or* $\{u_0, u_1, u_3, u_r\}$, *and in this case there is a bull* $u_r - cu_0u_1 - u_2$;
3. $r = 5$ *and* $N(c) \cap V(P) = \{u_0, u_1, u_2, u_3, u_5\}$, *and in this case there is a bull* $u_0 - cu_2u_3 - u_4$.

Proof. Since G contains no odd hole, c has two consecutive neighbors along P. If outcome 1 of the Lemma does not hold, then up to symmetry there exists an integer $i \in \{0, \ldots, r\}$ such that c sees u_i, u_{i+1} and misses u_{i+2}. Clearly $i \leq r - 3$.

Suppose i is odd. So $i \leq r-4$. We find a first bull $u_r - cu_iu_{i+1} - u_{i+2}$. Then $i = 1$, for otherwise we find a second bull $u_0 - cu_iu_{i+1} - u_{i+2}$ containing c. Then c misses every u_j with $5 \leq j \leq r-1$, for otherwise we find a second bull $u_j - cu_1u_2 - u_3$ containing c. Then c sees u_4 for otherwise $\{c, u_2, u_3, \ldots, u_r\}$ induces an odd hole. Then $r < 7$ for otherwise $\{c, u_4, u_5, \ldots, u_r\}$ induces an odd hole. So $r = 5$. But then we find a second bull $u_0 - cu_5u_4 - u_3$ containing c. Thus i is even.

Suppose $i = 0$. Then we find a first bull $u_r - cu_0u_1 - u_2$; and then c misses every c_j with $4 \leq j \leq r-1$, for otherwise we find a second bull $u_j - cu_0u_1 - u_2$ containing c. So we obtain outcome 2.

Suppose i is even and $i \geq 2$. Then we find a first bull $u_0 - cu_iu_{i+1} - u_{i+2}$. Then $i = r-3$, for otherwise we find a second bull $u_r - cu_iu_{i+1} - u_{i+2}$ containing c. Then c sees u_{i-1}, for otherwise we find a second bull $u_{i-1} - u_icu_{i+1} - u_{i+2}$ containing c. If $r = 5$ we have outcome 3. So suppose $r \geq 7$, so $i \geq 4$. Then c sees u_{i-2}, for otherwise we find a second bull $u_{i-2} - u_{i-1}u_ic - u_r$ containing c. But then we find a second bull $u_{i-2} - cu_iu_{i+1} - u_{i+2}$ containing c. This completes the proof of the lemma. □

A P_4 is a chordless path on four vertices. We call *double broom* the graph made of a P_4 (called the central P_4 of the double broom), plus two non-adjacent vertices a, b that see all vertices of the P_4, plus a vertex a' that sees only a and a vertex b' that sees only b. See Figure 2.

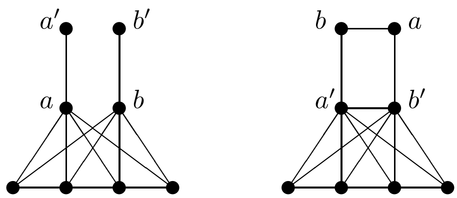

FIGURE 2. A double broom and its complement

Lemma 5. *Let G be a bull-reducible Berge graph. Let P be a chordless odd path of G of length at least 5, and let a, b, a', b' be four vertices of G such that aa' and bb' are edges, ab' and ba' are not edges, a, b see the two endpoints of P, and a', b' miss the two endpoints of P. Then G or $\overline{G}$ contains a double broom.*

Proof. Note that a, b do not lie on P. On the other hand, a', b' may be interior vertices of P. Put $P = u_0\text{-}u_1\text{-}\cdots\text{-}u_r$, with odd $r \geq 5$. Note that each of ab and $a'b'$ may be an edge or not. More precisely, if ab is an edge then $a'b'$ is an edge, for otherwise we find two intersecting bulls $a' - au_0b - b'$ and $a' - au_rb - b'$. Conversely, if $a'b'$ is an edge then ab is an edge, for otherwise $\{a', a, u_0, b, b'\}$ induces an odd hole.

We can apply Lemma 4 to P and each of a, b. If we have outcome 2 for one of a, b, say for a, then (regardless of symmetry) there is a bull containing a, u_0, u_r;

and then we do not have outcome 2 or 3 for b, for otherwise there would be a second bull containing one of u_0, u_r. If we have outcome 3 for both a, b, then we find two bulls containing u_2, u_3. Therefore we must have outcome 1 for at least one of a, b, say for b, that is, b sees all vertices of P. It follows that a' does not lie on P. We claim that a' misses every vertex of P. For suppose the contrary. Then, up to symmetry, a' sees u_i and misses u_{i-1} with $1 \leq i \leq (r-1)/2$, and we find a bull $a' - u_i u_{i-1} b - u_r$. Then a' sees u_{r-1}, for otherwise we find a second bull $a' - u_i u_{i-1} b - u_{r-1}$ containing b. But then we find a second bull $a' - u_{r-1} u_r b - u_0$ containing b, a contradiction. So the claim holds.

If we have outcome 2 for a, then $u_r - a u_0 u_1 - u_2$ and $a' - a u_0 u_1 - u_2$ are two intersecting bulls, a contradiction. If we have outcome 3 for a, then $u_0 - a u_2 u_3 - u_4$ and $a' - a u_2 u_3 - u_4$ are two intersecting bulls. So a sees all vertices of P, which restores the symmetry between a and b, and thus b' does not lie on P and misses every vertex of P. Now, if both $ab, a'b'$ are non-edges, then $\{u_0, u_1, u_2, u_3, a, b, a', b'\}$ induces a double broom in G, while if both are edges, the same subset induces a double broom in $\overline{G}$. This completes the proof of the lemma. □

Lemma 6. *Let G be a bull-reducible C_5-free graph that contains a double broom. Then G has a homogeneous set that contains the central P_4 of the double broom.*

Proof. Pick any double broom of G, and label its vertices $w_1, w_2, w_3, w_4, a, b, a', b'$ so that its edges are w_1w_2, w_2w_3, w_3w_4, aw_1, aw_2, aw_3, aw_4, bw_1, bw_2, bw_3, bw_4, aa', bb'. Vertices w_1, w_2, w_3, w_4 form the central P_4 of the double broom and we write $W = \{w_1, w_2, w_3, w_4\}$. We partition the vertices of $V(G) \setminus W$ as follows:

- Let T be the set of vertices of $V(G) \setminus W$ that see all of w_1, w_2, w_3, w_4.
- Let P be the set of vertices of $V(G) \setminus W$ that see at least one but not all of w_1, w_2, w_3, w_4.
- Let F be the set of vertices of $V(G) \setminus W$ that see none of w_1, w_2, w_3, w_4.

