Improved upper bounds on the crossing number, the 2-page crossing number and the rectilinear crossing number of the hypercube

Celina Miraglia Herrera de Figueiredo

COPPE, UFRJ

The same upper bound for both: the 2-page and the rectilinear crossing numbers of the *n*-cube

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WG 2013
39th International Workshop on Graph-Theoretic Concepts in Computer Science
June 19 - 21, 2013, Lübeck, Germany

Bounds for the crossing numbers of the *n*-cube

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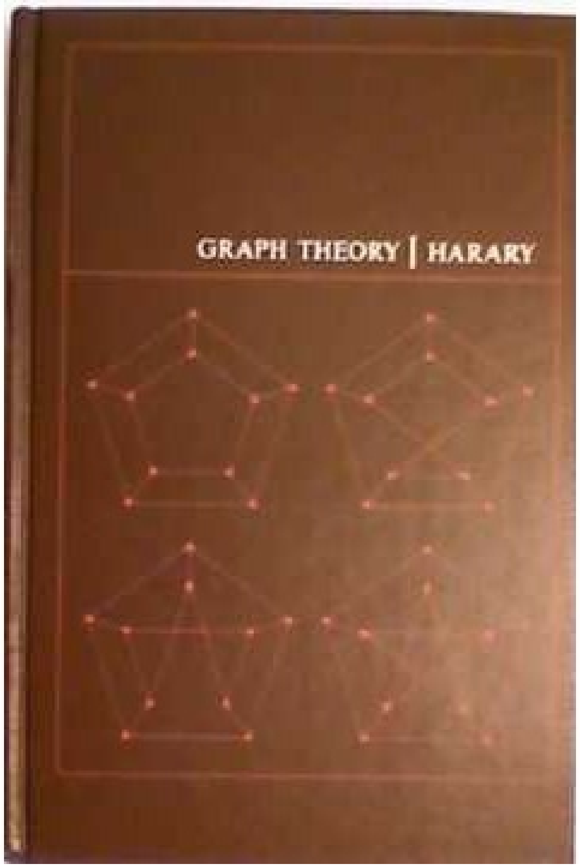
CNW'2014
6th Crossing Number Workshop June 11 - 15, 2014, Maribor, Slovenia

Crossing number challenge

Frank Harary



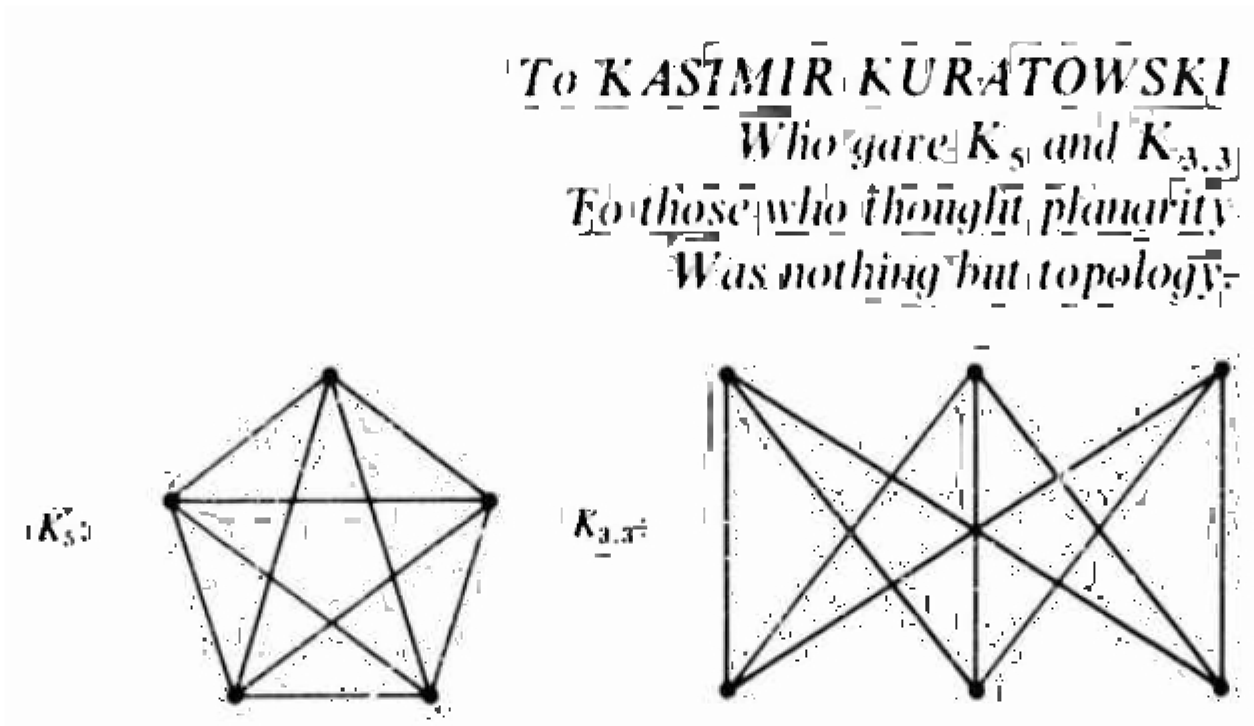

To KASIMIR KURATOWSKI
Who gave K_5 and $K_{3,3}$
To those who thought planarity
Was nothing but topology.
K_5:
$K_{3,3}$:

