NETWORK SURVIVABILITY AND POLYHEDRAL ANALYSIS

A. Ridha Mahjoub
LAMSADE, Université Paris-Dauphine, Paris, France

Contents

1.1. Polyhedral Approach

A Combinatorial Optimization (C.O.) problem is a problem of the form

$$\text{P} = \max\{c(F) = \textstyle\sum_{e \in F} c(e),\ F \in \mathbf{F}\}$$

where **F** is the set of solutions of P,
$\mathbf{F} \subset 2^E$ for a ground set E and $c(F)$ is the weight of F.

With $F \in \mathbf{F}$, we associate a {0,1} vector $x^F \in R^E$, called the incidence vector of F given by

$$x^F_i = \begin{cases} 1 & \text{if } i \in F \\ 0 & \text{if } i \in E \setminus F \end{cases}$$

A C.O. problem can be formulated as a 0-1 program.

Idea : Reducing the problem to a linear program.

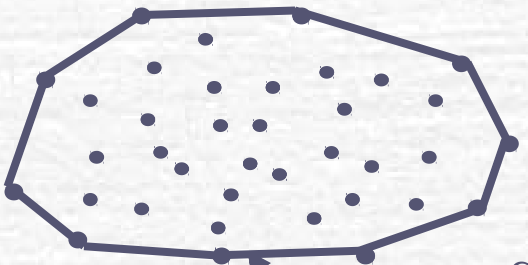
Max $\sum c_j x_j$

Subject to:

$\sum a_{ij} x_j \leq b_i \,,\; i = 1,\ldots,m$

$x_i \in \{0,1\} \,,\; i = 1,\ldots,n$

0-1 Program



A C.O. problem can be formulated as a 0-1 program.

Idea : Reducing the problem to a linear program.

Max $\sum c_j x_j$

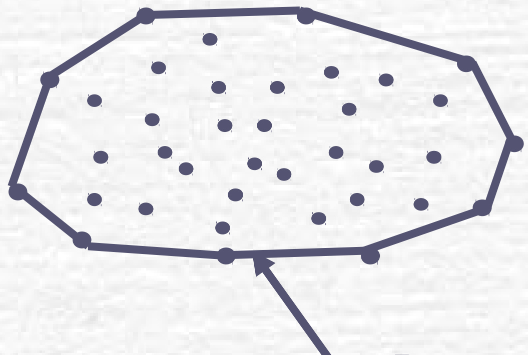
Subject to:

$\sum a_{ij} x_j \leq b_i \ , i=1,\ldots,m$

New Constraints

$x_i \geq 0, \ i=1,\ldots,n$

Linear Program



$$\text{P} \iff \max\{cx, \ x \in P(\text{P})\}$$

Polyhedral Approach:

Let P be a C.O. on a ground set E, $|E|=n$.

1. Represent the solutions of P as 0-1 vectors.
2. Consider these vectors as points of R^n, and define the convex hull *P(P)* of these points.
3. Characterize *P(P)* by a linear inequality system.
4. Apply linear programming for solving the problem.

This approach has been initiated by Edmonds in 1965 for the Matching Problem.

Step **3.** is the most difficult.

- If the problem is polynomial, generally it is possible to characterize the associated polytope!

- If the problem is NP-complete, there is a very little hope to get such a description.

Question: How to solve the problem when it is NP-complete.

A further difficulty:
The number of (necessary) constraints may be exponential.

The Traveling Salesman Problem

For 120 cities,
The number of (necessary) contraints is $\geq 10^{179}$
($\cong 10^{100}$ times the number of atoms in the globe)
(number of variables: 7140.)

To solve the TSP on 120 cities,
(Grötschel 1977), used only 96 constraints among the 10^{179} known constraints.

