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ON THE DEGREE OF TREES WITH GAME CHROMATIC NUMBER 4

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Abstract. The coloring game is played by Alice and Bob on a finite graph G. They take turns properly coloring the vertices with t colors. The goal of Alice is to color the input graph with t colors, and Bob does his best to prevent it. If at any point there exists an uncolored vertex without available color, then Bob wins; otherwise Alice wins. The game chromatic number $\chi_g(G)$ of G is the smallest number t such that Alice has a winning strategy. In 1991, Bodlaender showed the smallest tree T with $\chi_g(T)$ equal to 4, and in 1993 Faigle *et al.* proved that every tree T satisfies the upper bound $\chi_g(T) \leq 4$. The stars $T = K_{1,p}$ with $p \geq 1$ are the only trees satisfying $\chi_g(T) = 2$; and the paths $T = P_n$, $n \geq 4$, satisfy $\chi_g(T) = 3$. Despite the vast literature in this area, there does not exist a characterization of trees with $\chi_g(T) = 3$ or 4. We answer a question about the required degree to ensure $\chi_g(T) = 4$, by exhibiting infinitely many trees with maximum degree 3 and game chromatic number 4.

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1. Introduction

The *coloring game* is a two player non-cooperative game conceived by Steven Brams, firstly published in 1981 by Martin Gardner [7], as a *map-coloring game*, and reinvented in 1991 by Bodlaender [1], who studied the game in the context of graphs. The first application of a variation of this game to a non-game graph packing problem was presented in 2009 by Kierstead and Kostochka [9].

Let $G = (V, E)$ be a finite, simple, undirected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We recall that the distance $d(u, v)$ between two vertices u and v in a graph is the number of edges in a shortest path connecting them.

The two players, Alice and Bob, take turns properly coloring an uncolored vertex of graph G (that we call a move of the coloring game) by a color in a given color set with t colors. Alice's goal is to color the input graph with the t colors, and Bob does his best to prevent it. Alice wins when the graph is completely (properly) colored with t colors; otherwise, Bob wins. We say that Alice has a winning strategy with t colors when she has a sequence of moves that ensure that the graph can be completely (properly) colored with t colors regardless of

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Bob's moves along the game. Analogously, we say that Bob has a winning strategy with t colors when he has a sequence of moves that ensure that the graph can not be completely (properly) colored with t colors regardless of Alice's moves along the game.

The *game chromatic number* $\chi_g^a(G)$ (or simply $\chi_g(G)$) of G is the smallest number t of colors such that Alice has a winning strategy for the graph coloring game on G when she starts the game. In the literature, the authors only consider the case when Alice starts, but we find it useful to analyze the game when Bob starts as well. So, let $\chi_g^b(G)$ be the smallest number t of colors such that Alice has a winning strategy for the graph coloring game on G, when Bob starts the game.

Let G be a graph, Z a set of vertices of G, $c : Z \to \{1, \ldots, t\}$ a function that assigns to each vertex $v \in Z$ a color $c(v)$, and (G, Z, c) be the partially colored graph. We say that Alice (resp. Bob) plays on (G, Z, c), if Alice (resp. Bob) colors the uncolored vertices of $V(G) \setminus Z$. We introduce the auxiliary parameter $\chi_g^a(G, Z, c)$ (resp. $\chi_g^b(G, Z, c)$) as the smallest number t of colors such that Alice has a winning strategy for the graph coloring game on G, when Alice (resp. Bob) starts playing on a graph G with a previously colored set of vertices $Z \subseteq V(G)$. In order to simplify the notation, we omit the coloring c and write $\chi_g^a(G, Z)$ (resp. $\chi_g^b(G, Z)$) to mean $\chi_g^a(G, Z, c)$ (resp. $\chi_g^b(G, Z, c)$).

Clearly, for any graph G, we have that $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$, where $\chi(G)$ denotes the chromatic number and $\Delta(G)$ the maximum degree of graph G. So, we have that a complete graph K_n has $\chi_g(K_n) = n$, because $\chi(K_n) = \Delta(K_n)+1 = n$, and, analogously, an independent set S_n has $\chi_g(S_n) = 1$. Analyzing the game chromatic number of paths P_n, with n vertices, we can quickly check that $\chi_g(P_1) = 1$ and $\chi_g(P_2) = \chi_g(P_3) = 2$. For $n \geq 4$, we have that $\chi_g(P_n) = 3$ because, regardless of where Alice colors on her first turn, Bob can always assign a different color to a vertex at distance 2 from the vertex that Alice colored, forcing the third color. Using the same idea, we have that cycles C_n have $\chi_g(C_n) = 3$, and stars $K_{1,p}$ with $p \geq 1$ are the only connected graphs satisfying $\chi_g(G) = 2$.

The coloring game has been extensively studied for different graph classes in order to obtain better upper and lower bounds for $\chi_g(G)$: toroidal grids [10], cartesian products of some classes of graphs [2], planar graphs [11], outerplanar graphs [8], forests [3] and partial k-trees [12].

Bodlaender [1] showed an example of a tree with game chromatic number at least 4, and proved that every tree has game chromatic number at most 5. Faigle *et al.* [4] subsequently improved this bound by proving that every forest has game chromatic number at most 4.