Table 11.1

CONJECTURED VALUES FOR $\zeta(K_p)$

p	13	18	21	24	27	$9n + 7$
$\zeta(K_p)$	7	15	21	28	36	$(9n^2 + 13n + 2)/2$

Theorem 11.28 The crossing number of the complete graph satisfies the inequality

$$\nu(K_p) \leq \frac{1}{4}\left[\frac{p}{2}\right]\left[\frac{p-1}{2}\right]\left[\frac{p-2}{2}\right]\left[\frac{p-3}{2}\right]. \qquad (11.20)$$

Theorem 11.29 The crossing number of the complete bigraph satisfies the inequality

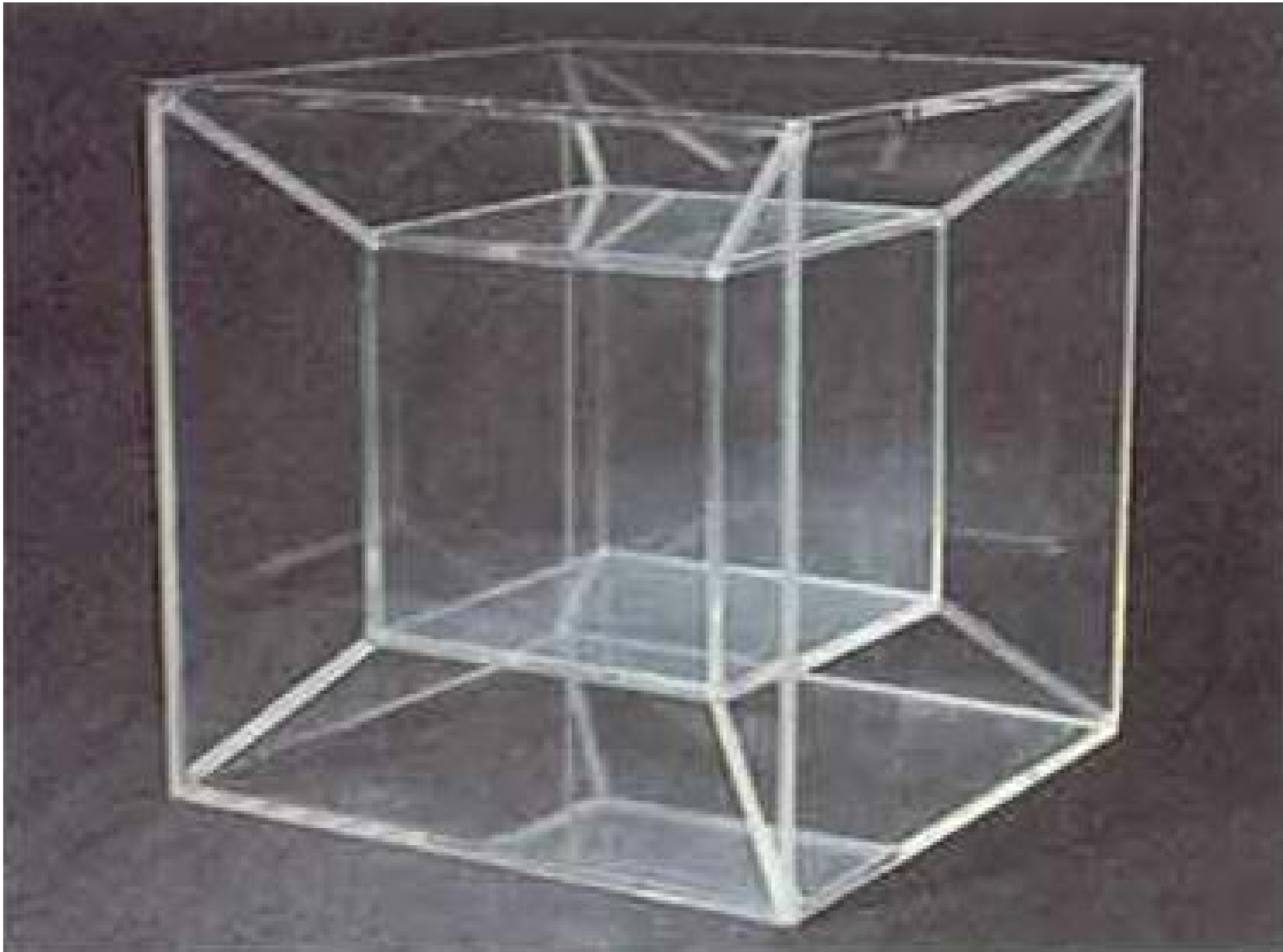
$$\nu(K_{m,n}) \leq \left[\frac{m}{2}\right]\left[\frac{m-1}{2}\right]\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]. \qquad (11.21)$$

T. Saaty showed that (11.20) is an equation for $p \leq 10$ while D. Kleitman proved equality in (11.21) for $m \leq 6$. These are the only known values of $\nu(K_p)$ and $\nu(K_{m,n})$. For the cubes, no one has even conjectured what is ν.

