Combinatorial Games in Graphs: theory and ludic

Ana Luísa Furtado (CEFET-RJ)
Celina de Figueiredo (COPPE/UFRJ)
Simone Dantas (IM/UFF)
Sylvain Gravier (Université Grenoble Alpes)

Semana PESC
October 2, 2018
Summary

1. Academic trajectory
2. Graphs and Combinatorial Games
3. Coloring Game
4. Nordhaus-Gaddum type inequalities
Summary

1. Academic trajectory
2. Graphs and Combinatorial Games
3. Coloring Game
4. Nordhaus-Gaddum type inequalities
Bachelor’s degree in mathematics - UFRJ (2007).
Academic trajectory

Bachelor’s degree in mathematics - UFRJ (2007).

Discrete math course.
Academic trajectory

Bachelor’s degree in mathematics - UFRJ (2007).

Discrete math course.
Academic trajectory

Master’s degree in mathematics education - PEMAT/UFRJ (2010).
Academic trajectory

Master’s degree in mathematics education - PEMAT/UFRJ (2010).

Academic trajectory

Master’s degree in mathematics education - PEMAT/UFRJ (2010).


Master’s thesis in linear algebra teaching.
Academic trajectory

Master’s degree in mathematics education - PEMAT/UFRJ (2010).


Master’s thesis in linear algebra teaching.
Academic trajectory

Master’s degree in mathematics education - PEMAT/UFRJ (2010).


Master’s thesis in linear algebra teaching.

Professor at CEFET-RJ’ (since 2010).
Academic trajectory

Master’s degree in mathematics education - PEMAT/UFRJ (2010).


Master’s thesis in linear algebra teaching.

Professor at CEFET-RJ’ (since 2010).
Academic trajectory

What comes next?
Academic trajectory

What comes next?

Course of Introduction to Computer Theory (PESC/UFRJ).
Academic trajectory

What comes next?

Course of Introduction to Computer Theory (PESC/UFRJ).
Academic trajectory

What comes next?

Course of Introduction to Computer Theory (PESC/UFRJ).
I started my doctorate at PESC/COPPE in September 2013.
I started my doctorate at PESC/COPPE in September 2013.

Thesis topic: Combinatorial Games in Graphs.
I started my doctorate at PESC/COPPE in September 2013.

Thesis topic: Combinatorial Games in Graphs.

Professors:
I started my doctorate at PESC/COPPE in September 2013.

Thesis topic: Combinatorial Games in Graphs.

Professors:
In January 2014, Professor Simone introduced me to Professor Sylvain.
In January 2014, Professor Simone introduced me to Professor Sylvain.
In January 2014, Professor Simone introduced me to Professor Sylvain.

New game: Timber Game in caterpillars.
Academic trajectory

In January 2015: qualification.
Academic trajectory

In January 2015: qualification.

In March 2015: Grenoble.
Academic trajectory

In January 2015: qualification.

In March 2015: Grenoble.
Academic trajectory

In January 2015: qualification.

In March 2015: Grenoble.
Academic trajectory

In Grenoble, in addition to working on Timber Game, we started working on Coloring Game.
In Grenoble, in addition to working on Timber Game, we started working on Coloring Game.

Working together with:
Academic trajectory

In Grenoble, in addition to working on Timber Game, we started working on Coloring Game.

Working together with:

- Clément Charpentier;
- Simon Schmidt.
- Math à modeler team.
Academic trajectory

Back to Brasil in October 2015, we continued to work on combinatorial games, and write articles and my Ph.D. thesis.
Academic trajectory

Back to Brasil in October 2015, we continued to work on combinatorial games, and write articles and my Ph.D. thesis.

Two years later...
Back to Brasil in October 2015, we continued to work on combinatorial games, and write articles and my Ph.D. thesis.

Two years later...
Academic trajectory

And two weeks ago I was at CNMAC, receiving the best thesis award from SBMAC.
Academic trajectory

And two weeks ago I was at CNMAC, receiving the best thesis award from SBMAC.
Summary

1. Academic trajectory
2. Graphs and Combinatorial Games
3. Coloring Game
4. Nordhaus-Gaddum type inequalities
Combinatorial games
Combinatorial games
Combinatorial games

Graphs and Combinatorial Games
Many researchers have been studying winning strategies in 2-player combinatorial games.