Despite the vast literature in this area, only in 2015, Dunn *et al.* [3] considered the distinction between forests with different game chromatic numbers, by investigating special cases. They characterized forests with game chromatic number 2, and suggested the characterization of forests with game chromatic number 3 and 4 as open problems, due to the difficulty concerning this subject. In our work, we contribute to their study by considering a special tree called caterpillar in order to define an infinite family of trees with game chromatic number 4.

A *caterpillar* $H = cat(k_1, k_2, \ldots, k_s)$ is a tree obtained from a central path v_1, v_2, v_3, $\ldots$, v_s (called *spine*) by joining k_i leaf vertices to v_i, for each $i = 1, \ldots, s$; and with number of vertices $n = s + \sum_{i=1}^{s} k_i$. We consider caterpillars with $k_1 = k_s = 0$. For $i = 2, \ldots, s-1$, if $k_i \geq 1$, then we say that the vertex v_i has k_i adjacent *leg leaves*.

Our motivation to focus on caterpillars relies on the fact that Bodlaender [1] proved the existence of a tree with $\chi_g(T) \geq 4$ by considering the caterpillar $H_d = cat(0, 2, 2, 2, 2, 0)$ depicted in Figure 1. Actually, Dunn *et al.* [3] proved that the caterpillar H_d is the smallest tree such that $\chi_g(T) = 4$, and is the unique tree with fourteen vertices and game chromatic number 4. Observe that this fundamental tree has maximum degree 4.

Since Faigle *et al.* [4] and Dunn *et al.* [3] analized the game on forests, a natural question that arises is: given a forest F with r tree connected components T_i, for $1 \leq i \leq r$, is it true that $\chi_g(F) = max\,\{\chi_g(T_i)|\ i = 1, \ldots, r\}$? The answer is no, and we present two examples that show how the auxiliary parameter χ_g^b is useful.

We refer to the forest $F = P_1 \cup P_4$ depicted in Figure 2. We claim that Alice wins the coloring game on F with 2 colors. Indeed, she starts coloring the unique vertex of the P_1 component, and this forces Bob to start coloring a vertex of the P_4 component. Regardless of which vertex Bob colors on his first turn, Alice can always

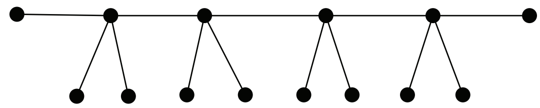

FIGURE 1. The caterpillar H_d satisfies $\chi_g(H_d) = 4$.

FIGURE 2. Forest $F = P_1 \cup P_4$ with $\chi_g(F) = \chi_g^b(P_4) = 2$, but $\chi_g(P_4) = 3$.

FIGURE 3. Forest $F = H_1 \cup H_1$ with $\chi_g(F) = \chi_g^b(H_1) = 4$, but $\chi_g(H_1) = 3$.

give the same color to a vertex at distance 2 from the vertex that Bob colored, *i.e.*, $\chi_g^b(P_4) = 2$. So, $\chi_g(F) = 2$, although $\chi_g(P_4) = 3$.

Now, let $H_1 = cat(0, 2, 2, 0, 2, 2, 0)$ with $\chi_g(H_1) = 3$ and $\chi_g^b(H_1) = 4$. We refer to the forest $F = H_1 \cup H_1$ depicted in Figure 3. Similarly, regardless of which vertex Alice colors on her first turn, Bob can always choose to start coloring on vertex v_4 of the component with no colored vertices, and this implies that $\chi_g(F) = 4$ as well. Hence, it is not possible to establish that $\chi_g(F) \leq max\,\{\chi_g(T_i)|\ i = 1, \ldots, j\}$ either.

Faigle *et al.* [4] proved that $\chi_g(T) \leq 4$, for trees T, and stated that this result can be extended to forests F, that is, $\chi_g(F) \leq 4$. Dunn *et al.* [3] asked whether the maximum degree is relevant to characterize trees and forests with game chromatic number equal to 3. Recall that the smallest tree with game chromatic number 4 has maximum degree 4. In a previous LAGOS extended abstract [6], we have defined necessary and sufficient conditions for a tree with maximum degree 4 to have game chromatic number 4.

We contribute to this maximum degree question by analyzing, in Section 2, caterpillars with maximum degree 3 and without vertex of degree 2. We define an infinite family that has game chromatic number 4 in Theorem 2.6, and extend this result to trees in Theorem 2.7. Our work establishes that the required maximum degree to ensure game chromatic number 4 is in fact 3, and corrects a previous CLAIO extended abstract [5]. Finally, we present our conclusion and further questions in Section 3.

2. Trees with game chromatic number 4

We recall that, by Faigle *et al.* [4], every caterpillar H, which is not a star, has $3 \leq \chi_g(H) \leq 4$. According to our notation, the star with $n-1$ leaves, for $n \geq 4$, is denoted $K_{1,n-1} = cat(0, n-3, 0)$.

It is an open challenge to characterize the caterpillars with game chromatic number respectively equal to 3 and to 4. In the present work, we manage to find the smallest maximum degree that ensures the existence an infinite family of caterpillars with $\chi_g(H)$ equal to 4.