EXERCISES

11.1 If a (p_1, q_1) graph and a (p_2, q_2) graph are homeomorphic, then

$$p_1 + q_2 = p_2 + q_1.$$



Master Dissertation: Crossing number of Product of graphs

- Marian Klesc
- Bruce Richter
- Ondrej Sýkora
- Imrich V’rto

Definitions

- The crossing number $\nu(G)$ of G is the minimum number of crossings in a drawing of G in the plane.
- The rectilinear crossing number $\overline{cr}(G)$ of G is the minimum number of crossings in a drawing of G in the plane with straight line segments.
- The 2-page crossing number $\nu_2(G)$ of G is the minimum number of crossings in a drawing of G into 2 semiplanes where the vertices of G belong to a straight line bounding the semiplanes.

Relationship between $\nu(G)$ $\overline{cr}(G)$ and $\nu_2(G)$.

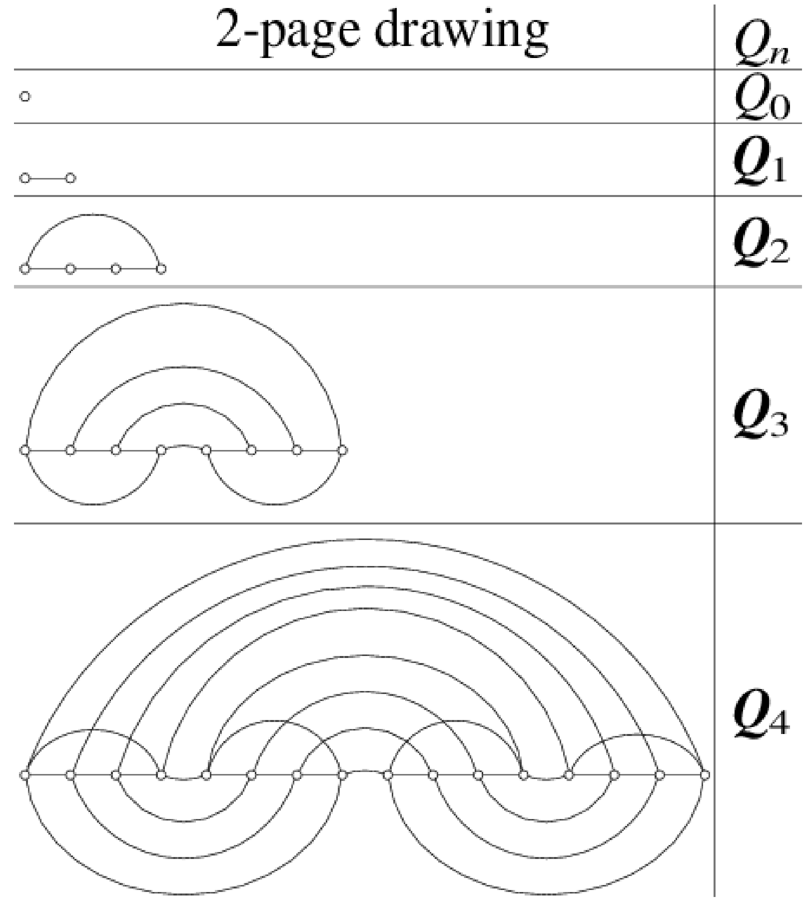
$$\nu(G) \leq \overline{cr}(G)$$

$$\nu(G) \leq \nu_2(G) \leq \nu_1(G)$$

$$\nu_2(K_n) \leq \overline{cr}(K_n)$$

Abrégo, Aichholzer, Merchant, Ramos, and Salazar'2012

n-cube



n-cube Q_n

$|V(Q_n)| = 2^n$

uv in $E(Q_n)$

iff

diff(u,v)=1.

$\nu_2(Q_n) \le \frac{125}{750}4^n - 2^{n-3}n^2 - 2^{n-4}3 + \frac{(-2)^n}{48}$ Madej, 1991

Exact Results

$n \leq 3$

$$\nu(Q_n) = \nu_2(Q_n) = \overline{cr}(Q_n) = 0$$

n=4, A. Dean & B. Richter'95

$$\nu(Q_n) = \nu_2(Q_n) = \overline{cr}(Q_n) = 8$$


Computers in Number Theory Conference held in Oxford'1969

Richard Guy, Paul Erdös & R. B. Eggleton's Conjecture

$$\mathrm{cr}(Q_n) \leq \frac{5}{32}4^n - \left\lfloor \frac{n^2+1}{2} \right\rfloor 2^{n-2}$$