We study the Timber Game and the Coloring Game in a caterpillar.

Moreover, we study the Nordhaus-Gaddum type inequality to the parameter of these game.
Graphs and Combinatorial Games

Goal

- Many researchers have been studying winning strategies in 2-player combinatorial games.
- We study the Timber Game and the Coloring Game in a caterpillar.
- Moreover, we study the Nordhaus-Gaddum type inequality to the parameter of these game.
Many researchers have been studying winning strategies in 2-player combinatorial games.

We study the Timber Game and the Coloring Game in a caterpillar.

Moreover, we study the Nordhaus-Gaddum type inequality to the parameter of these games.
Summary

1. Academic trajectory
2. Graphs and Combinatorial Games
3. Coloring Game
4. Nordhaus-Gaddum type inequalities
What is Coloring Game?

- The *coloring game* is a two player non-cooperative game conceived by Steven Brams.


- Reinvented in 1991 by Bodlaender, who studied the game in the context of graphs.
The *coloring game* is a two player non-cooperative game conceived by Steven Brams.


Reinvented in 1991 by Bodlaender, who studied the game in the context of graphs.
What is Coloring Game?

- The *coloring game* is a two player non-cooperative game conceived by Steven Brams.


- Reinvented in 1991 by Bodlaender, who studied the game in the context of graphs.
How to play?

Given $t$ colors, Alice and Bob take turns properly coloring an uncolored vertex.
How to play?

Given $t$ colors, Alice and Bob take turns properly coloring an uncolored vertex.

Alice X Bob
How to play?

Given $t$ colors, Alice and Bob take turns properly coloring an uncolored vertex.

Alice $\times$ Bob

minimizer $\times$ maximizer
How to play?
How to play?
How to play?
Who wins?

- Alice wins when the graph is completely colored with \( t \) colors; otherwise, Bob wins.

- The game chromatic number \( \chi_g(G) \) of \( G \) is the smallest number \( t \) of colors that ensures that Alice wins (when Alice starts the game).
Who wins?

- Alice wins when the graph is completely colored with $t$ colors; otherwise, Bob wins.

- The *game chromatic number* $\chi_g(G)$ of $G$ is the smallest number $t$ of colors that ensures that Alice wins (when Alice starts the game).
Who wins?

- Alice wins when the graph is completely colored with $t$ colors; otherwise, Bob wins.

- The *game chromatic number* $\chi_g(G)$ of $G$ is the smallest number $t$ of colors that ensures that Alice wins (when Alice starts the game).

\[
\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1
\]
Simple results

- $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$
  - $\chi_g(K_n) = n$
  - $\chi_g(S_n) = 1$

- $\chi_g(P_1) = 1$, $\chi_g(P_2) = \chi_g(P_3) = 2$

- For $n \geq 4$, we have that $\chi_g(P_n) = 3$

- $\chi_g(C_n) = 3$

- The stars $K_{1,p}$ with $p \geq 1$ are the only connected graphs satisfying $\chi_g(G) = 2$
Simple results

- $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$
  - $\chi_g(K_n) = n$
  - $\chi_g(S_n) = 1$

- $\chi_g(P_1) = 1$, $\chi_g(P_2) = \chi_g(P_3) = 2$

- For $n \geq 4$, we have that $\chi_g(P_n) = 3$

- $\chi_g(C_n) = 3$

- The stars $K_{1,p}$ with $p \geq 1$ are the only connected graphs satisfying $\chi_g(G) = 2$
Simple results

- $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$
  - $\chi_g(K_n) = n$
  - $\chi_g(S_n) = 1$

- $\chi_g(P_1) = 1$, $\chi_g(P_2) = \chi_g(P_3) = 2$

- For $n \geq 4$, we have that $\chi_g(P_n) = 3$

- $\chi_g(C_n) = 3$

- The stars $K_{1,p}$ with $p \geq 1$ are the only connected graphs satisfying $\chi_g(G) = 2$. 