Clearly the four sets W, T, P, F are pairwise disjoint and their union is $V(G)$. Note that $a, b \in T$ and $a', b' \in F$. We define some subsets of T as follows:

$$A = \{t \in T \mid ta' \in E, tb' \notin E\};$$
$$B = \{t \in T \mid ta' \notin E, tb' \in E\};$$
$$C = \{t \in T \mid ta' \in E, tb' \in E\}.$$

Note that A, B, C are pairwise disjoint and that $a \in A$, $b \in B$.

Claim 6.1. *There is no edge between A and B.*

Proof. For suppose there is an edge uv with $u \in A$, $v \in B$. Then $a' - uw_iv - b'$ is a bull, for every $i = 1, \ldots, 4$, so a' belongs to four bulls, a contradiction. □

Claim 6.2. *If $p \in P$, then:*

1. *There exist adjacent vertices $w_g, w_h \in W$ such that p sees w_g and misses w_h;*
2. *There exist nonadjacent vertices w_r, $w_s \in W$ such that p sees w_r and misses w_s.*

Proof. This follows directly from the definition of P and the fact that W induces a connected subgraph in G and in $\overline{G}$. □

Claim 6.3. *Every vertex of P sees all of $A \cup B \cup C$ and none of a', b'.*

Proof. Consider any $p \in P$ and $u \in A$. We first prove that p sees u. Suppose on the contrary that p misses u.

Case 1: There is a subpath w-w'-w'' of W such that p sees both w, w' and misses w''. If p misses a', then $p - w'w''u - a'$ is a bull, while if p sees a', then $a' - pww' - w''$ is a bull. In either case, p must miss b' for otherwise $b' - pww' - w''$ is a second bull containing p, a contradiction. Then p sees b or else $p - w'w''b - b'$ is a second bull containing p. But then $u - w'pb - b'$ is a second bull containing p. So p sees u.

Case 2: p sees exactly one of w_1, w_2 and misses w_4. Then $\{p, w_1, w_2, u, w_4\}$ induces a bull. This implies $pb \in E$, for otherwise $\{p, w_1, w_2, b, w_4\}$ induces a second bull containing p, a contradiction. Then p sees a', for otherwise $\{p, w_1, w_2, u, a'\}$ induces a second bull containing p. But then $\{a', p, w_g, b, w_4\}$, where $g \in \{1, 2\}$ is such that p sees w_g, induces a second bull containing p, a contradiction. So p sees u. The case where p sees exactly one of w_3, w_4 and misses w_1 is symmetric.

It is easy to see that if we are not in one of the above two cases, and up to symmetry, then p sees w_1, w_4 and misses w_2, w_3; but then $\{p, w_1, w_2, w_3, w_4\}$ induces a C_5, a contradiction. Thus we have proved that p sees u, and so p sees every vertex of $A \cup B$.

Now we prove that p misses both a' and b'. By Claim 6.2 there are two nonadjacent vertices $w_r, w_s \in W$ such that p sees w_r and misses w_s. Suppose that p sees a'. Then $a' - pw_rb - w_s$ is a bull. Now if p sees b', then $b' - pw_ra - w_s$ is a second bull containing p; while if p misses b', then $a' - pw_rb - b'$ is a second bull containing p, in either case a contradiction. So p misses a' and by symmetry it misses b'.

Finally, we prove that p sees every vertex $c \in C$. Recall that c sees both a', b'. By Claim 6.2, there are two adjacent vertices $w_g, w_h \in W$ such that p sees w_g and misses w_h. Then p sees c for otherwise we find two bulls $p - w_gw_hc - a'$ and $p - w_gw_hc - b'$ that contain p, a contradiction. Thus Claim 6.3 holds. □

Now we define subsets X, Z of F and a subset Y of $T \setminus (A \cup B \cup C)$ as follows:

- $x \in X$ if $x \in F$ and there exists in G a path p-x_1-$\cdots$-x_i, with $p \in P$, $i \geq 1$, $x_1, x_2, \ldots, x_i \in F$ and $x = x_i$. Any such path will be called a *forcing sequence* for x.
- $y \in Y$ if $y \in T \setminus (A \cup B \cup C)$ and there exists in $\overline{G}$ a path x-y_1-$\cdots$-y_j, with $x \in P \cup X$, $j \geq 1$, $y_1, y_2, \ldots, y_j \in T \setminus (A \cup B \cup C)$, and $y = y_j$. Note that if x is not in P there exists a forcing sequence p-x_1-$\cdots$-x_i for $x = x_i$. In this case the sequence p-x_1-$\cdots$-x_i-y_1-$\cdots$-y_j will be called a forcing sequence for y. In case $x \in P$ the sequence x-y_1-$\cdots$-y_j will be called a forcing sequence for y. In either case a forcing sequence for y can be denoted by x_0-$\cdots$-x_i-y_1-$\cdots$-y_j with $i \geq 0$ and $j \geq 1$.

- $z \in Z$ if $z \in F \setminus X$ and there exists in G a path y-z_1-$\ldots$-z_k, with $y \in Y$, $k \geq 1$, $z_1, z_2, \ldots, z_k \in F \setminus X$, and $z = z_k$. Note that there exists a forcing sequence x_0-x_1-$\cdots$-x_i-y_1-$\cdots$-y_j for $y = y_j$, with $i \geq 0$ and $j \geq 1$. The sequence x_0-x_1-$\cdots$-x_i-y_1-$\cdots$-y_j-z_1-$\cdots$-z_k will be called a forcing sequence for z.

