September 3, 2014
Ciclo de Seminários PESC

Duality for nonconvex optimization

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Outline

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The Primal Problem

$$\text{minimize } f(x) \quad \text{s.t.} \quad x \text{ in } K, \;\; h(x) = 0, \qquad (P)$$

- $f : \mathbb{R}^n \to \mathbb{R}$ lsc
- $h : \mathbb{R}^n \to \mathbb{R}^m$ continuous
- $K \subset \mathbb{R}^n$ compact

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The Dual Problem

The sharp Lagrangian (Rockafellar-Wets, 1997):

$$L(x,u,c) := f(x) - \langle u, h(x)\rangle + c\|h(x)\|.$$

The dual function: $q : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}$

$$q(u,c) = \inf_{x\in K} L(x,u,c)$$

with dual problem:

$$\text{maximize} \quad q(u,c) \quad \text{s.t} \quad (u,c) \text{ in } \mathbb{R}^m \times \mathbb{R}_+. \qquad (D)$$

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Duality Properties

Sharp Lagrangian is particular case of more general family of Augmented Lagrangians proposed by Rockafellar and Wets, 1997:

- Strong duality: dual and primal problem have same optimal value
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The subproblem

Fix (u, c) a dual variable:

$$\text{Find } \tilde{x} \in X(u,c) := \operatorname*{Argmin}_{x \in K} \left[f(x) + c\,\|h(x)\| - \langle u, h(x) \rangle \right]$$

Fact: $(-h(\tilde{x}), \|h(\tilde{x})\|) \in \partial q(u,c)$

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A subgradient direction

Update rule: Given current $z_k := (u_k, c_k)$, search along subgradient direction $g_k \in \partial q(z_k)$:

$$z_{k+1} = z_k + s_k \, g_k$$

where step-size $s_k > 0$. A subgradient of q at z_k is

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Choices of Step-size

Divergent series rule:

$$\sum_{k=0}^{\infty} s_k = \infty, \qquad \sum_{k=0}^{\infty} s_k^2 < \infty \qquad \text{(SLOW)}$$

If $Opt(P) = Opt(D) = \bar{q}$ is known:

$$s_k = \delta \frac{\bar{q} - q(u_k, c_k)}{\|g_k\|^2}, \qquad 0 < \delta < 2 \qquad \text{(BETTER)}$$

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Step 0 Choose (u_0, c_0) with $c_0 \geq 0$

Step k Given (u_k, c_k):

Find $x_k \in X(u_k, c_k)$. If $h(x_k) = 0$, STOP.

$$\text{Set} \quad \begin{cases} u_{k+1} := u_k - s_k h(x_k) \\ c_{k+1} := c_k + s_k \|h(x_k)\| \end{cases}$$

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$$\text{Set } \begin{cases} u_{k+1} = u_k - s_k h(x_k) \\ c_{k+1} = c_k + (s_k + \varepsilon_k)\|h(x_k)\| \end{cases}$$

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(Gasimov 2002; Gasimov & Ismayilova 2004):

$q_{k+1} > q_k$

Under $0 < \varepsilon_k < s_k = \dfrac{2\,(\bar{q} - q_k)}{5\,\|h(x_k)\|^2}$, one gets

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Under $s_k \geq \eta \dfrac{\bar{q} - q_k}{\|h(x_k)\|^2}$, $\eta > 0$, $\varepsilon_k > 0$, and

$$q_k \to \bar{q} \,, \qquad (u_k, c_k) \to (\bar{u}, \bar{c}) \,, \qquad \tilde{x}_k \to \bar{x}$$

where $\tilde{x}_k \in X(u_k, c_k + \beta)$ for $\beta > 0$

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(B. & Kaya 2007):

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(B. & Kaya & Mammadov, 2009):
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$$L(x_k, (u_k, c_k)) \leq q_k + r_k,$$

for r_k small and converging to zero,

and same convergence results hold.