.2. Separation and Optimization

With a linear system

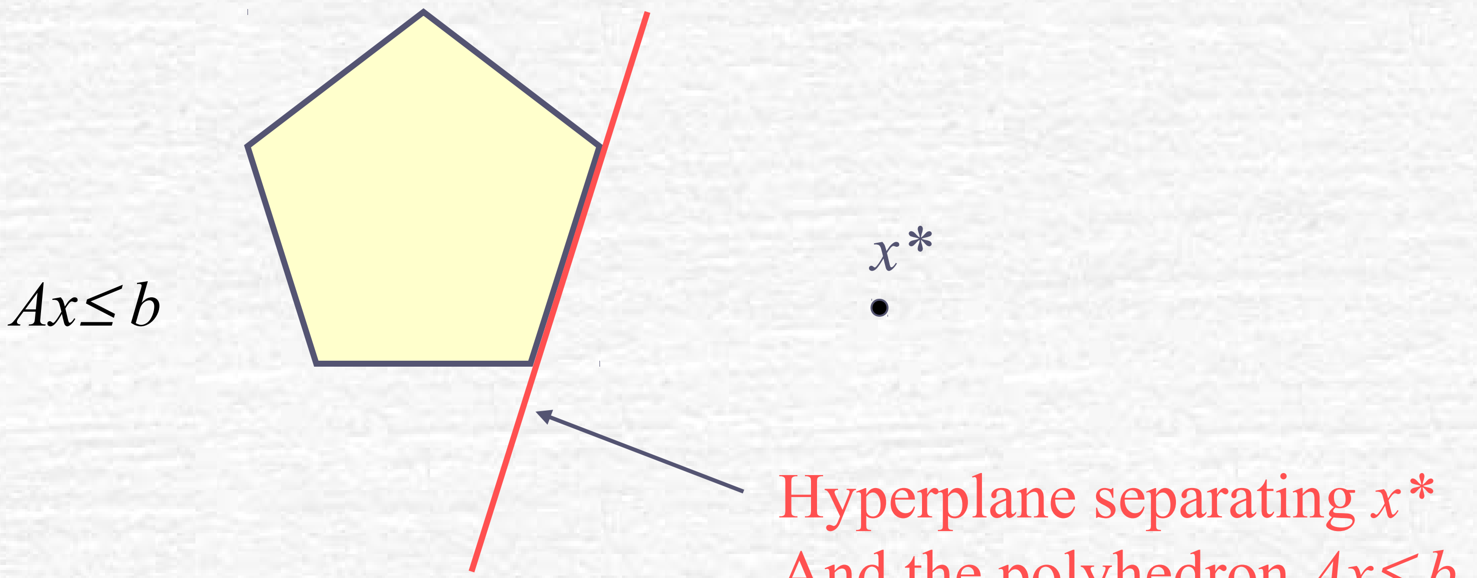
$$Ax \leq b$$

we associate the following problem:

Given a solution x^*, verify whether x^* satisfies $Ax \leq b$, and if not, determine a constraint of $Ax \leq b$ which is violated by x^*.

This problem is called the separation problem associated with $Ax \leq b$.

If x^* does not verify system $Ax \leq b$, then there is a hyperplane that separates x^* and the polyhedron $Ax \leq b$.



Theorem: (Grötschel, Lovász, Schrijver, 1981)

Given a linear program

$$P = \max\{cx,\ Ax \leq b\},$$

there is a polynomial time algorithm for P if and only if there is a polynomial time algorithm for the separation problem associated with $Ax \leq b$.

1.3. Cutting-plane method

1. Consider a linear program with a raisonable number of constraints among the constraints of $Ax \leq b$. Let

$$P_1 = \max\{cx,\ A_1x \leq b\}.$$

be this program.

2. Solve P_1. Let x^*_1 be the optimal solution of P_1.

   If x^*_1 is solution of P (in 0-1), STOP, x^*_1 is optimal solution of P .