Naturally we can consider for each $v \in X \cup Y \cup Z$ a shortest forcing sequence. Such sequences have notable properties which we express in the following claims.

Claim 6.4.

1. *If $x \in X$ and p-x_1-$\cdots$-x_i is a shortest forcing sequence for $x = x_i$ then it is a chordless path of G.*
2. *If $y \in Y$ and x_0-x_1-$\cdots$-x_i-y_1-$\cdots$-y_j is a shortest forcing sequence for $y = y_j$, with the above notation, then x_0-x_1-$\cdots$-x_i is a chordless path of G, x_i-y_1-$\cdots$-y_j is a chordless path of $\overline{G}$, and, if $i \geq 1$, each of $x_0, x_1, \ldots, x_{i-1}$ sees each of $y_1, \ldots, y_j$.*
3. *If $z \in Z$ and p-x_1-$\cdots$-x_i-y_1-$\cdots$-y_j-z_1-$\cdots$-z_k is a shortest forcing sequence for $z = z_k$, with the above notation, then p-x_1-$\cdots$-x_i is a chordless path of G, x_i-y_1-$\cdots$-y_j is a chordless path of $\overline{G}$, y_j-z_1-$\cdots$-z_k is a chordless path of G, each of $p, x_1, \ldots, x_{i-1}$ sees each of $y_1, \ldots, y_j$, and each of $p, x_1, \ldots, x_i, y_1, \ldots, y_{j-1}$ misses each of $z_1, \ldots, z_k$.*

Proof. The claim follows routinely from the definition of X, Y, Z and from the definition of a shortest forcing sequence. Details are omitted. □

Claim 6.5. *If $y \in Y$, a shortest forcing sequence for y contains at most two vertices of X.*

Proof. For suppose on the contrary that there exists a shortest forcing sequence $S = p$-x_1-$\cdots$-x_i-y_1-$\cdots$-y_j with $j \geq 1$ and $i \geq 3$. Then S satisfies the properties stated in Claim 6.4, part 2. Then for each $h = 1, \ldots, 4$ we find a bull $w_h - y_1 x_{i-2} x_{i-1} - x_i$ that contains y_1, so y_1 lies in four bulls, a contradiction. □

Claim 6.6. *If $z \in Z$, a shortest forcing sequence for z contains no vertex of X.*

Proof. For suppose on the contrary that $S = p$-x_1-$\cdots$-x_i-y_1-$\cdots$-y_j-z_1-$\cdots$-z_k is a shortest forcing sequence for $z = z_k$ with $i \geq 1$. Recall that S satisfies the properties stated in Claim 6.4, part 3. By Claim 6.2, there are nonadjacent vertices $w_r, w_s \in W$ such that p sees w_r and misses w_s. By the preceding claim we have $i \leq 2$. Suppose $i = 1$. Then $w_s - y_1 w_r p - x_1$ is a bull. If $j = 1$, then $z_1 - y_1 w_r p - x_1$ is a second bull containing p; if $j = 2$, then $z_1 - y_2 x_1 p - y_1$ is a second bull containing p; if $j \geq 3$, then $z_1 - y_j y_{j-2} p - y_{j-1}$ is a second bull containing p; in either case we have a contradiction. So $i = 2$. Then $w_s - y_1 p x_1 - x_2$ is a bull. If $j = 1$, then $z_1 - y_1 p x_1 - x_2$ is a second bull containing p; if $j = 2$, then $z_1 - y_2 x_2 x_1 - y_1$ is a second bull containing x_1; if $j \geq 3$, then $z_1 - y_j y_{j-2} p - y_{j-1}$ is a second bull containing p; in either case we have a contradiction. Thus the claim holds. □

Claim 6.7. *If $z \in Z$, a shortest forcing sequence for z contains at most two vertices of Y.*

Proof. For let $S = p\text{-}y_1\text{-}\cdots\text{-}y_j\text{-}z_1\text{-}\cdots\text{-}z_k$ be a shortest forcing sequence for $z = z_k$. The sequence S satisfies the properties stated in Claim 6.4, part 3, and it contains no vertex of X by Claim 6.6. Suppose that $j \geq 3$. Then $z_1 - y_j y_{j-2} w_h - y_{j-1}$ is a bull that contains z_1 for each $h = 1, \ldots, 4$, a contradiction. So $j \leq 2$, and the claim holds. □

Let H be the set of vertices that form the connected component of $G \setminus (T \setminus Y)$ that contains W.

Claim 6.8. *$H = W \cup P \cup X \cup Y \cup Z$.*

Proof. Put $H' = W \cup P \cup X \cup Y \cup Z$. First we prove that $H' \subseteq H$. Clearly, $W \subseteq H$. We also have $P \cup X \cup Y \subset H$ since every vertex of $P \cup X \cup Y$ is linked to W by a path in $G \setminus (T \setminus Y)$. Consequently $Z \subset H$, since every vertex of Z is linked to Y by a path in $G \setminus (T \setminus Y)$. So we have $H' \subseteq H$.

Conversely, let $h \in H$. Recall that $V(G)$ is partitioned into the four sets W, P, T, F. If $h \in W \cup P$ then $h \in H'$. If $h \in T$, then, by the definition of H, we have $h \in Y$. If $h \in F$, then, by the definition of H, there exists a path in $G \setminus (T \setminus Y)$ from h to W. Along this path, let v be the first vertex, starting from h, that is not in F. Then v must be in $P \cup W \cup Y$. If $v \in P \cup W$, then $h \in X$. If $v \in Y$, then $h \in Z$. So we have $H \subseteq H'$, and the claim holds. □

Claim 6.9. *Every vertex of H sees all of $T \setminus (A \cup B \cup C \cup Y)$.*