We refer to Figure 4, where vertex v_i is simply labeled i. In the coloring game on a caterpillar, a player is forced to use four colors if, during the game, there exists an induced subgraph isomorphic to a *claw*, the caterpillar $cat(0, 1, 0)$, with its leaves colored with different colors, as depicted in Figure 4a. Thus, Bob's strategy is to obtain a previously partially colored claw subgraph as in Figure 4b to start coloring on it.

Let $C = cat(0, 1, 0)$ be a claw and $Z = \{v_1, v_3 \mid c(v_1) \neq c(v_3)\}$ be a previously colored set of vertices of C. We claim that $\chi_g^a(C, Z) = 3$ and $\chi_g^b(C, Z) = 4$. We refer to Figure 4b. First, we observe that, since vertex v_2 can not be properly colored with the two colors previously given to v_1 and to v_3, at least three colors are necessary.

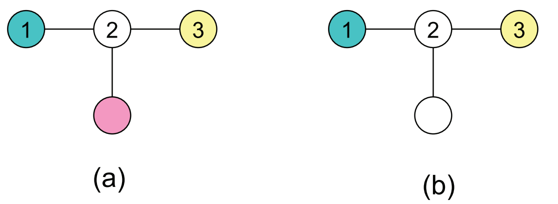

FIGURE 4. (a) The partially colored claw forces the game chromatic number to be 4; (b) claw-situation.

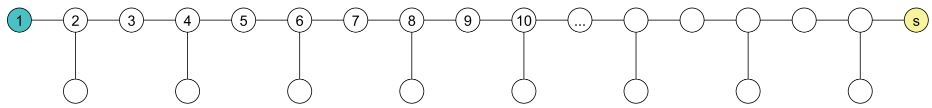

FIGURE 5. The partially colored caterpillar $\tilde{H}_s$, with s odd and $s \geq 5$.

If Bob starts playing on (C, Z), then he colors the unique leg leaf (adjacent to v_2) with a third color, forcing a fourth color in v_2. If Alice starts playing on (C, Z), then she colors v_2 with a third color, which ensures that she is able to win the game with 3 colors.

We define the *claw-situation* as an ordered pair (C, Z), where $C = cat(0,1,0)$ is a claw and $Z = \{v_1, v_3 \mid c(v_1) \neq c(v_3)\}$ is a previously colored set of vertices (see Fig. 4b). The claw-situation generalizes the key tool implicitly used by Bodlaender to build the smallest tree with game chromatic number 4. We strengthen the tool in order to define an infinite family of trees with game chromatic number 4 and maximum degree 3.

2.1. Caterpillar without vertices of degree 2 and with $\Delta = 3$

The goal of this subsection is to exhibit an infinite family of caterpillars H without vertices of degree 2 and maximum degree $\Delta(H) = 3$, with game chromatic number 4 (Thm. 2.6). We observe that such a family shows that a tree is not required to have a vertex of degree 4 in order to have game chromatic number 4.

We begin this subsection, showing auxiliary lemmas that allow vertices of degree 2.

Let H be a caterpillar with $\Delta(H) = 3$ and λ_i the leg leaf adjacent to v_i, for $i = 2, \ldots, s-1$. Given a partially colored caterpillar (H, Z) with $Z = \{v_1, v_s\}$, we are interested in identifying Alice's moves for which Bob has a winning strategy for the game with 3 colors, establishing $\chi_g^b(H, Z) = 4$. We show that Alice can not move at distance of any parity of some colored vertices. The idea is to study which Alice's moves, on vertices at odd or even distances of specific colored vertices, lead to Bob's victory.

In the next lemma, we consider $(\tilde{H}_s, Z)$ such that:

(i) $\tilde{H}_s = cat(k_1, \ldots, k_s)$ with s odd and $s \geq 5$;
(ii) $k_i = 0$, for i odd, and $k_i = 1$, for i even, $1 \leq i \leq s$;
(iii) $Z = \{v_1, v_s\}$.

See an example of $(\tilde{H}_s, Z)$ in Figure 5. We observe that there exists no constraint about the colors of vertices v_1 and v_s, *i.e.*, $c(v_1) \neq c(v_s)$ or $c(v_1) = c(v_s)$. Note that, in $\tilde{H}_s$, the distance between the colored vertices v_1 and v_s is even.

Lemma 2.1. *Let $\tilde{H}_s = cat(k_1, \ldots, k_s)$, with s odd and $s \geq 5$, be a caterpillar such that $k_i = 0$, for i odd, and $k_i = 1$, for i even, $1 \leq i \leq s$. If $Z = \{v_1, v_s\}$ then $\chi_g^b(\tilde{H}_s, Z) = 4$.*

Proof. Bob starts the game on $(\tilde{H}_s, Z)$ coloring vertex v_3 such that $c(v_3) \neq c(v_1)$, and produces a claw-situation (C, Z) with $V(C) = \{v_1, v_2, v_3, \lambda_2\}$ and $Z = \{v_1, v_3\}$. So, to prevent the fourth color, Alice is forced to color

FIGURE 6. (a) $\tilde{H}_7$-situation. The partially colored caterpillars in subfigures (b), (c), (d) and (e) have a copy of the game on (H', Z') depicted in subfigure (a).

v_2 (with $c(v_2) \neq c(v_1) \neq c(v_3)$). The game continues as follows: on his i-th turn, Bob colors vertex v_{2i+1} with $c(v_{2i+1}) \neq c(v_{2i-1})$, and produces a claw-situation. Next, on her i-th turn, Alice is forced to color v_{2i}, to prevent the fourth color.