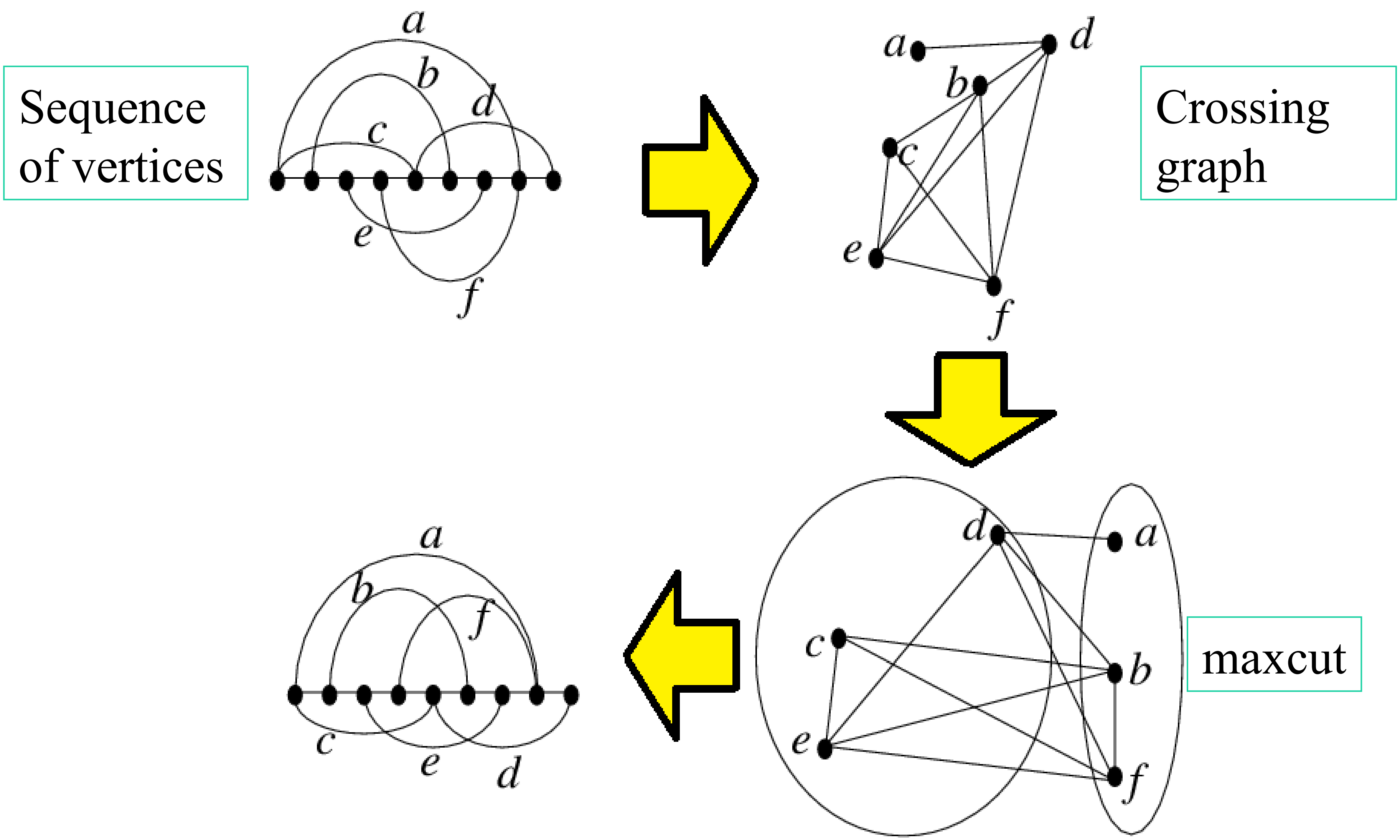
Ondrej Sýkora and Imrich Vrt’o - 2002

Imrich Vrt’o - 2012

From computational results to a drawing

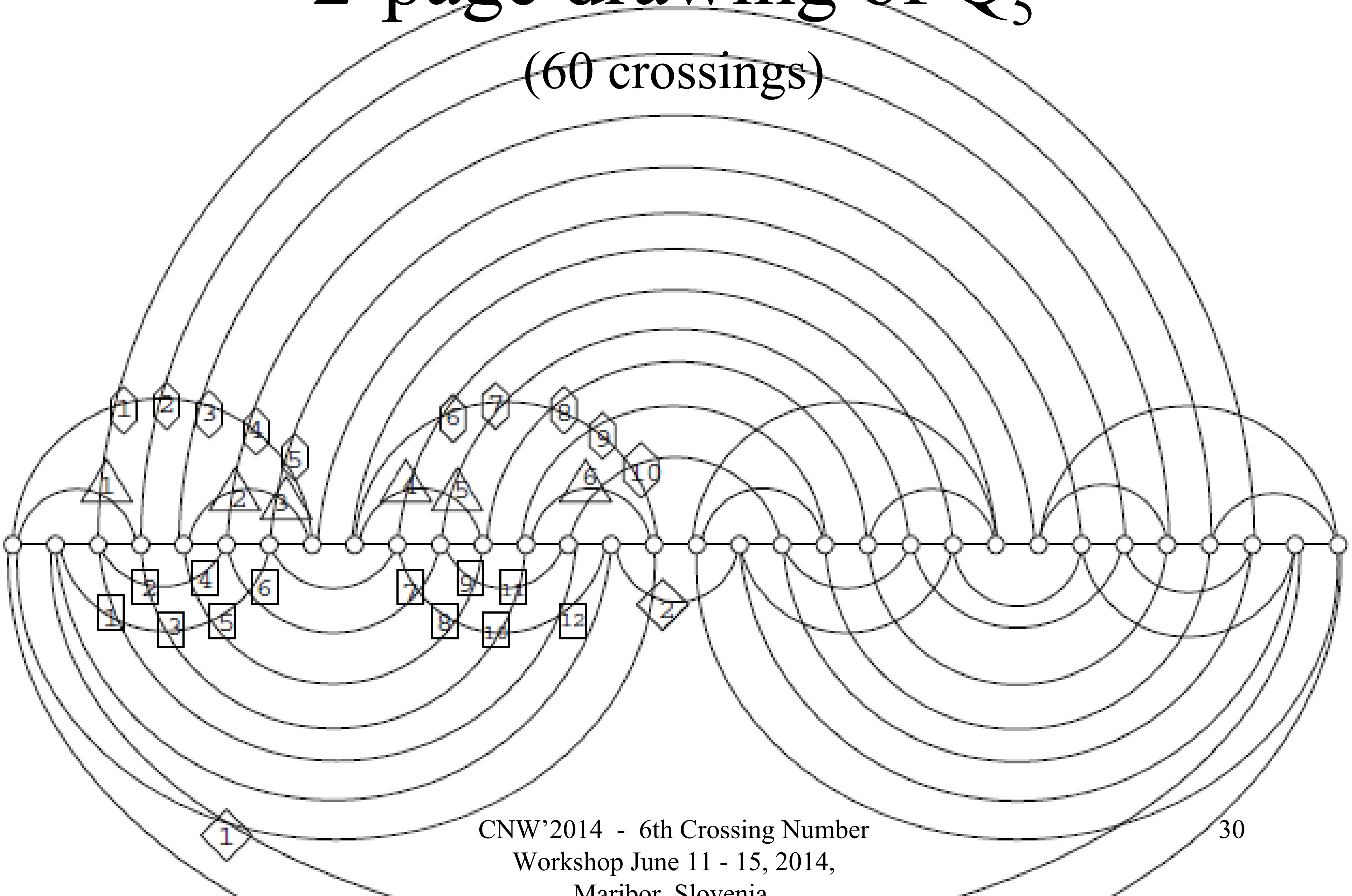