Simple results

- $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$
  - $\chi_g(K_n) = n$
  - $\chi_g(S_n) = 1$

- $\chi_g(P_1) = 1$, $\chi_g(P_2) = \chi_g(P_3) = 2$

- For $n \geq 4$, we have that $\chi_g(P_n) = 3$

- $\chi_g(C_n) = 3$

- The stars $K_{1,p}$ with $p \geq 1$ are the only connected graphs satisfying $\chi_g(G) = 2$
Simple results

- \( \chi(G) \leq \chi_g(G) \leq \Delta(G) + 1 \)
  - \( \chi_g(K_n) = n \)
  - \( \chi_g(S_n) = 1 \)

- \( \chi_g(P_1) = 1, \chi_g(P_2) = \chi_g(P_3) = 2 \)

- For \( n \geq 4 \), we have that \( \chi_g(P_n) = 3 \)

- \( \chi_g(C_n) = 3 \)

- The stars \( K_{1,p} \) with \( p \geq 1 \) are the only connected graphs satisfying \( \chi_g(G) = 2 \)
Simple results

- $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$
  - $\chi_g(K_n) = n$
  - $\chi_g(S_n) = 1$

- $\chi_g(P_1) = 1$, $\chi_g(P_2) = \chi_g(P_3) = 2$

- For $n \geq 4$, we have that $\chi_g(P_n) = 3$

- $\chi_g(C_n) = 3$

- The stars $K_{1,p}$ with $p \geq 1$ are the only connected graphs satisfying $\chi_g(G) = 2$
Simple results

- $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$
  - $\chi_g(K_n) = n$
  - $\chi_g(S_n) = 1$

- $\chi_g(P_1) = 1$, $\chi_g(P_2) = \chi_g(P_3) = 2$

- For $n \geq 4$, we have that $\chi_g(P_n) = 3$

- $\chi_g(C_n) = 3$

- The stars $K_{1,p}$ with $p \geq 1$ are the only connected graphs satisfying $\chi_g(G) = 2$
Different graph classes studied

- planar graphs: $7 \leq \chi_g(P) \leq 17$;

- outerplanar graphs: $6 \leq \chi_g(O) \leq 7$;

- toroidal grids: $\chi_g(TG) = 5$;

- partial $k$-trees: $\chi_g(P) \leq 3k + 2$;

- the cartesian products of some classes of graphs: for example, $\chi_g(T_1 \square T_2) \leq 12$;
Different graph classes studied

- planar graphs: $7 \leq \chi_g(P) \leq 17$;

- outerplanar graphs: $6 \leq \chi_g(O) \leq 7$;

- toroidal grids: $\chi_g(TG) = 5$;

- partial $k$-trees: $\chi_g(P) \leq 3k + 2$;

- the cartesian products of some classes of graphs: for example, $\chi_g(T_1 \Box T_2) \leq 12$;
Different graph classes studied

- planar graphs: $7 \leq \chi_g(P) \leq 17$;
- outerplanar graphs: $6 \leq \chi_g(O) \leq 7$;
- toroidal grids: $\chi_g(TG) = 5$;
- partial $k$-trees: $\chi_g(P) \leq 3k + 2$;
- the cartesian products of some classes of graphs: for example, $\chi_g(T_1 \square T_2) \leq 12$;
Different graph classes studied

- planar graphs: $7 \leq \chi_g(P) \leq 17$;

- outerplanar graphs: $6 \leq \chi_g(O) \leq 7$;

- toroidal grids: $\chi_g(TG) = 5$;

- partial $k$-trees: $\chi_g(P) \leq 3k + 2$;

- the cartesian products of some classes of graphs: for example, $\chi_g(T_1 \Box T_2) \leq 12$;
Different graph classes studied

- planar graphs: $7 \leq \chi_g(P) \leq 17$;
- outerplanar graphs: $6 \leq \chi_g(O) \leq 7$;
- toroidal grids: $\chi_g(TG) = 5$;
- partial $k$-trees: $\chi_g(P) \leq 3k + 2$;
- the cartesian products of some classes of graphs: for example, $\chi_g(T_1 \square T_2) \leq 12$;
Literature for trees

- Bodlaender (1991): $\chi_g(T) \leq 5$.

- Faigle et al. (1993): $\chi_g(F) \leq 4$.

- Dunn et al. (2015): criteria for determining $\chi_g(F)$, for a forest without vertex of degree 3, in polynomial time.
Literature for trees

- Bodlaender (1991): $\chi_g(T) \leq 5$.