On his $\frac{s-3}{2}$-th turn, Bob colors v_{s-2} such that $c(v_{s-2}) \neq c(v_{s-4})$ and $c(v_{s-2}) \neq c(v_s)$, and produces two claw-situations, that is, (C, Z) and (C', Z') with $V(C) = \{v_{s-4}, v_{s-3}, v_{s-2}, \lambda_{s-3}\}$ and $Z = \{v_{s-4}, v_{s-2}\}$; $V(C') = \{v_{s-2}, v_{s-1}, v_s, \lambda_{s-1}\}$ and $Z' = \{v_{s-2}, v_s\}$. Again, Alice is forced to color v_{s-3} (resp. v_{s-1}). Now, Bob colors λ_{s-1} (resp. λ_{s-3}) and obtain a claw such that all vertices adjacent to v_{s-1} (resp. v_{s-3}) have different colors. Hence, Bob wins the game and the result follows. □

Let H and H' be two caterpillars, and let H'' be a subgraph of H isomorphic to H'. Let Z and Z' be two previously colored sets with $Z \subseteq V(H)$ and $Z' \subseteq V(H')$. We say that a game on (H, Z) has a *copy of the game* on (H', Z') (or simply a *copy* of (H', Z')), if there exists an isomorphism $\phi : V(H') \to V(H'')$ that preserves the colors of the vertices of H' (preserves both the colored and also the uncolored vertices of H') in the vertices of H''. Formally, $\phi : V(H') \to V(H'')$ is a bijection, satisfying the following conditions:

(1) for any $v, w \in V(H')$, $vw \in E(H')$ if and only if $\phi(v)\phi(w) \in E(H'')$;
(2) $c(\phi(v)) = c(v)$, for each $v \in V(H')$.

In particular, if we define $Z'' = \{\phi(v) \mid v \in Z'\}$, then we have that $Z'' \subseteq Z$, *i.e.*, H may have additional colored vertices that are not in Z''.

We refer to Figure 6a for an example of (H', Z') with $H' = \tilde{H}_7$ and $Z' = \{v_1, v_7 \mid c(v_1) \neq c(v_7)\}$. The remaining caterpillars of Figure 6 have a copy of the game on (H', Z'). We observe that, for every s, the distance in $\tilde{H}_s$ between the only two colored vertices of Z' is $s - 1$.

Now we study another caterpillar and consider the case where, during the game, a copy of the game on $(\tilde{H}_s, Z)$ appears on Bob's turn.

Lemma 2.2. *Let H be a caterpillar $cat(k_1, \ldots, k_{\overline{s}})$, with $\overline{s} \geq 5$, such that $k_1 = k_{\overline{s}} = 0$, and $k_i = 1$, for $2 \leq i \leq \overline{s} - 1$. If, during the game on (H, Z), a copy of $(\tilde{H}_s, Z')$ appears on Bob's turn, then $\chi_g^b(H, Z) = 4$.*

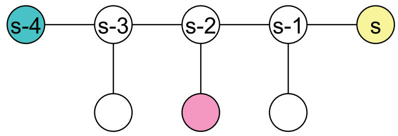

FIGURE 7. Example of *Case 2* of Lemma 2.2 where $c(\lambda_{s-2})$, $c(v_{s-4})$ and $c(v_s)$ are pairwise different.

Proof. Let H'' be the subgraph of H isomorphic to $\tilde{H}_s$. We define $V(H'') = \{v_1, \ldots, v_s, \lambda_2, \ldots, \lambda_{s-1}\}$. Similarly to the proof of Lemma 2.1, until the $\frac{s-5}{2}$-th Bob's turn, regardless of whether Alice plays or whether there exists colored vertices in $V(H) \setminus V(H'')$, the game proceeds as follows. On his i-th turn, Bob properly colors v_{2i+1} such that $c(v_{2i+1}) \neq c(v_{2i-1})$, producing a claw-situation. Next, on her i-th turn, Alice is forced to color v_{2i}, to prevent the fourth color. On his $\frac{s-3}{2}$-th turn, Bob has two options:
Case 1. either λ_{s-2} is uncolored, or $c(\lambda_{s-2})$, $c(v_{s-4})$ and $c(v_s)$ are not pairwise different. In this case Bob properly colors v_{s-2} with a color such that $c(v_{s-2}) \neq c(v_{s-4})$ and $c(v_{s-2}) \neq c(v_s)$, producing two claw-situations. Again, Alice is forced to color v_{s-3} (resp. v_{s-1}). Now, Bob colors λ_{s-1} (resp. λ_{s-3}) and obtain a claw such that all vertices adjacent to v_{s-1} (resp. v_{s-3}) have different colors.
Case 2. $c(\lambda_{s-2})$, $c(v_{s-4})$ and $c(v_s)$ are pairwise different (see Fig. 7). Bob colors λ_{s-3} such that $c(\lambda_{s-3}) = c(\lambda_{s-2})$. So, Alice can not avoid a claw-situation without producing another one on Bob's turn.