   If not, solve the separation roblem associated with $Ax \leq b$ and x^*_1

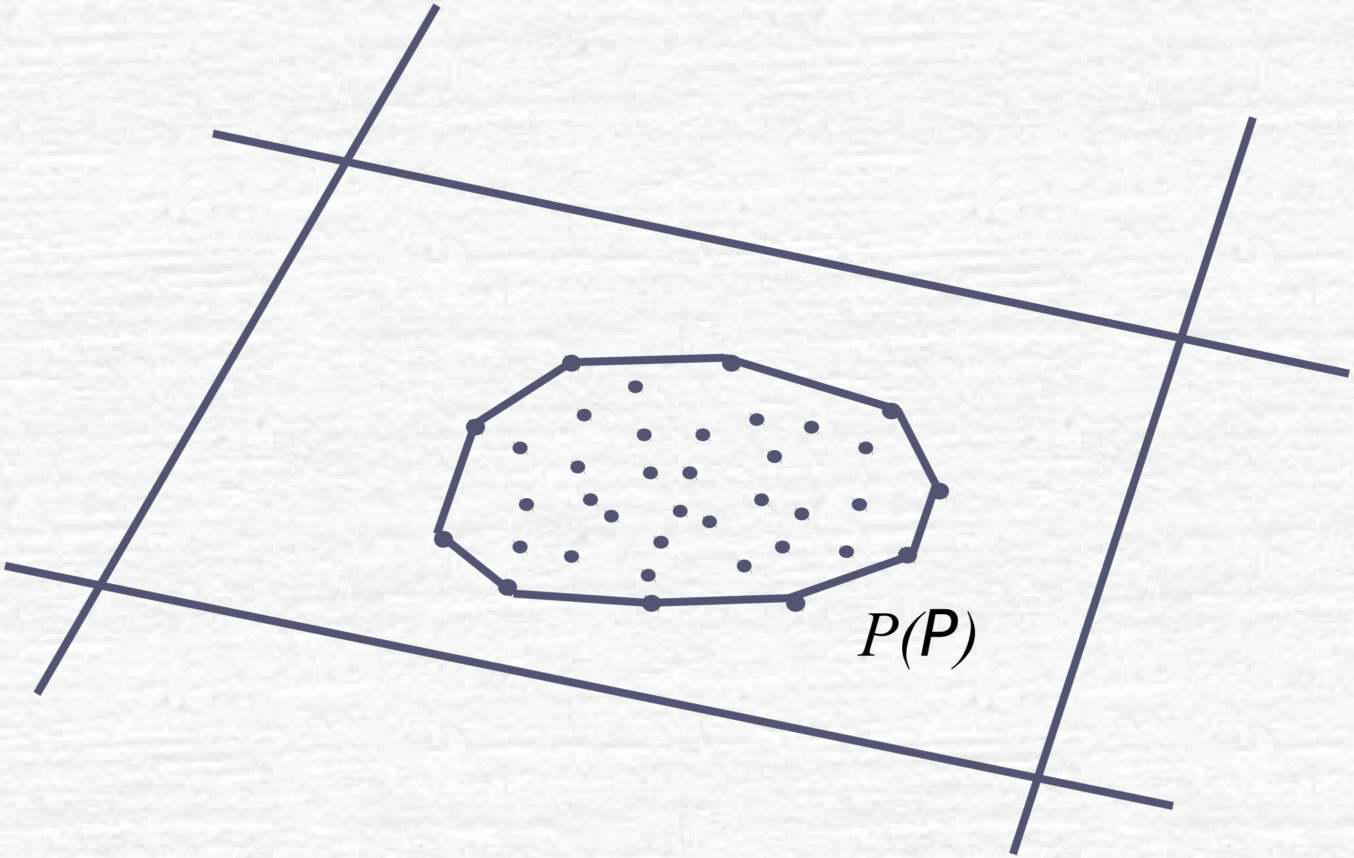
   Let $\alpha_1 x \leq \beta_1$ be a constraint violated by x^*_1.

