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A STRONGLY POLYNOMIAL-TIME ALGORITHM FOR THE STRICT HOMOGENEOUS LINEAR-INEQUALITY FEASIBILITY PROBLEM

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Summary

The problem

Linear feasibility problem:

Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.
To obtain $x \in V := \{x \in \mathbb{R}^n \;:\; x \geq 0,\; Ax \geq b\}$.

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Linear programming:

(LP) $\max c^T x$
s. to $Ax \leq b$, $x \geq 0$

(LD) $\min b^T y$
s. to $A^T y \geq c$, $y \geq 0$

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Feasibility associated problem:

To obtain $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$: $Ax \leq b$, $x \geq 0$, $A^T y \geq c$, $y \geq 0$, $c^T x = b^T y$.

Some applications

Proton therapy planning: Chen, Craft, Madden, Zhang, Kooy and Herman, 2010;

Set theoretic estimation: Combettes, 1993;

Image reconstruction in computerized tomography: Herman, 2009;

Radiation therapy: Herman and Chen,2008;

Image reconstruction: Herman, Lent and Lutz, 1978.

Main methods

Elimination method: Fourier, 1824; Motzkin, 1936; Kuhn, 1956.

Relaxation methods for linear equations: Kaczmarz, 1937; Cimmino, 1938.

Extension to linear inequalities: Agmon, 1954; Motzkin and Schoenberg, 1954; Merzlyakov, 1963.

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Exponential complexity of relaxation methods: Todd, 1979; Goffin, 1982.

Main methods

Projection algorithms: Bauschke and Borwein, 1996: convergence and rate of convergence.

Intermittent: Bauschke and Borwein, 1996

Cyclic: Gubin, Polyak and Raik, 1967; Herman, Lent and Lutz, 1978

Block: Censor, Altschuler and Powlis, 1988

Weighted: Eremin, 1969.

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Least-squares algorithm: Censor and Elfving, 1982.

Subgradient algorithms: Bauschke and Borwein, 1996; Eremin, 1969, Polyak, 1987; Shor, 1985: closely related to projection methods.

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Center methods:

Based on geometric concepts:

Center of gravity of a convex body: Levin, 1965 and Newman, 1965.

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Center of the max-volume ellipsoid inscribing the body: Tarasov, Khachiyan and Erlikh, 1988.

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Based on analytic concepts

Generic center in the body that maximizes a distance function: Lieu and Huard, 1966, Huard, 1967.

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Minimum square approach: Ho and Kashyap, 1965, $Ax > 0$, through $\min ||Ax - b||_2$, $b > 0$, exponential convergence.

Strongly polynomial-time algorithm:

Linear programming in fixed dimension: Megiddo, 1984.

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Linear programming: Chubanov, 2014, $\min\{c^T x \mid Ax = b,\ x \geq \mathbf{0}\}$, integer data.

$$(P) \quad \text{Find } \; x \in V \subset R^n, \text{with } V := \{x \in R^n : x > 0, \; Ax > 0\},$$

$A \in R^{m \times n}$, with rows a_i^T, $i = 1, ..., m$.

Our aim: to find a point of V or to show that $V = \emptyset$.

Strict homogeneous feasibility and associated problem

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Our aim: to find a point of V or to show that $V = \emptyset$.

Theorem 3.2 of Gaddum, 1952: In order that $Ax \geq 0$ has a solution, it is necessary and sufficient that the system $AA^T y \geq 0$ has a non-negative solution.

Related problem

$$(QL) \quad \min \frac{\rho}{2} \sum_{i=1}^{n} x_i^2 - \mu \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{m} \ln y_i$$

subject to $a_i^T x = y_i, \; i = 1, ...m,$

$(x, y) > 0,$

$\rho > 0$ and $\mu > 0$ are parameters.

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Lemma

Only one of the following statements is true:
1. $V = \emptyset$, therefore (QL) is not feasible.
2. (QL) is feasible.