- Faigle et al. (1993): $\chi_g(F) \leq 4$.

- Dunn et al. (2015): criteria for determining $\chi_g(F)$, for a forest without vertex of degree 3, in polynomial time.
Literature for trees

- Bodlaender (1991): $\chi_g(T) \leq 5$.

- Faigle et al. (1993): $\chi_g(F) \leq 4$.

- Dunn et al. (2015): criteria for determining $\chi_g(F)$, for a forest without vertex of degree 3, in polynomial time.
Our problem

- Due to the difficulty concerning this subject, the problem of characterizing forests with $\chi_g(F) = 3$ remains open.

- In our work, we contribute to this study by analyzing the caterpillar.
Our problem

- Due to the difficulty concerning this subject, the problem of characterizing forests with $\chi_g(F) = 3$ remains open.

- In our work, we contribute to this study by analyzing the *caterpillar*.
A caterpillar $cat(k_1, k_2, \ldots, k_s)$ is a tree which is obtained from a central path $v_1, v_2, v_3, \ldots, v_s$ (called spine), and by joining $v_i$ to $k_i$ new vertices, $i = 1, \ldots, s$.

Figure: $cat(0, 2, 1, 1, 0, 3, 0)$. 

Why caterpillars?

- Example presented in Bodlaender (1991) to prove the existence of a tree $H_d$ with $\chi_g(H_d) \geq 4$:

- Dunn et al. (2015) proved that this caterpillar is the smallest tree such that $\chi_g(T) = 4$.

- We are interested in characterizing when $\chi_g(H)$ is 3 or 4.
Why caterpillars?

- Example presented in Bodlaender (1991) to prove the existence of a tree $H_d$ with $\chi_g(H_d) \geq 4$:

- Dunn et al. (2015) proved that this caterpillar is the smallest tree such that $\chi_g(T) = 4$.

- We are interested in characterizing when $\chi_g(H)$ is 3 or 4.
Why caterpillars?

- Example presented in Bodlaender (1991) to prove the existence of a tree $H_d$ with $\chi_g(H_d) \geq 4$:

- Dunn et al. (2015) proved that this caterpillar is the smallest tree such that $\chi_g(T) = 4$.

- We are interested in characterizing when $\chi_g(H)$ is 3 or 4.
Why caterpillars?

- Example presented in Bodlaender (1991) to prove the existence of a tree $H_d$ with $\chi_g(H_d) \geq 4$:

- Dunn et al. (2015) proved that this caterpillar is the smallest tree such that $\chi_g(T) = 4$.

- We are interested in characterizing when $\chi_g(H)$ is 3 or 4.
What did we do?

- We have determined:
  - two sufficient conditions for $\chi_g(H) = 4$ for any caterpillar $H$;
  - two necessary conditions for $\chi_g(H) = 4$ for any caterpillar $H$.

- Caterpillars
What did we do?

- We have determined:
  - two sufficient conditions for $\chi_g(H) = 4$ for any caterpillar $H$;
  - two necessary conditions for $\chi_g(H) = 4$ for any caterpillar $H$.

- Caterpillars
  - with maximum degree 3;
  - without vertex of degree 2;
  - without vertex of degree 3;
  - with vertices of degree 1, 2, 3 and 4.
What did we do?

- We have determined:
  - two sufficient conditions for $\chi_g(H) = 4$ for any caterpillar $H$;
  - two necessary conditions for $\chi_g(H) = 4$ for any caterpillar $H$.

- Caterpillars
  - with maximum degree 3;
  - without vertex of degree 2;
  - without vertex of degree 3;
  - with vertices of degree 1, 2, 3 and 4.
What did we do?

- We have determined:
  - two sufficient conditions for $\chi_g(H) = 4$ for any caterpillar $H$;
  - two necessary conditions for $\chi_g(H) = 4$ for any caterpillar $H$.

- Caterpillars
  - with maximum degree 3;
  - without vertex of degree 2;
  - without vertex of degree 3;
  - with vertices of degree 1, 2, 3 and 4.
What did we do?

- We have determined:
  - two sufficient conditions for $\chi_g(H) = 4$ for any caterpillar $H$;
  - two necessary conditions for $\chi_g(H) = 4$ for any caterpillar $H$.