Proof. Consider any $t \in T \setminus (A \cup B \cup C \cup Y)$. So t sees all of W by the definition of T. In addition, t sees all of $P \cup X \cup Y$, for otherwise t would be in Y. Now suppose that t misses a vertex z of Z. There exists a shortest forcing sequence S for z, and by Claims 6.6 and 6.7 we have $S = p\text{-}y_1\text{-}\cdots\text{-}y_j\text{-}z_1\text{-}\cdots\text{-}z_k$ with $z = z_k$ and with $j \in \{1, 2\}$. We may also choose z such that k is as small as possible, so t sees all vertices of $S \setminus z_k$. Let w_g, w_h be two adjacent vertices of W such that p sees w_g and misses w_h. Suppose $j = 1$. Then we find a first bull $p - w_g w_h y_1 - z_1$. If $k = 1$, then $p - t w_h y_1 - z_1$ is a second bull containing p; if $k = 2$, then $p - t y_1 z_1 - z_2$ is a second bull containing p; if $k \geq 3$, then $p - t z_{k-2} z_{k-1} - z_k$ is a second bull containing p; in either case there is a contradiction. So $j = 2$. Then we find a first bull $y_1 - w_g p y_2 - z_1$. If $k = 1$, then $y_1 - t p y_2 - z_1$ is a second bull containing y_1; if $k = 2$, then $y_1 - t y_2 z_1 - z_2$ is a second bull containing y_1; If $k \geq 3$, then $y_1 - t z_{k-2} z_{k-1} - z_k$ is a second bull containing y_1; in either case there is a contradiction. Thus the claim holds. □

Claim 6.10. *Every vertex of X sees all of $A \cup B \cup C$ and none of a', b'.*

Proof. Consider any $x \in X$. By the definition of X, there exists a shortest forcing sequence $S = p\text{-}x_1\text{-}\cdots\text{-}x_i$ for $x = x_i$, with $i \geq 1$, $p \in P$, and $x_1, \ldots, x_{i-1} \in X$. Then S satisfies the properties stated in Claim 6.4, part 1, i.e., S is a chordless path. Let w_r, w_s be nonadjacent vertices of W such that p sees w_r and misses w_s. We argue by induction on i.

Assume $i = 1$. Let $u \in A \cup C$, and suppose that x misses u. We find a first bull $w_s - u w_r p - x$. Then x sees b, for otherwise $w_s - b w_r p - x$ is a second bull containing x. Then x misses a', for otherwise $a' - xpb - w_s$ is a second bull containing x. But then $a' - uw_rp - x$ is a second bull containing x. Hence x sees every vertex of $A \cup C$. Analogously, x sees every vertex of B. Suppose that x sees a'. So we find a first bull $a' - xpb - w_s$. Then x misses b', for otherwise we find a second bull $b' - xpa - w_s$ containing x. But then $b' - bpx - a'$ is a second bull containing x. Hence x misses a', and analogously, x misses b'.

Now assume $i \geq 2$. So vertices x_{i-1} and x_{i-2} are defined, with $x_{i-1} \in X$ and $x_{i-2} \in P \cup X$. Let $u \in A \cup C$, and suppose that x_i misses u. By the induction hypothesis u sees x_{i-1} and x_{i-2}, and we obtain a first bull $x_i - x_{i-1}x_{i-2}u - w_s$. Then x_i sees b, for otherwise $x_i - x_{i-1}x_{i-2}b - w_s$ is a second bull containing x_i. Then x_i misses a', for otherwise $a' - x_ix_{i-1}b - w_s$ is a second bull containing x_i. But then $x_i - x_{i-1}x_{i-2}u - a'$ is a second bull containing x_i. Hence, x_i sees every $u \in A \cup C$. Analogously, x_i sees every vertex of B. Suppose that x_i sees a'. By the induction hypothesis, x_{i-1} sees b. Hence $a' - x_ix_{i-1}b - w_r$ and $a' - x_ix_{i-1}b - w_s$ are two intersecting bulls, a contradiction. Hence, x_i misses a', and analogously, x_i misses b'. Thus the claim holds. □

Claim 6.11. *Every vertex of Y sees all of $A \cup B \cup C$ and none of a', b'.*

Proof. Consider any $y \in Y$. By the definition of Y, there exists a shortest forcing sequence $S = x_0\text{-}\cdots\text{-}x_i\text{-}y_1\text{-}y_2\text{-}\cdots\text{-}y_j$ for $y = y_j$, with $j \geq 1$, and by Claim 6.5 we have $i \leq 2$. Since $Y \subseteq T \setminus (A \cup B \cup C)$, y misses a' and b'. Consider any $u \in A \cup C$. Pick a vertex w as follows: If $i = 0$ then $x_i \in P$ and x_i sees a vertex $w \in W$. If $i > 0$ then we take $w = x_{i-1}$. By Claims 6.3 and 6.10, x_i and w see both u, b and miss both a', b'. Also w sees all of $y_1, \ldots, y_j$ by the definition of Y. We prove by induction on j that y sees u. Suppose the contrary.

Assume $j = 1$. So we find a first bull $a' - ux_iw - y$. If u sees b', we find a second bull $b' - ux_iw - y$, a contradiction. So u misses b', so $u \in A$, so u misses b by Claim 6.1. Then y sees b, for otherwise we find a second bull $b' - bx_iw - y$ containing y. But then we find a second bull $b' - byw - u$ containing y. Hence, y sees every $u \in A \cup C$. Analogously, y sees every vertex of B.

Assume $j \geq 2$. By the induction hypothesis, y_{j-1} sees u and b. Then we find a first bull $a' - uy_{j-1}w - y$. If u sees b', we find a second bull $b' - uy_{j-1}w - y$, a contradiction. So u misses b', so $u \in A$, so u misses b by Claim 6.1. Then y sees b, for otherwise we find a second bull $b' - by_{j-1}w - y$ containing y. But then we find a second bull $b' - byw - u$ containing y. Hence, y sees every $u \in A \cup C$. Analogously, y sees every vertex of B. Thus the claim holds. □

Claim 6.12. *Every vertex of Z sees all of $A \cup B \cup C$ and none of a', b'.*

Proof. Consider any $z \in Z$. By Claims 6.6 and 6.7, there exists a shortest forcing sequence $S = p\text{-}y_1\text{-}\cdots\text{-}y_j\text{-}z_1\text{-}\cdots\text{-}z_k$ for $z = z_k$ with $1 \leq j \leq 2$; and S satisfies the properties given in Claim 6.4, part 3. Consider any $u \in A \cup B \cup C$. So u sees all of W and, by the preceding claims, u sees all of $p, y_1, \ldots, y_j$. As usual there exist adjacent vertices $w_g, w_h \in W$ such that p sees w_g and misses w_h. We prove that z sees u and misses a', b' by induction on k.