- C. Buchheim and L. Zheng‘2006

From a computational result to a 2-page drawing



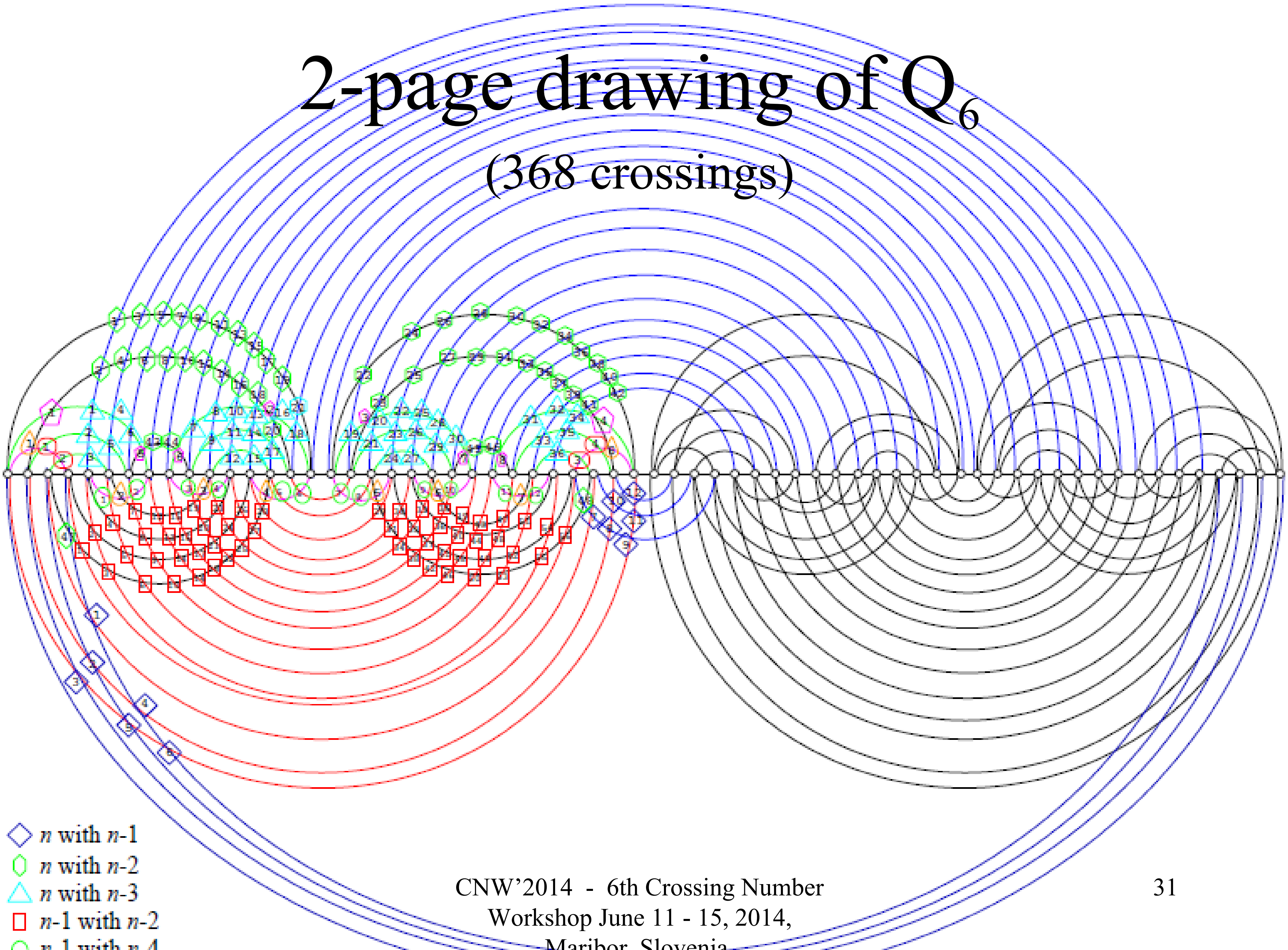
2-page drawing of Q_5

(60 crossings)