- Caterpillars
  - with maximum degree 3;
    - without vertex of degree 2;
    - without vertex of degree 3;
    - with vertices of degree 1, 2, 3 and 4.
What did we do?

- We have determined:
  - two sufficient conditions for $\chi_g(H) = 4$ for any caterpillar $H$;
  - two necessary conditions for $\chi_g(H) = 4$ for any caterpillar $H$.
- Caterpillars
  - with maximum degree 3;
  - without vertex of degree 2;
  - without vertex of degree 3;
  - with vertices of degree 1, 2, 3 and 4.
What did we do?

- We have determined:
  - two sufficient conditions for $\chi_g(H) = 4$ for any caterpillar $H$;
  - two necessary conditions for $\chi_g(H) = 4$ for any caterpillar $H$.

- Caterpillars
  - with maximum degree 3;
  - without vertex of degree 2;
  - without vertex of degree 3;
  - with vertices of degree 1, 2, 3 and 4.
What did we do?

- We have determined:
  - two sufficient conditions for $\chi_g(H) = 4$ for any caterpillar $H$;
  - two necessary conditions for $\chi_g(H) = 4$ for any caterpillar $H$.

- Caterpillars
  - with maximum degree 3;
  - without vertex of degree 2;
  - without vertex of degree 3;
  - with vertices of degree 1, 2, 3 and 4.
Theorem (Furtado et al., 2017)

Let $H$ be the caterpillar $\text{cat}(k_1, ..., k_s)$ with $\Delta(H) = 3$. We have that $H$ has $\chi^a_g(H), \chi^b_g(H) \leq 3$. Moreover, let $F$ be the forest where each connected component is a caterpillar and $\Delta(F) = 3$. We have that $F$ has $\chi^a_g(F) \leq 3$. 
Caterpillar without vertex of degree 2

Theorem (Furtado et al., 2017)

Let $H$ be the caterpillar without vertex of degree 2. We have that
\[ \chi^a_g(H) = \chi^b_g(H) = 4 \] if, and only if, $H$ is caterpillar $\text{cat}(k_1, \ldots, k_s)$, such that $k_1 = k_s = 0$, $k_i \neq 0$, $\forall i \in \{2, \ldots, s - 1\}$, and there are at least four vertices of degree at least 4.
Caterpillar without vertex of degree 3

Let *Family Q* be the set of caterpillars $H_d$, $H_{33}$, $H_{[\alpha]} \cup H_{[\beta]}$, $H_{[\alpha][\beta]}$ and $H_{[\alpha]3[\beta]}$. 
Caterpillar without vertex of degree 3

Let *Family Q* be the set of caterpillars $H_d$, $H_{33}$, $H_{[\alpha]} \cup H_{[\beta]}$, $H_{[\alpha][\beta]}$ and $H_{[\alpha]3[\beta]}$.

*Figure*: Caterpillars (a) $H_{33}$ (b) $H_{[3]}$ (c)$H_{[3][4]}$ (d)$H_{[3]3[4]}$. 

COPPE UFRJ
Caterpillar without vertex of degree 3

Theorem

A caterpillar $H$ without vertex of degree 3 has $\chi_g(H) = 4$ if, and only if, $H$ has a caterpillar of Family $Q$ as an induced subcaterpillar.
Caterpillar with vertices of degree 1, 2, 3 and 4

Let \( \text{Family } Q' \) be the set of caterpillars \( \{ H'_\alpha \cup H'_\beta, H'_\alpha \cup H_3, H_3 \cup H_3, H'_{22} \text{ and } H'_{[\alpha][\beta]}, H'_{23} \} \).
Caterpillar with vertices of degree 1, 2, 3 and 4

Let *Family Q'* be the set of caterpillars \{H'_{[\alpha]} \cup H'_{[\beta]}, H'_{[\alpha]} \cup H_3, H_3 \cup H_3, H_{22}'\text{ and } H'_{[\alpha][\beta]}, H'_{23}\}.