Assume $k = 1$. If $j = 1$, we find a bull $z - y_1 w_h w_g - p$. Then z sees u for otherwise we find a second bull $z - y_1 w_h u - p$ containing z. So z sees all of $A \cup B \cup C$. Then z misses a', for otherwise we find a second bull $a' - zy_1 b - p$ containing z. Likewise z misses b'. If $j = 2$, we find a bull $z - y_2 p w_g - y_1$. Then z sees u, for otherwise we find a second bull $z - y_2 p u - y_1$ containing z. So z sees all of $A \cup B \cup C$. Then z misses a', for otherwise we find a second bull $a' - zy_2 b - y_1$ containing z. Likewise z misses b'. So the claim holds when $k = 1$.

Assume $k \geq 2$. By the induction hypothesis, u sees all of $z_1, \ldots, z_{k-1}$. If $j = 1$, we find a bull $p - w_g w_h y_1 - z_1$. Then z sees u, for otherwise we find a second bull $p - uz' z_{k-1} - z$ containing z, where $z' = z_{k-2}$ if $k \geq 3$ and $z' = y_1$ if $k = 2$. If $j = 2$, we find a bull $z_1 - y_2 p w_g - y_1$. Then z sees u, for otherwise we find a second bull $z - z_{k-1} z' u - y_1$ containing y_1, where $z' = z_{k-2}$ if $k \geq 3$ and $z' = y_2$ if $k = 2$. So z sees all of $A \cup B \cup C$. In either case ($j = 1$ or 2), z misses a', for otherwise we find a second bull $a' - zz_{k-1} b - p$ containing p. Likewise z misses b'. Thus the claim holds. □

Claim 6.13. *H is a homogeneous set.*

Proof. Since H is a component of $G \setminus (T \setminus Y)$, it suffices to prove the property that every vertex $v \in H$ sees every vertex $t \in T \setminus Y$. Claim 6.9 establishes this property when $t \in T \setminus (A \cup B \cup C \cup Y)$. Suppose $t \in A \cup B \cup C$. Then when $v \in W$ the property follows from the definition of A, B, C; and when $v \in P, X, Y, Z$ the property follows respectively from Claims 6.3, 6.10, 6.11 and 6.12. Thus the claim holds. □

This completes the proof of Lemma 6. □

3. Even pairs

Recall that a graph is *weakly triangulated* if G and $\overline{G}$ contain no hole of length at least 5. In the case of weakly triangulated the desired result is already known as it was proved by Hayward, Hoàng and Maffray [12] in a stronger form. Say that two non-adjacent vertices form a *2-pair* if every chordless path joining them has length 2.

Theorem 7 ([12]). *Let G be a weakly triangulated graph that is not a clique. Then G has a 2-pair.*

Now we are ready to prove our main result, which we state again:

Theorem 8. *Let G be a bull-reducible Berge graph with at least two vertices. Then either G or $\overline{G}$ has an even pair.*

Proof. We prove Theorem 8 by induction on the number of vertices of the graph G. First, suppose that G and $\overline{G}$ contain no hole of length at least 5. Then G is weakly triangulated. In that case the result follows from Theorem 7. So suppose that G is not weakly triangulated. Suppose that G has a homogeneous set. By the induction hypothesis, the subgraph H induced by this set has two vertices a, b that form an even pair in H or in $\overline{H}$. Since every vertex of $G \setminus H$ either sees both a, b or misses both a, b, it follows that a, b also form an even pair in G or in $\overline{G}$.

Now suppose that G has no homogeneous set and that one of $G, \overline{G}$ contains a hole of length at least 5. By Lemma 3, G and $\overline{G}$ contain no wheel. By Lemma 6, G and $\overline{G}$ contain no double broom. Let l be the number of vertices of a shortest hole of length at least 5 in G or $\overline{G}$. By symmetry, we may assume that G contains a hole of length l. Note that $l \geq 6$ and l is even since G is Berge. So $V(G)$ contains l pairwise disjoint and non-empty subsets $V_1, \ldots, V_l$ such that, for each $i = 1, \ldots, l$ (with subscript arithmetic modulo l), every vertex of V_i sees every vertex of $V_{i-1} \cup V_{i+1}$ and misses every vertex of $V_{i+2} \cup V_{i+3} \cup \cdots \cup V_{i-3} \cup V_{i-2}$. We write $V^* = V_1 \cup V_2 \cup \cdots \cup V_l$. We can choose these sets so that V^* is maximal. Given these subsets, we define some further subsets:

- Let A_1 be the set of vertices of $V(G) \setminus V^*$ that see all of $V_2 \cup V_4 \cup \cdots \cup V_l$ and miss all of $V_1 \cup V_3 \cup \cdots \cup V_{l-1}$;
- Let A_2 be the set of vertices of $V(G) \setminus V^*$ that see all of $V_1 \cup V_3 \cup \cdots \cup V_{l-1}$ and miss all of $V_2 \cup V_4 \cup \cdots \cup V_l$;
- For each $i = 1, \ldots, l$, let X_i be the set of vertices of $V(G) \setminus (V^* \cup A_1 \cup A_2)$ that see all of $V_{i-1} \cup V_{i+1}$ and miss all of $V_{i-2} \cup V_{i+2}$;
- Let $Z = V(G) \setminus (V^* \cup A_1 \cup A_2 \cup X_1 \cup \cdots \cup X_l)$.

Clearly, the sets $V_1, \ldots, V_l, A_1, A_2, X_1, \ldots, X_l, Z$ are pairwise disjoint and their union is $V(G)$. Let us now establish some useful properties of these sets. In the following claims, for each $i = 1, \ldots, l$, we let v_i be an arbitrary vertex of V_i.