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Choices for errors r_k

I: $r_k = \tau$, $k = 0, 1, \ldots$ (MSG)

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Murtagh and Saunders (1983)

$$\begin{cases} \min & f_0(x) = (x_1 - 1)^2 + (x_1 - x_2)^2 \\ & \qquad + (x_2 - x_3)^3 + (x_3 - x_4)^4 + (x_4 - x_5)^4 \\ \\ s.\ t. & f_1(x) = x_1 + x_2^2 + x_3^3 - 3\sqrt{2} - 2 = 0 \\ & f_2(x) = x_2 - x_3^2 + x_4 - 2\sqrt{2} + 2 = 0 \\ & f_3(x) = x_1\, x_5 - 2 = 0 \end{cases}$$

(Floudas et al. 1999)

t_{CPU}: CPU time in seconds
n_L: Total number of (Lagrangian) function evaluations
n_{IMSG}: Number of IMSG iterations

	I	II	III	IV	V
t_{CPU}	0.58	0.43	0.39	0.45	0.45
n_L	6400	3631	2990	4065	3873
n_{IMSG}	7	7	7	7	7

Table : Performance of IMSG: Problem 1, with $c_0 = 1$, $u_0 = (0,\ 1,\ 1)$, $\hat{q} = 0.1$ (upper estimate for $\bar{q}$), $\alpha = 1$, $\delta = 0.1$; $\tau = 10^{-10}$, and $a = 0.9$.

Quadratic Integer Programming

$$
\begin{cases}
\min & f_0(x) = a^T x + \frac{1}{2} x^T Q x \\
\text{s. t.} & -1 \le x_1 x_2 + x_3 x_4 \le 1 \\
& -3 \le x_1 + x_2 + x_3 + x_4 \le 2 \\
& x_i \in \{-1, 1\}
\end{cases}
$$

where

$$
a^T = [6 \;\; 8 \;\; 4 \;\; -2], \qquad Q = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 2 & -1 \end{bmatrix}.
$$

$$g_1(x) := x_1 x_2 + x_3 x_4$$
$$g_2(x) := x_1 + x_2 + x_3 + x_4$$

$$\begin{cases} \min & f_0(x) = a^T x + \frac{1}{2} x^T Q x \\ \\ \text{s. t.} & f_1(x) = \max(0, g_1(x) - 1) = 0 \\ & f_2(x) = \max(0, -(g_1(x) + 1)) = 0 \\ & f_3(x) = \max(0, g_2(x) - 2) = 0 \\ & f_4(x) = \max(0, -(g_2(x) + 3)) = 0 \\ & f_5(x) = \sum_{i=1}^{4} |(x_i - 1)(x_i + 1)| = 0 \end{cases}$$

	I	II	III	IV	V
t_{CPU}	0.79	0.57	0.37	0.35	0.37
n_L	7112	1506	1255	1217	1469
n_{IMSG}	7	22	9	8	8

Table : Performance of IMSG: Problem 2, with $c_0 = 1$, $u_0 = (-1,\ -1,\ -1,\ -1)$, $\hat{q} = -19$, $\alpha = 1$, $\delta = 0.05$; $\tau = 10^{-10}$, and $a = 0.7$.

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Stepsize needs knowledge of optimal value

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- $\{h(x_k)\}$ converges to zero
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$$x \in X_r(u, c) := \{x \in K : f(x) + c\,\sigma(h(x)) - \langle u, h(x)\rangle \leq q(u, c) + r\},$$

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P^* primal solutions & D^* dual solutions:

$$X(u,c) = \{x \in K \ : \ f(x) + c\,\sigma(h(x)) - \langle u, h(x)\rangle = q(u,c)\}$$

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If $d(u) < \infty$, TFAE:

(i) $d(u)$ is exact penalty parameter

(ii) $q(\cdot, \cdot)$ differentiable at $(u, d(u))$

If $0 < d(u)$ (i)(ii) are equivalent to:

(iii) Whenever (u_k, c_k) converges to $(u, d(u))$ and x_k is such that $x_k \in X_{r_k}(u_k, c_k)$, with r_k converging to zero, then every accumulation point of x_k solves (P).

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$$M(w) := \sup_{x \in K, \|h(x)\| \leq R} \langle w, \frac{h(x)}{\sigma(h(x))} \rangle < \infty$$

then TFAE:

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