Figure: Caterpillars (a) $H'_{6}$ (b) $H'_{3}$ (c)$H'_{22}$ (d)$H'_{[6][3]}$ (e)$H'_{23}$. 
Theorem (Furtado et al., 2017)

Let $H$ be a caterpillar with vertices of 1, 2, 3 and 4. If $H$ has a caterpillar of Family $Q'$ as a induced subcaterpillar, then $\chi_g(H) = 4$. 
Summary

<table>
<thead>
<tr>
<th>$\Delta(H)$</th>
<th>$\chi_g(H) = 1$</th>
<th>$\chi_g(H) = 2$</th>
<th>$\chi_g(H) = 3$</th>
<th>$\chi_g(H) = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$P_1$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>-</td>
<td>$P_2$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>$P_3$</td>
<td>$P_n, n \geq 4$</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>star</td>
<td>not a star</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>star</td>
<td>see next Figure</td>
<td>see next Figure</td>
</tr>
</tbody>
</table>
Summary

**Figure:** Caterpillars with $\Delta(H) = 4$ and $\chi_g(H) = 4$. 

- H without vertex of degree 2
- H without vertex of degree 3
- H with exactly four vertices of degree 4 except $H_d$
- $H_d$
- Family Q except $H_d$
- Family Q'
- ...

*Coloring Game*
Theorem (Furtado et al., 2017)

Let $F$ be a forest composed by $r$ trees $T_1$, ..., $T_r$. Assume that $\chi_g^a(T_1) \leq \chi_g^a(T_2) \leq ... \leq \chi_g^a(T_r)$, and, if there exist two trees with the same game chromatic number, then $T_i$ and $T_j$ are ordered in a way that $\chi_g^b(T_i) \leq \chi_g^b(T_j)$, for $i < j$. We have that:

1. If $\chi_g^b(T_r) > \chi_g^a(T_r)$, then $\chi_g(F) = \chi_g^a(T_r)$;
2. If $\chi_g^b(T_r) = \chi_g^b(T_{r-1}) > \chi_g^a(T_{r-1})$, then $\chi_g(F) = \chi_g^b(T_r)$;
3. If $\chi_g^a(T_r) = \chi_g^b(T_r)$, then $\chi_g(F) = \chi_g^a(T_r) = \chi_g^b(T_r)$;
4. If $\chi_g^b(T_r) < \chi_g^a(T_r)$ and $\sum_{i=1}^{r-1} |V(T_i)|$ is even, then $\chi_g(F) = \chi_g^a(T_r)$;
5. If $\chi_g^b(T_r) < \chi_g^a(T_r)$ and $\sum_{i=1}^{r-1} |V(T_i)|$ is odd, then $\chi_g(F) = \max \{\chi_g^a(F \setminus T_r), \chi_g^b(T_r)\}$. 

$\chi_g(F)$
Summary

1. Academic trajectory
2. Graphs and Combinatorial Games
3. Coloring Game
4. Nordhaus-Gaddum type inequalities
What are Nordhaus-Gaddum type inequalities?

- Nordhaus and Gaddum (1956) showed lower and upper bounds on the sum of the chromatic number of a graph and its complement:

\[ 2\sqrt{n} \leq \chi(G) + \chi(G^c) \leq n + 1. \]

These bounds are best possible for infinitely many values of \( n \).

- Survey by Aouiche and Hansen (2013): 360 articles.

- To the best of our knowledge, the only Nordhaus-Gaddum type inequality existing for invariants related to games on graphs is by Alon et al. (2002) and concerns the game domination number.
What are Nordhaus-Gaddum type inequalities?

Nordhaus and Gaddum (1956) showed lower and upper bounds on the sum of the chromatic number of a graph and its complement:

**Theorem (Nordhaus and Gaddum, 1956)**

*If $G$ is a graph of order $n$, then $2\sqrt{n} \leq \chi(G) + \chi(\bar{G}) \leq n + 1$. These bounds are best possible for infinitely many values of $n$.***

- Survey by Aouiche and Hansen (2013): 360 articles.
- To the best of our knowledge, the only Nordhaus-Gaddum type inequality existing for invariants related to games on graphs is by Alon et al. (2002) and concerns the game domination number.
Nordhaus and Gaddum (1956) showed lower and upper bounds on the sum of the chromatic number of a graph and its complement:

**Theorem (Nordhaus and Gaddum, 1956)**

*If \( G \) is a graph of order \( n \), then \( 2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1 \). These bounds are best possible for infinitely many values of \( n \).*

Survey by Aouiche and Hansen (2013): 360 articles.