Claim 8.1. *For $i = 1, \ldots, l$, if $X_i \neq \emptyset$ then $l = 6$ and every vertex of X_i has a neighbor in V_{i+3}. Moreover, if a vertex of X_i sees all of V_{i+3} then it has a neighbor in V_i.*

Proof. For simpler notation put $i = 3$. Let x be any vertex of X_3. So x sees all of $V_2 \cup V_4$ and misses all of $V_1 \cup V_5$. Then x must have a neighbor in $V_6 \cup \cdots \cup V_l$, for otherwise we could add x to V_3, which would contradict the maximality of V^*. Let h be the smallest index such that x has a neighbor y in V_h with $6 \leq h \leq l$. If $h \geq 7$, the set $\{x, v_4, \ldots, v_{h-1}, y\}$ induces a hole of length $h-2$, with $5 \leq h-2 \leq l-2$, which contradicts G being Berge (if h is odd) or the definition of l (if h is even). So $h = 6$. Suppose $l \geq 8$. Then we can apply Lemma 2 to the hole induced by $\{v_1, v_2, v_3, v_4, v_5, y, \ldots, v_l\}$ and to x, which implies that x sees every v_j with even $j \neq 6$ and misses every v_j with odd j. Then applying Lemma 2 to the hole induced by $\{v_1, \ldots, v_l\}$ implies that x also sees every $v_6 \in V_6$. But then we have $x \in A_1$, which contradicts the definition of X_3. Thus the first part of the claim holds.

To prove the second part, let x be a vertex of X_3 that sees all of V_6. Thus $l = 6$. So x sees all of $V_2 \cup V_4 \cup V_6$ and misses all of $V_1 \cup V_5$. By Lemma 2, if x has no neighbor in V_3 then x must be in A_1, which contradicts the definition of X_3. So x has a neighbor in V_3. Thus the claim holds. □

Claim 8.2. *For $i = 1, \ldots, l$, there is no P_4 in $V_i \cup X_i$.*

Proof. For if there is a P_4 in $V_i \cup X_i$, then its four vertices together with v_{i-1}, v_{i-2}, v_{i+1}, v_{i+2} induce a double broom, a contradiction. □

Claim 8.3. *For $i = 1, \ldots, l$, if i is odd there is no edge between $V_i \cup X_i$ and A_1; and if i is even there is no edge between $V_i \cup X_i$ and A_2.*

Proof. Up to symmetry and for simpler notation we may take $i = 3$ and suppose that there exists an edge da with $d \in V_3 \cup X_3$ and $a \in A_1$. The definition of A_1 implies $d \in X_3$ and so, by Claim 8.1, we have $l = 6$ and d has a neighbor $u_6 \in V_6$. If d has a neighbor $u_3 \in V_3$ then we find two bulls $u_3 - dau_6 - v_5$ and $u_3 - dau_6 - v_1$ containing d, a contradiction. So d has no neighbor in V_3, and so, by Claim 8.1, d has a non-neighbor $w_6 \in V_6$. Then we find two bulls $v_3 - v_4da - w_6$ and $v_3 - v_2da - w_6$ containing d, a contradiction. Thus the claim holds. □

Claim 8.4. *For $i = 1, \ldots, l$, there is no edge between $V_i \cup X_i$ and $V_{i+2} \cup X_{i+2}$.*

Proof. Put $i = 3$, and suppose that there is an edge xy with $x \in V_3 \cup X_3$ and $y \in V_5 \cup X_5$. Since x has a neighbor in $V_5 \cup X_5$ we have $x \notin V_3$, so $x \in X_3$; and then, by Claim 8.1, we have $l = 6$ and x has a neighbor $u_6 \in V_6$. Likewise, y is in X_5 and has a neighbor $u_2 \in V_2$. If x has a non-neighbor $w_6 \in V_6$ and y has a non-neighbor $w_2 \in V_2$ then $\{x, y, w_6, v_1, w_2\}$ induces a C_5, a contradiction. So we may assume, up to symmetry, that x sees all of V_6. Then, by Claim 8.1, x has a neighbor $w_3 \in V_3$. So we find a first bull $w_3 - xyu_6 - v_1$. If y has a neighbor $w_5 \in V_5$, then we find a second bull $w_5 - yxu_2 - v_1$ containing x, a contradiction. So y has no neighbor in V_5, and, by Claim 8.1, y has a non-neighbor $w_2 \in V_2$. But then we find a second bull $v_1 - w_2w_3x - y$, a contradiction. Thus the claim holds. □

Claim 8.5. *For $i = 1, \ldots, l$, let x be a vertex that has a neighbor and a non-neighbor in $V_i \cup X_i$. If x has a neighbor in V_{i-1}, then it misses all of V_{i+2}. Likewise, if it has a neighbor in V_{i+1}, then it misses all of V_{i-2}.*

Proof. Put $i = 3$ and let a, b respectively be a neighbor and a non-neighbor of x in $V_3 \cup X_3$. Recall that a, b see all of $V_2 \cup V_4$ and miss all of $V_1 \cup V_5$. Suppose up to symmetry that x has neighbors $u_2 \in V_2$ and $u_5 \in V_5$. Then x sees every $v_4 \in V_4$, for otherwise $\{x, u_2, b, v_4, u_5\}$ induces an odd hole. Then Lemma 2, applied to x and the hole induced by $\{v_1, u_2, v_3, v_4, u_5, v_6, \ldots, v_l\}$ for every $v_3 \in V_3$, $v_6 \in V_6$, $v_1 \in V_1$, and the fact that G contains no wheel, implies that $l = 6$ and that x sees every vertex of $V_6 \cup V_4$ and none of $V_1 \cup V_3$. So $x \in A_1 \cup X_5$; and since x has a neighbor, we have $x \in X_5$; but then the edge xa contradicts Claim 8.4. Thus the claim holds. □

Claim 8.6. *For $i = 1, \ldots, l$, there is no chordless odd path of G of length at least 5 whose two endpoints are in $V_i \cup X_i$.*

Proof. For suppose that there is such a path P. Then its two endpoints see both v_{i-1}, v_{i+1} and miss both v_{i-2}, v_{i+2}, and so we can apply Lemma 5 in G to P and vertices $v_{i-1}, v_{i+1}, v_{i-2}, v_{i+2}$, which implies that G or $\overline{G}$ contains a double broom, a contradiction. □