Hence, in both cases, Bob wins the game and the result follows. □

Let H be a caterpillar with $cat(k_1, \ldots, k_{\overline{s}})$, $s \geq 5$, $k_1 = k_{\overline{s}} = 0$ and $k_i = 1$, for $2 \leq i \leq \overline{s} - 1$. By Lemma 2.2, if a vertex $v \in V(H)$ is colored, then Alice avoids coloring a vertex w such that there exists a copy of the game on $(\tilde{H}_s, Z')$, s odd and $s \geq 5$, with $Z' = \{v, w\}$. This means that Alice avoids coloring a vertex w that is at even distance to v, and we call $(\tilde{H}_s, Z')$ as an $\tilde{H}_s$*-situation*, or simply $\tilde{H}$*-situation*. We refer to Figure 6a for an $\tilde{H}_7$-situation with $Z' = \{v_1, v_7\}$. Also, in Figures 6b,c,d,e we have an $\tilde{H}_7$-situation, because these partially colored caterpillars have a copy of the game on $(\tilde{H}_7, Z')$.

The following lemma corrects a previous CLAIO extended abstract [5].

Lemma 2.3. *Let (H, Z) be a game such that H is a caterpillar $cat(k_1, \ldots, k_{\overline{s}})$, with $\overline{s} \geq 10$, such that $k_1 = k_{\overline{s}} = 0$ and $k_i = 1$, for $2 \leq i \leq \overline{s} - 1$, and let Z be a previously colored vertex set of H. Suppose that (H, Z) has a copy of a game on (H', Z') where H' is a $cat(k_1, \ldots, k_{10})$ such that $k_1 = k_{10} = 0$ and $k_i = 1$, for $2 \leq i \leq 9$, and $Z' = \{v'_1, v'_{10} \mid c(v'_1) \neq c(v'_{10})\}$, then $\chi_g^b(H, Z) = 4$.*

Proof. Let H'' be the subgraph of H isomorphic to H'. We define $V(H'') = \{v_1, \ldots, v_{10}, \lambda_2, \ldots, \lambda_9\}$. So, H'' is a caterpillar $cat(k_1, \ldots, k_{10})$ such that $k_1 = k_{10} = 0$ and $k_i = 1$, for $2 \leq i \leq 9$; and $Z'' = \{v_1, v_{10} \mid c(v_1) \neq c(v_{10})\}$. By Lemma 2.2, we assume that Alice's strategy, on the game on (H, Z), is to avoid claw-situations and $\tilde{H}$-situations on Bob's turns, since these situations imply winning strategies for Bob. So, we have the following possibilities and, in each of them, we present tables containing the vertices that are forbidden for Alice, *i.e.*, vertices that lead to these situations.
Bob's first turn: Bob starts coloring λ_3 with $c(v_1)$, as in Figure 8.

Based on Table 1, the unique possibility for Alice is to color λ_9, such that $c(\lambda_9) = c(v_{10})$ on her first turn.
Bob's second turn: Bob colors λ_8, with $c(\lambda_8) = c(\lambda_9) = c(v_{10})$, as in Figure 9.

Based on Table 2, we have two possibilities for Alice's second turn:
Case 1. Alice colors v_4 with $c(\lambda_3)$. So, Bob colors λ_6 with the same color, as in Figure 10. We refer to Table 3 for the analysis of Alice's third turn.
Case 2. Alice colors λ_2 with $c(v_1)$. So, Bob colors λ_4 with $c(\lambda_4) = c(\lambda_3)$, as in Figure 11. We refer to Table 4 for the analysis of Alice's third turn.

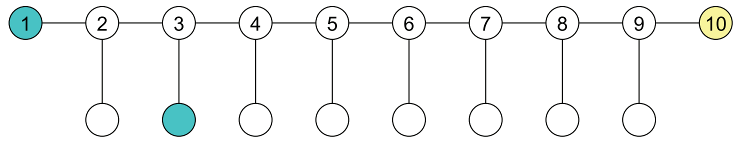

FIGURE 8. Bob colors λ_3 with $c(v_1)$ on his first turn.

TABLE 1. Analysis of Alice's first turn that leads to Bob's victory. Alice is forced to color vertex λ_9 with $c(\lambda_9) = c(v_{10})$.

Forbidden moves	Reason
v_2, v_3	claw-situation
$v_4, v_6, \lambda_5, \lambda_7$	$\tilde{H}$-situation (even distance to v_{10})
$v_5, v_7, v_9, \lambda_4, \lambda_6, \lambda_8$	$\tilde{H}$-situation (even distance to v_1)
v_8	$\tilde{H}$-situation (even distance to λ_3)
λ_2 or $v \in V(H) \setminus V(H'')$	$\tilde{H}$-situation ($d(\lambda_3, v_{10})$ is even)

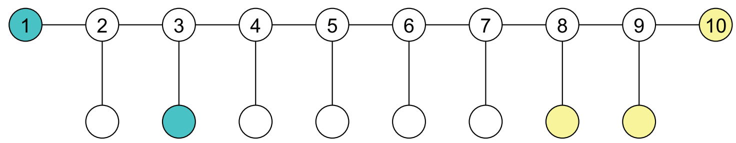

FIGURE 9. Bob colors λ_8 with $c(v_{10})$ on his second turn.