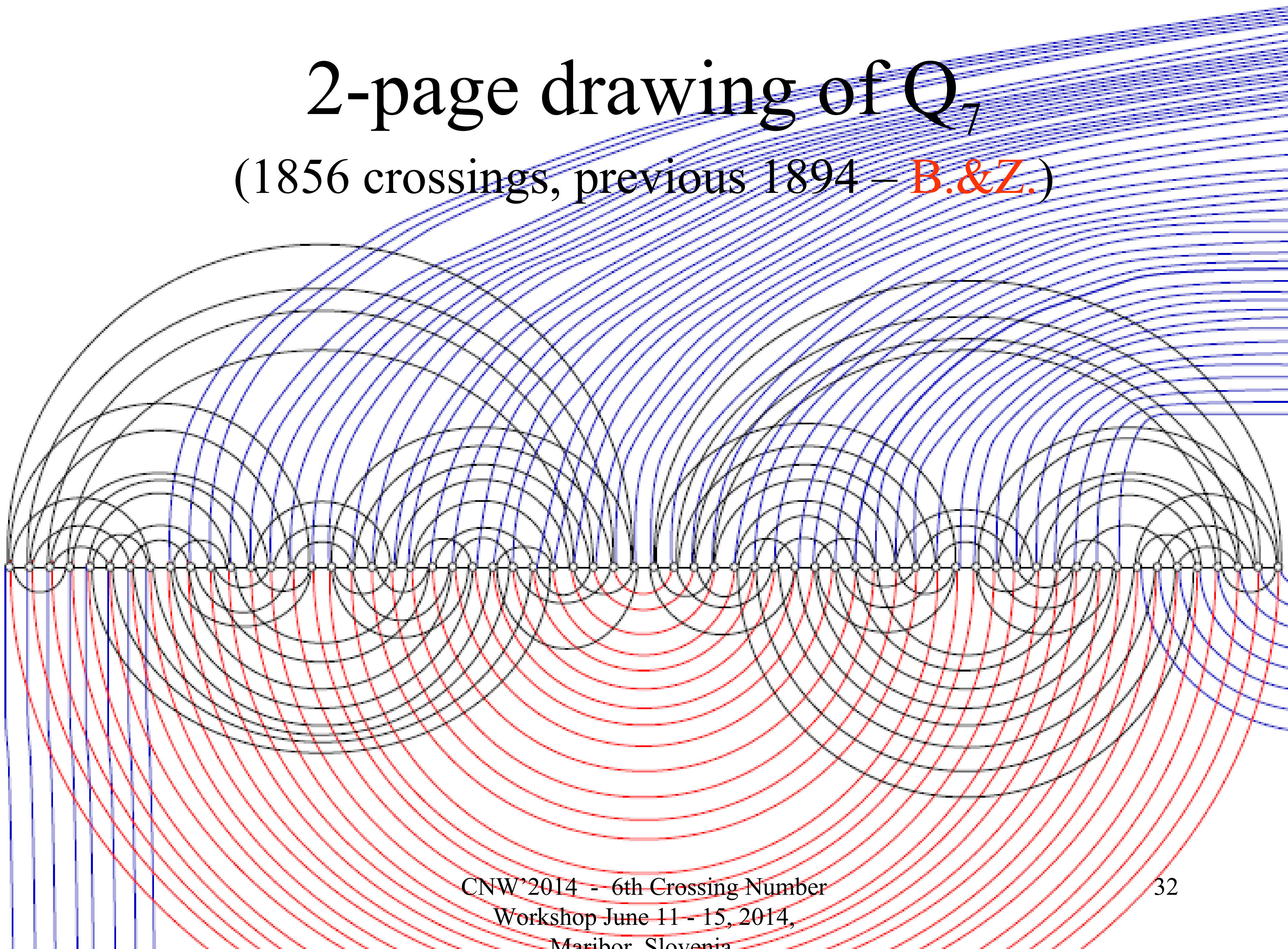
2-page drawing of Q_6

(368 crossings)



2-page drawing of Q_7

(1856 crossings, previous 1894 – B.&Z.)