Claim 8.7. *For $i = 1, \ldots, l$, there is no chordless odd path in $\overline{G}$ of length at least 5 whose two endpoints are in $V_i \cup X_i$.*

Proof. For suppose that there is such a path Q in $\overline{G}$. Then, in $\overline{G}$, its two endpoints see both v_{i-2}, v_{i+2} and miss both v_{i-1}, v_{i+1}, and so we can apply Lemma 5 in $\overline{G}$ to Q and vertices $v_{i-1}, v_{i+1}, v_{i-2}, v_{i+2}$, which implies that G or $\overline{G}$ contains a double broom, a contradiction. □ □

Claim 8.8. *For $i = 1, \ldots, l$, suppose that there exists a chordless path x-a-b-y in G with $a, b \in V_i \cup X_i$. Then one of x, y is in $V_i \cup X_i$.*

Proof. Put $i = 3$, and suppose that x sees v_2, v_4. By Claim 8.5, x misses all of $V_1 \cup V_5$. If x has a non-neighbor $w_2 \in V_2$, we find two intersecting bulls $v_1 - w_2ba - x$ and $w_2 - axv_4 - v_5$. So x sees all of V_2; likewise x sees all of V_4. So $x \in V_3 \cup X_3 \cup A_2$; actually, since x sees a and by Claim 8.3, we have $x \in V_3 \cup X_3$. So the claim holds in this case. It holds similarly if y sees v_2, v_4.

Suppose now that x does not see both v_2, v_4, and the same for y. At least one of x, y must see at least one of v_2, v_4, for otherwise we find two intersecting bulls $x - av_2b - y$ and $x - av_4b - y$. So assume x sees v_2 and misses v_4. By Claim 8.5, x misses v_5, and so we find a bull $x - abv_4 - v_5$. Then y sees v_4, for otherwise we find a second bull $x - av_4b - y$ containing a. Then y misses v_1 by Claim 8.5 and v_2 by the preceding paragraph. But then we find a second bull $y - bav_2 - v_1$ containing a. Thus the claim holds. □

Claim 8.9. *For $i = 1, \ldots, l$, suppose that there exists a chordless path a-u-v-b in G with $a, b \in V_i \cup X_i$. Then one of u, v is in $V_i \cup X_i$.*

Proof. Put $i = 3$. So a, b see all of $V_2 \cup V_4$ and miss all of $V_1 \cup V_5$.

First consider the case where one u, v, say u, has a neighbor in each of V_2, V_4. Let $u_2 \in V_2, u_4 \in V_4$ be neighbors of u. By Claim 8.5, u misses all of $V_1 \cup V_5$. Suppose that u has a non-neighbor $w_2 \in V_2$. Then we find a first bull $w_2 - auu_4 - v_5$. Vertex v sees w_2, for otherwise $\{w_2, a, u, v, b\}$ induces an odd hole. Then, by Claim 8.5, v misses all of V_5. Vertex v sees v_1, for otherwise we find a second bull $v_1 - u_2au - v$ containing a. Then, by Claim 8.5, v misses all of V_4. But then we find a second bull $v_5 - u_4au - v$ containing a. So u sees all of V_2, and similarly u sees all of V_4. So u is in $V_3 \cup X_3 \cup A_1$; and the definition of V_3, X_3 and Claim 8.3 imply $u \in V_3 \cup X_3$. So in this case the claim holds.

In the remaining case, we may assume that u misses all of V_4, and so v sees all of V_4 (for otherwise $\{w_4, a, u, v, b\}$ induces an odd hole for any $w_4 \in V_4 \setminus N(v)$), and so v misses all of V_2, and so u sees all of V_2. By Claim 8.5, u misses all of V_5, and v misses all of V_1. If u misses any $w_1 \in V_1$, we find two intersecting bulls $w_1 - v_2ua - v_4$ and $w_1 - v_2au - v$, a contradiction. So u sees all of V_1. Likewise, v sees all of V_5. By Lemma 2 applied to u and to the hole induced by $\{v_1, \ldots, v_l\}$, and since u sees v_1, v_2 and misses v_4, v_5, we have $N(u) \cap \{v_6, \ldots, v_l\} \subseteq \{v_l\}$.

Likewise we have $N(v) \cap \{v_6, \ldots, v_l\} \subseteq \{v_6\}$. Suppose $l \geq 8$. If u misses v_l and v misses v_6 then $\{v_1, u, v, v_5, v_6, \ldots, v_l\}$ induces a hole of odd length $l-1$. If u sees v_l and v sees v_6 then $\{u, v, v_6, \ldots, v_l\}$ induces a hole of odd length $l-3$. If u sees v_l and v misses v_6, then $\{u, v, v_5, v_6, \ldots, v_l\}$ induces an even hole of length $l-2$, a contradiction to the definition of l. A similar contradiction occurs if u misses v_l and v sees v_6. So we must have $l = 6$. Then every v_6 sees one of u, v, for otherwise $\{v_1, u, v, v_5, v_6\}$ induces an odd hole. Up to symmetry let us assume that v has a neighbor $u_6 \in V_6$. Then v misses every $v_3 \in V_3$, for otherwise $\{v, v_3, v_2, v_1, u_6\}$ induces an odd hole. Suppose that v also has a non-neighbor $w_6 \in V_6$. Then, u sees w_6, for otherwise $\{w_6, v_1, u, v, v_5\}$ induces an odd hole; and u misses every $v_3 \in V_3$, for otherwise $\{u, v_3, v_4, v_5, w_6\}$ induces an odd hole; but then $\{v_2, u, v, v_4, v_3\}$ induces an odd hole, a contradiction. Thus v sees all of V_6. Now the fact that v sees all of $V_4 \cup V_5 \cup V_6$ and misses all of $V_1 \cup V_3$ implies that v is in $V_5 \cup X_5$; but then the edge vb contradicts Claim 8.4. Thus the claim holds. □

Claim 8.10. *If for some $i = 1, \ldots, l$, the set $V_i \cup X_i$ is not a clique then it contains an even pair of G or an even pair of $\overline{G}$.*