TABLE 2. Analysis of Alice's second turn that leads to Bob's victory. Alice is forced to either color vertex v_4 with $c(\lambda_3)$ or color vertex λ_2 with $c(v_1)$.

Forbidden moves	Reason
v_2, v_3, v_9	claw-situation
$v_5, v_7, \lambda_4, \lambda_6$	$\tilde{H}$-situation (even distance to v_1)
$v_6, v_8, \lambda_5, \lambda_7$	$\tilde{H}$-situation (even distance to λ_3)
$v \in V(H) \setminus V(H'')$	$\tilde{H}$-situation ($d(v_1, \lambda_8)$ is even)

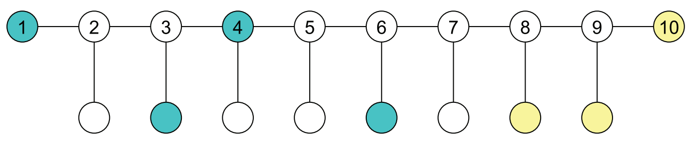

FIGURE 10. Alice's third turn according to *Case 1*.

TABLE 3. Analysis of Alice's third turn (*Case 1*) that leads to Bob's victory.

Forbidden moves	Reason
v_5, v_6, v_7, v_9	claw-situation
v_8, λ_7	$\tilde{H}$-situation (even distance to v_4)
$v_2, v_3, \lambda_2, \lambda_4, \lambda_5$ or $v \in V(H) \setminus V(H'')$	$\tilde{H}$-situation ($d(\lambda_6, \lambda_8)$ is even)

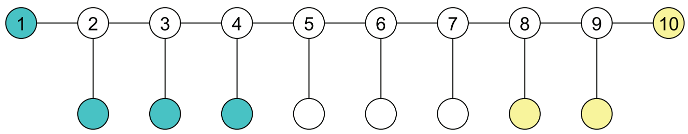

FIGURE 11. Alice's third turn according to *Case 2*.

TABLE 4. Analysis of Alice's third turn (*Case 2*) that leads to Bob's victory.

Forbidden moves	Reason
v_2, v_3, v_4, v_9	claw-situation
v_5, λ_6	$\tilde{H}$-situation (even distance to λ_8)
$v_6, v_8, \lambda_5, \lambda_7$	$\tilde{H}$-situation (even distance to λ_3)
v_7	$\tilde{H}$-situation (even distance to λ_4)
$v \in V(H) \setminus V(H'')$	$\tilde{H}$-situation ($d(\lambda_4, \lambda_8)$ is even)

In both cases, according to Tables 3 and 4, Alice's third turn leads to a claw-situation or a $\tilde{H}$-situation. This ends the proof. □

Lemma 2.4. *Let (H, Z) be a game such that H is a caterpillar $cat(k_1, \dots, k_{\overline{s}})$, with $\overline{s} \geq 12$, such that $k_1 = k_{\overline{s}} = 0$ and $k_i = 1$, for $2 \leq i \leq \overline{s} - 1$, and let Z be a previously colored vertex set of H. Suppose that (H, Z) has a copy of a game on (H', Z') where H' is a $cat(k_1, \dots, k_s)$, with s even and $s \geq 12$, such that $k_1 = k_s = 0$ and $k_i = 1$, for $2 \leq i \leq s - 1$, and $Z' = \{v'_1, v'_s\}$, then $\chi_g^b(H, Z) = 4$.*

Proof. We proceed by induction on s. Let H'' be the subgraph of H isomorphic to H'. We define $V(H'') = \{v_1, \dots, v_s, \lambda_2, \dots, \lambda_{s-1}\}$. So, for $s = 12$, H'' is a caterpillar $cat(k_1, \dots, k_{12})$ such that $k_1 = k_{12} = 0$ and $k_i = 1$, for $2 \leq i \leq 11$, and $Z'' = \{v_1, v_{12}\}$. Bob starts coloring vertex v_3 such that $c(v_3) \neq c(v_1)$ and $c(v_3) \neq c(v_{12})$. Hence, we have a claw-situation and a $cat(k_1, \dots, k_{10})$ such that $k_1 = k_{10} = 0$ and $k_i = 1$, for $2 \leq i \leq 9$, and a previously colored set $\{v_3, v_{12}\}$. So, independently of the vertex colored on Alice's first turn, Bob starts coloring in one of these partially colored subgraphs, taking advantage either of the claw-situation or of Lemma 2.3. Thus, $\chi_g^b(H, Z) = 4$.