Upper bound: $\nu_2(Q_n)$ and $\overline{cr}(Q_n)$

$\nu_2(Q_n) \leq \frac{125}{750} 4^n - 2^{n-3} n^2 - 2^{n-4} 3 + \frac{(-2)^n}{48}$ Madej, 1991

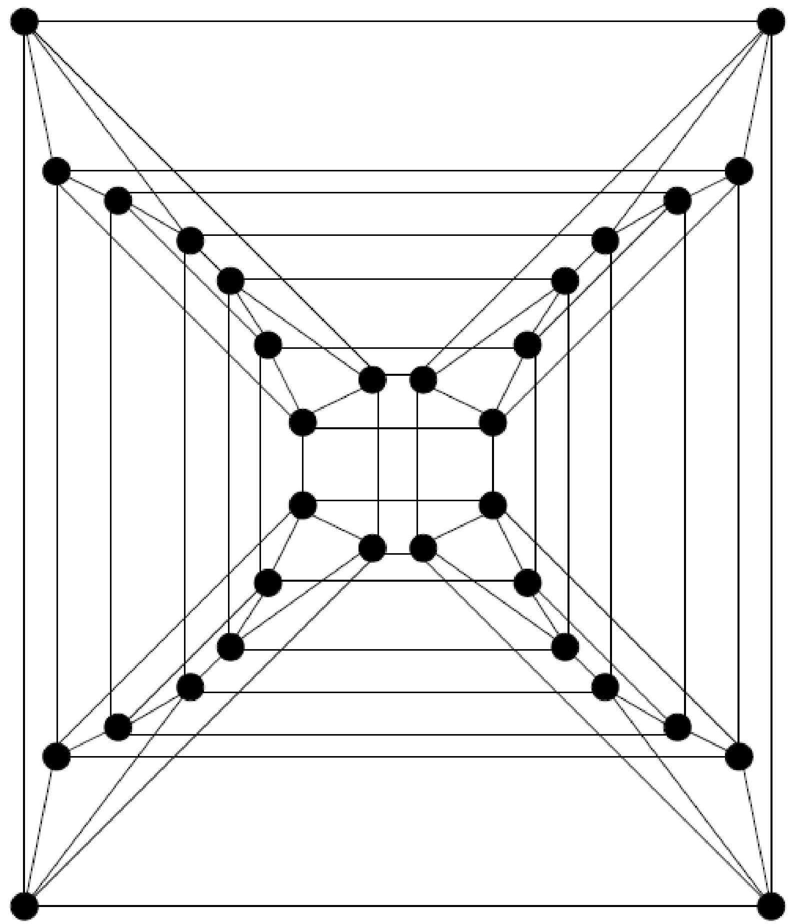
$$\overline{cr}(Q_n), \; \nu_2(Q_n) \leq \frac{125}{768} 4^n - \frac{2^{n-3}}{3} \left(3n^2 + \frac{9+(-1)^{n+1}}{2} \right)$$

$\nu(Q_n) \leq \frac{125}{800} 4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2}$ Faria, Figueiredo, Sýkora & Vrt'o WG'2003.