Proof. Put $i = 3$. For any two vertices $a, b \in V_3 \cup X_3$, put $N_{in}(a, b) = N(a) \cap N(b) \cap (V_3 \cup X_3)$. Choose a pair $\{a, b\}$ of non-adjacent vertices of $V_3 \cup X_3$ that maximizes the size of $N_{in}(a, b)$ (such a pair exists since $V_3 \cup X_3$ is not a clique). If the claim does not hold, $\{a, b\}$ is not an even pair of G, so there exists a chordless odd path of G with endpoints a, b. By Claim 8.6 this path has length 3, so we can write it as a-u-v-b. By Claim 8.9, we may assume up to symmetry that $u \in V_3 \cup X_3$. Consider any $d \in N_{in}(a, b)$. Then d sees u, for otherwise u-a-d-b is a P_4 in $V_3 \cup X_3$, which contradicts Claim 8.2. So we have $N_{in}(a, b) \subseteq N_{in}(u, b)$, and the choice of $\{a, b\}$ implies $N_{in}(a, b) = N_{in}(u, b)$. We claim that $\{a, u\}$ is an even pair of $\overline{G}$. For suppose that there exists a chordless odd path Q in $\overline{G}$ with endpoints a, u. By Claim 8.7, Q has length 3. So we can write $Q = a$-x-y-u in $\overline{G}$, which means that in G we have a chordless path y-a-u-x. By Claim 8.8, one of x, y is in $V_3 \cup X_3$. By symmetry we may assume that $x \in V_3 \cup X_3$. Then x misses b, for otherwise we have $x \in N_{in}(u, b) \setminus N_{in}(a, b)$. Then x sees every $d \in N_{in}(a, b)$, for otherwise x-u-d-b is a P_4 in $V_3 \cup X_3$, which contradicts Claim 8.2. But then we have $N_{in}(a, x) \supseteq N_{in}(a, b) \cup \{u\}$, which contradicts the choice of $\{a, b\}$. Thus the claim holds. □

Claim 8.11. *If for some $i = 1, \ldots, l$, the set $V_i \cup X_i$ induces a clique of size at least 2 then any two vertices of $V_i \cup X_i$ form an even pair of $\overline{G}$.*

Proof. For suppose that there is a chordless odd path Q in $\overline{G}$ with endpoints a, b in $V_i \cup X_i$. By Claim 8.7, Q has length 3, so we can write $Q = a$-x-y-b in $\overline{G}$, and so we have a chordless path y-a-b-x in G. By Claim 8.8, one of x, y is in $V_i \cup X_i$; but this contradicts the fact that $V_i \cup X_i$ is a clique. Thus the claim holds. □

Claim 8.12. *Suppose that for every $i = 1, \ldots, l$, the set $V_i \cup X_i$ has size 1. Then $\{v_i, v_{i+2}\}$ is an even pair of G for every i.*

Proof. For suppose on the contrary and up to symmetry that $\{v_1, v_3\}$ is not an even pair; so there is a chordless odd path $P = x_0\text{-}x_1\text{-}\cdots\text{-}x_r$ with $v_1 = x_0$, $v_3 = x_r$ and $r \geq 3$. Since $V(P) \cup \{v_2\}$ cannot induce an odd hole (when $r = 3$), and by Lemma 4 (when $r \geq 5$), and up to symmetry, we may assume that v_2 sees x_1. If x_1 sees v_l, then x_1 misses v_{l-1} by Lemma 2, and we have $x_1 \in V_1 \cup X_1$, a contradiction. So x_1 misses v_l, and we find a bull $v_l - v_1x_1v_2 - v_3$. Then v_2 misses x_{r-1}, for otherwise by symmetry we find a second bull $v_4 - v_3x_{r-1}v_2 - v_1$. If $r = 3$, then v_l sees x_2, for otherwise we find a second bull $v_l - v_1v_2x_1 - x_2$ containing v_2; but then $\{v_l, v_1, v_2, v_3, x_2\}$ induces an odd hole. So $r \geq 5$. Since v_2 misses x_{r-1}, we have outcome 2 or 3 of Lemma 4, and in either case Lemma 4 states that there is a second bull containing v_2, a contradiction. Thus the claim holds. □

Claims 8.10, 8.11 and 8.12 complete the proof of the theorem. □

4. Comments

For any integer $k \geq 0$, let $\mathcal{B}_k$ be the class of graphs in which every vertex belongs to at most k bulls. So $\mathcal{B}_0$ is the class of bull-free graphs, and $\mathcal{B}_1$ is the class of bull-reducible graphs. One can consider the following statements:

Statement A_k: For every Berge graph G in $\mathcal{B}_k$ with at least two vertices, either G or $\overline{G}$ has an even pair.

Statement A'_k: For every Berge graph G in $\mathcal{B}_k$ that contains no antihole, either G is a clique or G has an even pair.

Statement A''_k: For every Berge graph G in $\mathcal{B}_k$ that contains no antihole, G is perfectly orderable.

Statements A_0 and A'_0 are theorems proved in [9]. Statement A''_0 is a theorem proved in [11]. Statement A_1 is the main result in this article. Statements A'_1 and A'_2 are theorems, as they can be obtained easily as corollaries of the main result in [17]. On the other hand, consider the graph H_{12} with 12 vertices $v_1, \ldots, v_{12}$ such that $v_1\text{-}v_2\text{-}\cdots\text{-}v_8\text{-}v_1$ is a hole, vertex v_9 is adjacent to v_1, v_2, v_{11}, vertex v_{10} is adjacent to v_3, v_4, v_{12}, vertex v_{11} is adjacent to v_5, v_6, v_9, and vertex v_{12} is adjacent to v_7, v_8, v_{10}. Then it is easy to see that H_{12} is a Berge graph (it is actually the line-graph of a bipartite graph), it contains no antihole, it is in $\mathcal{B}_5$, and H_{12} and its complement have no even pair. So H_{12} is a counterexample to statements A_k, A'_k for any $k \geq 5$. Moreover, the graph "E" in [13, p. 142, Fig. 7.1] is a counterexample to A''_3. We do not have a proof or a counterexample for any of the remaining statements A_2, A_3, A_4, A'_3, A'_4 and A''_1, A''_2.

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