To the best of our knowledge, the only Nordhaus-Gaddum type inequality existing for invariants related to games on graphs is by Alon et al. (2002) and concerns the game domination number.
Nordhaus and Gaddum (1956) showed lower and upper bounds on the sum of the chromatic number of a graph and its complement:

**Theorem (Nordhaus and Gaddum, 1956)**

If $G$ is a graph of order $n$, then $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$. These bounds are best possible for infinitely many values of $n$.

Survey by Aouiche and Hansen (2013): 360 articles.

To the best of our knowledge, the only Nordhaus-Gaddum type inequality existing for invariants related to games on graphs is by Alon et al. (2002) and concerns the game domination number.
Theorem (Furtado et al., 2017)

For any graph $G$ of order $n$, we have that $2\sqrt{n} \leq \chi_g(G) + \chi_g(\overline{G}) \leq \left\lceil \frac{3n}{2} \right\rceil$. Moreover, the bounds are best possible asymptotically:

1. for infinitely many values of $n$, there are graphs $G$ of order $n$ with $\chi_g(G) + \chi_g(\overline{G}) = \left\lceil \frac{4n}{3} \right\rceil - 1$;
2. for infinitely many values of $n$, there are graphs $G$ of order $n$ with $\chi_g(G) + \chi_g(\overline{G}) = 2\sqrt{2n} - 1$. 


Nordhaus-Gaddum type inequalities to $\chi_g(G) + \chi_g(\overline{G})$

Theorem (Furtado et al., 2017)

For any graph $G$ of order $n$, we have that $2\sqrt{n} \leq \chi_g(G) + \chi_g(\overline{G}) \leq \lceil \frac{3n}{2} \rceil$. Moreover, the bounds are best possible asymptotically:

1. for infinitely many values of $n$, there are graphs $G$ of order $n$ with $\chi_g(G) + \chi_g(\overline{G}) = \left\lceil \frac{4n}{3} \right\rceil - 1$;
2. for infinitely many values of $n$, there are graphs $G$ of order $n$ with $\chi_g(G) + \chi_g(\overline{G}) = 2\sqrt{2n} - 1$.

The lower bound follows from Theorem of Nordhaus and Gaddum (1965) and the inequality $\chi(G) \leq \chi_g(G)$.
We determine the Nordhaus-Gaddum type inequalities to

- the number of $P$-positions of a caterpillar (Timber Game);
- the game coloring number of any graph $G$ (Marking Game).

Marking Game is a “colorblind” version of the coloring game.

All bounds are tight, except the upper bound for the number of $P$-positions of a caterpillar.
We determine the Nordhaus-Gaddum type inequalities to

- the number of $P$-positions of a caterpillar (Timber Game);
- the game coloring number of any graph $G$ (Marking Game).

Marking Game is a “colorblind” version of the coloring game.

All bounds are tight, except the upper bound for the number of $P$-positions of a caterpillar.
We determine the Nordhaus-Gaddum type inequalities to

- the number of $P$-positions of a caterpillar (Timber Game);
- the game coloring number of any graph $G$ (Marking Game).

Marking Game is a “colorblind” version of the coloring game.

All bounds are tight, except the upper bound for the number of $P$-positions of a caterpillar.
Nordhaus-Gaddum type inequalities to other games

- We determine the Nordhaus-Gaddum type inequalities to
  - the number of $P$-positions of a caterpillar (Timber Game);
  - the game coloring number of any graph $G$ (Marking Game).

- *Marking Game* is a “colorblind” version of the coloring game.

- All bounds are tight, except the upper bound for the number of $P$-positions of a caterpillar.
Nordhaus-Gaddum type inequalities to other games

- We determine the Nordhaus-Gaddum type inequalities to
  - the number of $P$-positions of a caterpillar (Timber Game);
  - the game coloring number of any graph $G$ (Marking Game).

- *Marking Game* is a “colorblind” version of the coloring game.

- All bounds are tight, except the upper bound for the number of $P$-positions of a caterpillar.
Why games?

Figure: Salon International de la Culture et des jeux mathématiques, Paris, 2015.

Figure: Festival da Matemática, Rio de Janeiro, 2017.
THANK YOU!
Antena Brasileira de Matemática

http://www.antenabrasil.uff.br/

maths à modeler

http://mathsamodeler.ujf-grenoble.fr/