By induction hypothesis, we assume that the result is valid when (H, Z) has a copy of a game on (H', Z') where $H' = cat(k_1, \dots, k_{s-2})$, with $s - 2$ even and $s - 2 \geq 12$, and $Z' = \{v_1, v_{s-2}\}$. Again, since s is even, we prove that the result is valid when (H, Z) has a copy of the game on (H', Z') where $H' = cat(k_1, \dots, k_s)$, with s even and $s \geq 12$, and $Z' = \{v_1, v_s\}$. Similarly, Bob starts coloring vertex v_3 such that $c(v_3) \neq c(v_1)$ and $c(v_3) \neq c(v_s)$. Now, we have a claw-situation and copy of H' of the inductive hypothesis with $Z'' = \{v_3, v_s\}$. Independently of the vertex colored on Alice's first turn, Bob starts coloring in one of these partially colored claw or H' subgraphs, and the result follows either by a claw-situation or by the inductive hypothesis. □

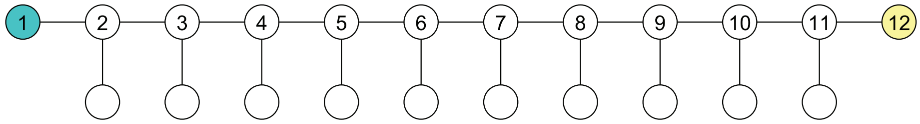

FIGURE 12. The smallest partially colored caterpillar that is a 12-situation.

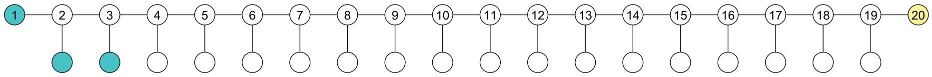

FIGURE 13. Game (H', Z') after Alice and Bob's first turns.

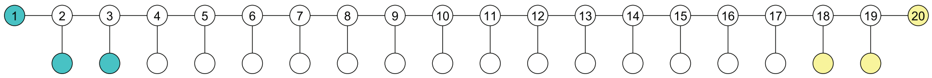

FIGURE 14. Game (H', Z') after Alice and Bob's second turns.

Let H be a caterpillar $cat(k_1, \ldots, k_s)$, with $s \geq 12$, $k_1 = k_s = 0$ and $k_i = 1$, for $2 \leq i \leq s-1$. By Lemma 2.4, if a vertex $v \in V(H)$ is colored, then Alice avoids coloring a vertex $w \in V(H)$ such that there exists a copy of the game on (H', Z), where $H' = cat(k_1, \ldots, k_{s'})$, with s even and $s' \geq 12$, such that $k_1 = k_{s'} = 0$ and $k_i = 1$, for $2 \leq i \leq s'-1$ with $Z = \{v, w\}$. This means that Alice avoids coloring a vertex w that is at an odd distance greater or equal to 11 from v, and we call (H, Z) as a 12-*situation*. In Figure 12, we present the smallest partially colored caterpillar that is a 12-situation, with distance 11 between the colored vertices. Also, Figure 14 has an example of a 12-*situation* with $Z' = \{\lambda_3, \lambda_{18}\}$ and $d(\lambda_3, \lambda_{18}) = 17$.

Lemma 2.5. *Let (H, Z) be a game such that H is a caterpillar $cat(k_1, \ldots, k_{\overline{s}})$, with $\overline{s} \geq 20$, such that $k_1 = k_{\overline{s}} = 0$ and $k_i = 1$, for $2 \leq i \leq \overline{s}-1$. Suppose that (H, Z) has a copy of a game on (H', Z') where H' is a $cat(k_1, \ldots, k_{20})$, such that $k_1 = k_{20} = 0$, $k_i = 1$, for $2 \leq i \leq 19$ and $Z' = \{v'_1, v'_{20} \mid c(v'_1) \neq c(v'_{20})\}$, then $\chi_g^a(H, Z) = 4$.*

Proof. Let H'' be the subgraph of H isomorphic to H'. We define $V(H'') = \{v_1, \ldots, v_{20}, \lambda_2, \ldots, \lambda_{19}\}$. Observe that if Alice starts in $V(H) \setminus V(H'')$, then Bob colors v_9, producing an $\tilde{H}$-situation and a 12-situation, and therefore a winning strategy for Bob, which implies $\chi_g^a(H, Z) = 4$.

Else, Alice starts in $V(H'')$, by coloring vertex λ_2 or λ_{19}. Indeed, note that if a vertex $w \in V(H'')$ is such that $d(w, v_1)$ (resp. $d(w, v_{20})$) is odd, then $d(w, v_{20})$ (resp. $d(w, v_1)$) is even. Moreover, one of these distances is greater than or equal to 10. So, if Alice colors v_i, $2 \leq i \leq 19$, or λ_i, $3 \leq i \leq 18$, then we have an $\tilde{H}$-situation or a 12-situation.

So without loss of generality, we may suppose that Alice colors on her first turn vertex λ_2, such that $c(\lambda_2) = c(v_1)$. Next, Bob colors in his first turn vertex λ_3, such that $c(\lambda_3) = c(v_1) = c(\lambda_2)$, as in Figure 13.