Crossing Number Workshop’2013

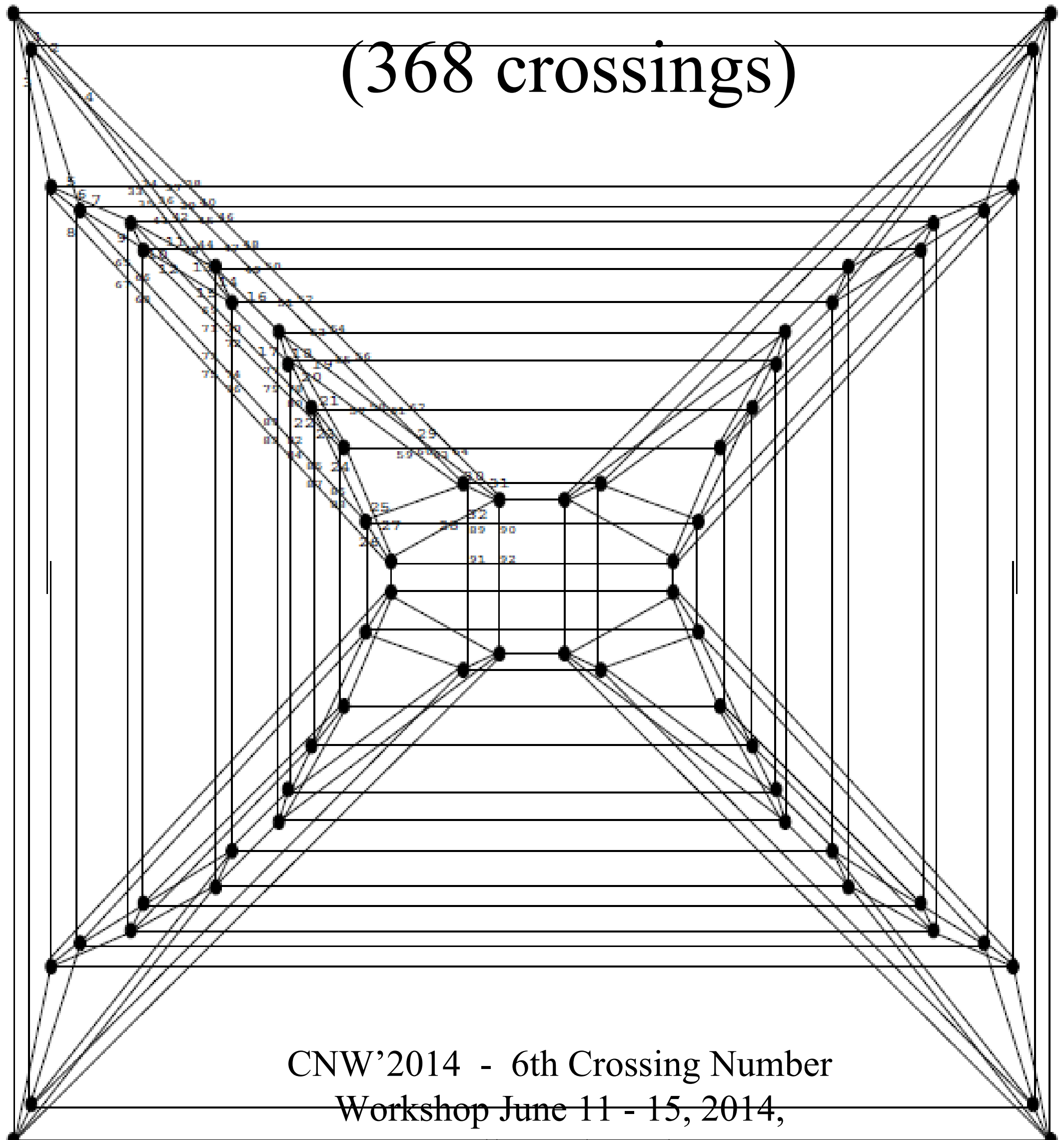
rectilinear drawing of Q_5

(60 crossings)



Rectilinear drawing of Q_6

(368 crossings)



$$\frac{125}{768}4^n - \frac{2^{n-3}}{3}\left(3n^2 + \frac{9+(-1)^{n+1}}{2}\right)$$

Q_n	Crossing number current best upper bound	This paper	Madej
n	$\frac{4^n 5}{32} - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2}$	$\frac{4^n 125}{768} - \frac{2^{n-3}}{3}(3n^2 + \frac{9+(-1)^{n+1}}{2})$	$\frac{4^n}{6} - 2^{n-3}n^2 - 2^{n-4}3 + \frac{(-2)^n}{48}$
5	56	60	64
6	352	368	384
7	1760	1856	1920
8	8192	8576	8832
9	35712	37376	38400
10	151040	157696	161792
11	624128	651264	667648
12	2547712	2656256	2721792
13	10311680	10747904	11010048
14	41541632	43286528	44335104
15	166846464	173834240	178028544

One More Open Problem

$$\nu_2(K_n) \leq \overline{cr}(K_n)$$

Abrégo, Aichholzer, Merchant, Ramos, and Salazar'2012

$$\nu_2(Q_n) \overset{?}{\leq} \overline{cr}(Q_n)$$

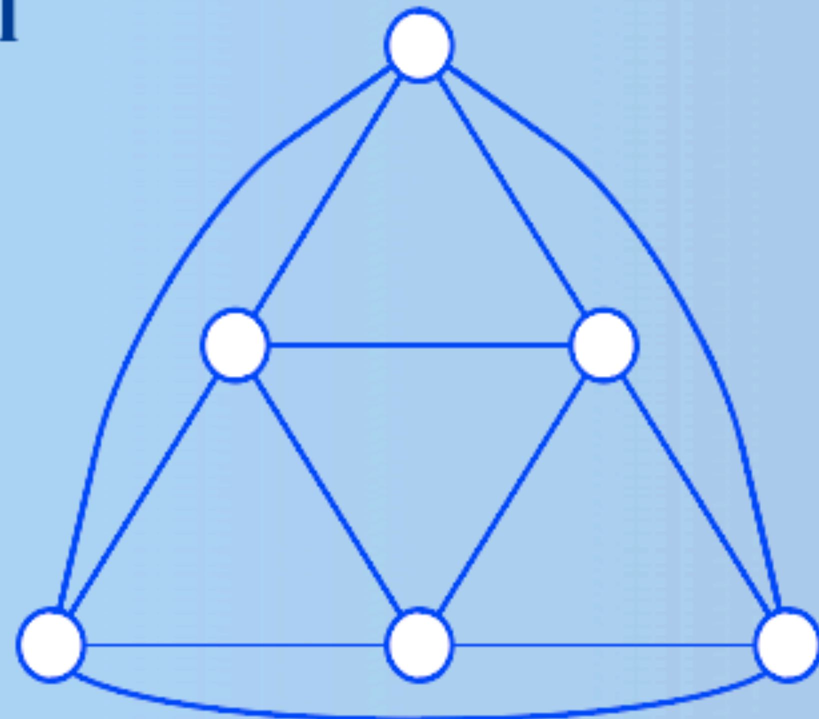
VIII Latin-American Algorithms, Graphs and Optimization Symposium

LAGOS 2015

Praia das Fontes, Beberibe, Ceará, Brazil

http://www.lia.ufc.br/lagos2015

May, 11-15, 2015



Invited Speakers

Béla Bollobás (Cambridge University, UK)
Gérard Cornuéjols (Carnegie Mellon, USA)
Frédéric Havet (INRIA Sophia-Antipolis, France)
Sulamita Klein (UFRJ, Rio de Janeiro, Brazil)
Frédéric Maffray (G-SCOP Grenoble, France)
Miguel Pizaña (UAM, Mexico)
Bruce Reed (McGill University, Canada)
Ola Svensson (EPFL, Switzerland)

Important Dates

November, 25, 2014 - submission deadline
February, 03, 2015 - notification of acceptance
February, 06, 2015 - registration opens