Related problem

KKT equations: $s \in \mathbb{R}^m$ is the Lagrange multiplier vector associated with the equalities.

$$\rho x_j - \frac{\mu}{x_j} - (A^T s)_j = 0, j = 1, ..., n \quad (1)$$

$$-\frac{1}{y_i} + s_i = 0, i = 1, ..., m \quad (2)$$

$$a_i^T x = y_i, i = 1, ..., m \quad (3)$$

(QL) is a convex-linear problem, KKT conditions are necessary and sufficient to determine its solution, if one exists.

Related problem

$$(1) \Rightarrow x_j(s) = \frac{1}{2\rho}[(A^T s)_j + \sqrt{(A^T s)_j^2 + 4\rho\mu}] > 0, \quad j = 1, ..., n.$$

(2) and $(3) \Rightarrow y_i = \frac{1}{s_i} = a_i^T x(s) = \frac{1}{2\rho} \sum_{j=1}^{n} a_{ij}[(A^T s)_j + \sqrt{(A^T s)_j^2 + 4\rho\mu}\,] > 0, \; i = 1, ..., m.$

Lemma

Let ρ and μ be positive parameters. Suppose (QL) is feasible. Then there exists a dual variable $s \in \mathbb{R}^m_{++}$ satisfying

$$(NLS) \quad F_i(s) := \frac{1}{s_i} - \frac{1}{2\rho} \sum_{j=1}^{n} a_{ij}[(A^T s)_j + \sqrt{(A^T s)_j^2 + 4\rho\mu}] = 0, \; i = 1, ..., m,$$

which is a (dual) solution to the KKT equations.

(NLS) is a square system in the variable s.

Hypothesis 1: Given small $\varepsilon > 0$, let

$$(H1) \quad \mu = \mu_1 = \frac{\varepsilon}{4\rho}.$$

(QL) can be interpreted as a perturbation of its linear version.

(NLS) $\Rightarrow s^* > 0 \Rightarrow (x^* > 0,\ Ax^* > 0)$.

a) $2\rho x_i^* > 0, 2\rho x_i^* \neq O(\varepsilon)$, for some $i = 1, ..., n$, then we are done.
b) $2\rho x_i^* > 0, 2\rho x_i^* \neq O(\varepsilon)$, for some $i = 1, ..., n$, then we are done.

The coefficient 2ρ prevents a false null entry of the solution, if ρ is large.

Theorem

Assume that $2\rho x_i^* > 0, 2\rho x_i^* = O(\varepsilon)$, for all $i = 1, ..., n$, μ satisfying (H1) and $\rho > 0$ given. Then, in the feasibility problem (P), within an ϵ approximation, $V = \emptyset$. Otherwise, in case a), any solution of (NLS) presents a positive solution to the feasibility problem.

Sketch of the proof

$$x_j(s) = \frac{1}{2\rho}[(A^Ts)_j + \sqrt{(A^Ts)_j^2 + \epsilon}] \approx 0 \Rightarrow (A^Ts)_j \leq O(\varepsilon).$$

i. $(A^Ts)_j = O(\varepsilon),\ j = 1, ..., n$
   or
ii. $(A^Ts)_j < 0$, for at least some $j = 1, ..., n$, $(A^Ts)_j = O(\varepsilon)$, for the remaining entries.

We approximate by:

i'. $(A^Ts)_j = 0,\ j = 1, ..., n$ (apply Gordan's alternative Lemma (1873))
   or
ii'. $A^Ts = c \leq 0,\ c \neq 0$, for some vector c (apply Farkas' Lemma (1901))

An Iterative Banach Procedure

Remark: We drop the set index $l = 1, ..., m$.

A general Banach fixed point method:

$$s_i^{k+1} = s_i^k - H_i(s^k)F_i(s^k) =: \Psi_i(s^k)$$

1. **Inclusion property**: $B \subset \mathbb{R}^m$, $s^k \in B$, $\forall k \in \mathbb{N} \Leftrightarrow \Psi_i(s) \in B$.
2. **Rate of convergence**: $|\partial \Psi_i(s)/\partial s_l| \leq \tau < 1$, $\forall i, l = 1, ..., m$.