In Alice's second turn, the unique possibility for Alice is to color λ_{19}, such that $c(\lambda_{19}) = c(v_{20})$. Indeed, if Alice plays in $V(H) \setminus V(H'')$, then we have an $\tilde{H}$-situation since $d(\lambda_3, v_{20}) = 18$. Moreover, Alice does not color v_2 due to a claw-situation. By Lemma 2.3, Alice does not color v_{11}, since $d(v_{11}, \lambda_3) = d(v_{11}, v_{20}) = 9$ and $c(\lambda_3) \neq c(v_{20})$. For the remaining vertices $w \in V(H'') \setminus \{v_2, v_{11}\}$, since $d(\lambda_3, v_{20})$ is even, every vertex w is either at odd distance from λ_3 and v_{20}, or at even distance from λ_3 and v_{20}. Moreover, $d(w, \lambda_3) \geq 10$ or $d(w, v_{20}) \geq 10$, which implies an $\tilde{H}$-situation or a 12-situation.

So, Alice colors on her second turn λ_{19}, such that $c(v_{20}) = c(\lambda_{19})$. Next, Bob colors on his second turn λ_{18}, such that $c(\lambda_{18}) = c(v_{20}) = c(\lambda_{19})$, as in Figure 14.

Finally, we prove that any vertex colored on Alice's next turn implies $\chi_g^a(H, Z) = 4$. As we argued before, Alice does not color a vertex in $V(H) \setminus V(H'')$, since $d(\lambda_3, \lambda_{18}) = 17$ is a 12-situation. Moreover, Alice does not color v_2 nor v_{19} due to a claw-situation. For the remaining vertices $w \in V(H'') \setminus \{v_2, v_{19}\}$, since $d(\lambda_3, \lambda_{18})$ is odd, if $d(w, \lambda_3)$ (resp. $d(w, \lambda_{18})$) is odd, then $d(w, \lambda_{18})$ (resp. $d(w, \lambda_3)$) is even, and this implies an $\tilde{H}$-situation or a 12-situation, which ends the proof. □

Theorem 2.6. *Let H be a caterpillar $cat(k_1, \ldots, k_s)$, with $k_1 = k_s = 0$ and $k_i = 1$, for $2 \leq i \leq s-1$. If $s \geq 40$, then $\chi_g^a(H) = \chi_g^b(H) = 4$.*

Proof. If Alice starts the game by coloring a spine vertex v_i, then Bob colors another spine vertex v_j such that $d(v_i, v_j) = 19$, and the result $\chi_g^a(H) = 4$ follows by Lemmas 2.4 and 2.5.

If Alice starts the game by coloring a leg leaf λ_i, then Bob colors a spine vertex v_{i-1} or v_{i+1} with a different color of λ_i. Now, to avoid a claw-situation, Alice is forced to color the spine vertex v_i, and the result $\chi_g^a(H) = 4$ follows as in the previous case.

On the other hand, if Bob starts the game, then Bob colors v_{20}. Depending on Alice's second move, Bob colors next v_1 or v_{40}. The result $\chi_g^b(H) = 4$ follows by Lemmas 2.1, 2.4, and 2.5. □

We observe that Bob's strategies in the previous lemmas can be also applied to an arbitrary tree. Theorem 2.7 considers a tree that contains the caterpillar $H = cat(k_1, \ldots, k_s)$, with $k_1 = k_s = 0$, $k_i = 1$ for $2 \leq i \leq s-1$, and $s \geq 40$ as an induced subgraph.

Theorem 2.7. *Let T be a tree that has caterpillar $H = cat(k_1, \ldots, k_s)$, with $k_1 = k_s = 0$, $k_i = 1$ for $2 \leq i \leq s-1$, and $s \geq 40$, as induced subgraph. If v_1 and v_s are the only vertices of H adjacent to vertices in $V(T) \setminus V(H)$, then $\chi_g^a(T) = \chi_g^b(H) = 4$.*

Proof. For trees, it is well known that $\chi_g^a(T) \leq 4$. So, to establish the equality $\chi_g^a(T) = 4$ we consider two cases. If Alice's first move is in $V(H)$, then it is enough to note that Bob's strategy in Theorem 2.6 can be applied in T. If Alice's first move is in $V(T) \setminus V(H)$ or if Bob starts the game, then Bob colors vertex v_{20}. Depending on Alice's second move, Bob colors v_1 or v_{40}. The result follows by Lemma 2.5. □

3. Final remarks

Our work establishes that the required maximum degree to ensure game chromatic number 4 is in fact 3. The smallest known tree with maximum degree 3 and game chromatic number 4 has 78 vertices, according to our Theorem 2.6. We highlight that the smallest tree with game chromatic number 4 is caterpillar H_d (Fig. 1), which has 4 vertices with degree 4 and has 14 vertices.

The definition of an infinite family of trees with game chromatic number 4 gives in fact additional trees with the same game chromatic number. For instance, according to Theorem 2.7, the existence of a caterpillar $cat(k_1, \ldots, k_s)$, with $k_1 = k_s = 0$, $k_i = 1$ for $2 \leq i \leq s-1$ and $s \geq 40$, as an induced subgraph, is enough to ensure that the game chromatic number is 4, on an arbitrary tree without degree restrictions on the remaining vertices of the graph.

It is a challenging open problem to characterize among the trees with maximum degree 3 the ones that have game chromatic number 4. We believe that our approach to consider that Bob starts the game on a partially colored tree (T, Z) and the auxiliary parameter $\chi_g^b(T, Z)$ is enough to face this challenge.

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