A class of methods - a bad example

Let $s \in [1,2]^m$. Denote

$$G_i(s) = \sum_{j=1}^{n} a_{ij}[(A^T s)_j + \sqrt{(A^T s)_j^2 + \epsilon} \ ]$$

Define:

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$$s_i^{k+1} = s_i^k - H_i(s_i^k)F_i(s^k) = \Psi_i(s^k),$$

where $H_i(s_i) > 0$, and $F_i(s)$ is given in (NLS):

$$F_i(s) = \frac{1}{2\rho}G_i(s) - \frac{1}{s_i}.$$

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$$F_i(s) = \frac{1}{2\rho}G_i(s) - \frac{1}{s_i}.$$

The iterative function $\Psi_i(s)$ writes:

$$\Psi_i(s) = s_i - H_i(s_i)[\frac{1}{2\rho}G_i(s) - \frac{1}{s_i}] = s_i - \frac{1}{2\rho}T_i(s) + \frac{H_i(s_i)}{s_i},$$

where $T_i(s) = H_i(s_i)G_i(s) = s_iQ_i(s_i)G_i(s)$, with $Q_i(s_i)$ a decreasing function.

A class of methods - a bad example

If $Q_i(s_i) = e^{-\alpha s_i}$, $\alpha > 0$, then

Inclusion property:

$$\Psi_i(s) = s_i - \frac{\rho}{2} T_i(s) + e^{-\alpha s_i} \leq 2$$

$$\Rightarrow (2 - s_i - e^{-\alpha s_i})\rho \geq -\frac{1}{2} T_i(s)$$

$$\Rightarrow (\delta - e^{-\alpha s_i})\rho \geq -\frac{1}{2} T_i(s)$$

$$s \in [1, 2-\delta]^m \Rightarrow \Psi_i(s) \in [1, 2], \text{ for small } \delta > 0, \; \alpha > -\frac{\ln \delta}{(2-\delta)}.$$

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Rate of convergence:

$$\tau > 1 + (e/2)\delta \ln \delta.$$

Iterative procedure

Define

$$\Psi_i(s) = s_i + s_i \ln(\alpha s_i + \beta) F_i(s),$$

where α and β are conveniently chosen,

$F_i(s) = \frac{1}{2\rho} G_i(s) - \frac{1}{s_i}, \qquad G_i(s) = \sum_{j=1}^{n} a_{ij}[(A^T s)_j + \sqrt{(A^T s)_j^2 + \epsilon}]$

Dual Variable Algorithm (DVA)

Input:
$\alpha,\ \beta,\ \rho,\ \epsilon$ positive parameters;
tol is the accuracy;
$1 \leq s_i^0 \leq 2,\ i = 1, ..., m$ is the initial vector iteration;
Compute:
$s_i^1 = \Psi_i(s^0),\ i = 1, ..., m$
begin
for $k \geq 0$
$s_i^{k+1} = \Psi_i(s^k),\ i = 1, ..., m$
$t_k := ||s^{k+1} - s^k||_\infty$
if $t_k \leq$ tol, then **stop**.

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The convergence of this algorithm ensures that $F_i(s^k) \to 0$, or, equivalently, we approach a solution to the system (NLS).

Convergence theory

Some estimations (they use the fact that $s_i \in [1, 2]$):

$$|G_i(s)| = \left| \sum_{j=1}^{n} a_{ij}[(A^T s)_j + \sqrt{(A^T s)_j^2 + \epsilon} \right| \leq M,$$

$$M = 4 \max_{i=1,\ldots,m} \sum_{j=1}^{n} \sum_{k=1}^{m} |a_{ij}||a_{kj}|.$$

$$\left| \frac{\partial G_i}{\partial s_l}(s) \right| \leq \Omega := \max_{i,l=1,\ldots,m} \{|a_i^T a_l| + \sum_{j=1}^{n} |a_{ij} a_{lj}|\}, \ \forall i, l$$

$$\max_{s \in [1,2]^m} \left| \frac{\partial T_i(s)}{\partial s_i} \right| \leq D =: M \left(|\ln(\alpha + \beta)| + \frac{2\alpha}{2\alpha + \beta} \right) + 2|\ln(\alpha + \beta)|\Omega$$

Remark: If a_i are normalized: $||a_i||_2 = 1$, then $M \leq 4mn$ and $\Omega \leq 1 + n$.

Assuring inclusion

Proposition 1

Let $\beta_2 < \beta < \beta_1$, $\alpha < 1/2$, $\rho \geq \max\{\rho_1, \rho_2\}$, with

$$\beta_1 = 1 - 2\alpha, \; \beta_2 = e^{-\delta} - (2 - \delta)\alpha$$

$$\rho_1 = M$$

$$\rho_2 = \frac{M|\ln(\alpha + \beta)|}{\delta + \ln[(2 - \delta)\alpha + \beta]}.$$

Then $\ln(\alpha s_i + \beta) < 0$, for all $s_i \in [1, 2]$, $\Psi_i(s) \in [1, 2]$, for all $s \in [1, 2 - \delta]^m$.

Convergence rate

$|s_i^{k+1} - s_i^k| \leq \tau \|s^k - s^{k-1}\|_\infty$, for some $0 < \tau < 1$.

From algorithm DVA, we have

$$|s_i^{k+1} - s_i^k| = |\Psi_i(s^k) - \Psi_i(s^{k-1}| \leq \max_{s \in [1,2]^m} \max_{i,l=1,\ldots,m} \left| \frac{\partial \Psi_i(s)}{\partial s_l} \right| \|s^k - s^{k-1}\|_\infty.$$

Convergence rate

Proposition 2

Suppose valid the hypothesis of Proposition 1. Let $\alpha < 1/2$, $\beta = (\beta_1 + \beta_2)/2$, $\tau > 1 - \alpha$ (for δ sufficiently small), ρ satisfies $\rho \geq \max\{\rho_3, \rho_4, \rho_5\}$, where

$$\rho_3 = \left(\tau - 1 + \frac{\alpha}{2\alpha + \beta}\right)^{-1} \frac{D}{2},$$

$$\rho_4 = (1 + \tau - \frac{\alpha}{\alpha + \beta})^{-1} \frac{D}{2}$$

and

$$\rho_5 = \frac{|\ln(\alpha + \beta)|\Omega}{\tau}, \text{ with } \beta = \frac{1}{2}[1 - (4 - \delta)\alpha + e^{-\delta}].$$

Then

$$\left|\frac{\partial \Psi_i(s)}{\partial s_l}\right| \leq \tau, \quad \forall s \in [1,2]^m, \forall i, l = 1, ..., m.$$

Corollary 1

Fix $\tau = 0.6$,

$$\frac{1 + e^{-\delta}}{4.4 - \delta} < \alpha < \frac{1}{2},$$

β as above, and ρ lower bounds computed similarly, with the fixed value for τ.
Then

$$\left|\frac{\partial \Psi_i(s)}{\partial s_l}\right| \leq 0.6, \quad \forall s \in [1, 2]^m, \forall i, l = 1, ..., m.$$

Convergence Theorem

We assume the previous results.

Banach fixed-point theorem

Let $s^0 \in [1,2]^m$ be given. Then the sequence $\{s^k\}$ produced by Algorithm DVA converges, and its limit s^* is unique. We also have the following estimation:

$$\|s^* - s^k\|_\infty \leq \frac{\tau^k}{1-\tau}\|s^1 - s^0\|_\infty \leq \frac{\tau^k}{1-\tau}.$$

Proof: See Banach (1922), Kantorovich and Akilov (1959).

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Theorem

Let the error be 10^{-p}, for $p \geq 1$, between solutions s^* and s^K. The Algorithm DVA then produces a solution with at most

$$\frac{1}{\log \tau}[\log(1-\tau) - p]$$

iterations and $O(m^2(n+p))$ arithmetic operations.

Sketch of the proof:

Number of iterations

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$$\Rightarrow O(m^2(n+p)) \text{ arithmetic operations.}$$

Strict non-homogeneous feasibility problem

The problem:

$$(\mathcal{F}) \text{ Find } x \in \mathbb{R}^n, Ax + b > 0, x > 0,$$

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

Homogeneous equivalent problem:

$$(\mathcal{F}_h) \text{ Find } x \in \mathbb{R}^n, Ax + bx_{n+1} > 0,\ (x, x_{n+1}) > 0.$$

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Our current research is directed to considering linear programming.

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