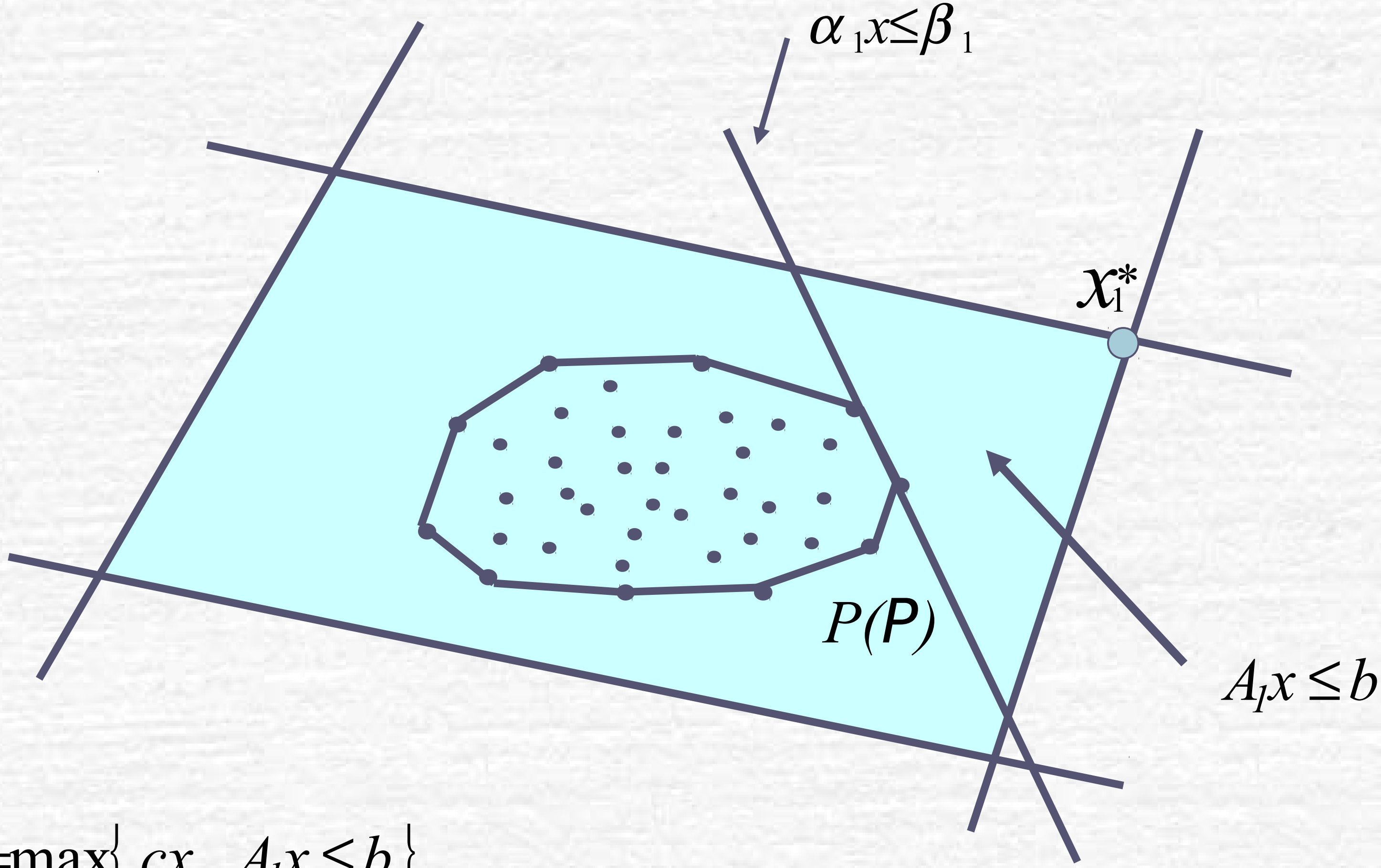
3. Add $\alpha_1 x \leq \beta_1$ to P_1. Let

$$P_2 = \max\{cx,\ A_1x \leq b,\ \alpha_1 x \leq \beta_1\}.$$

   Solve P_2. If x^*_2 is solution of P, STOP. If not, determine a contraint violated by x^* and so on

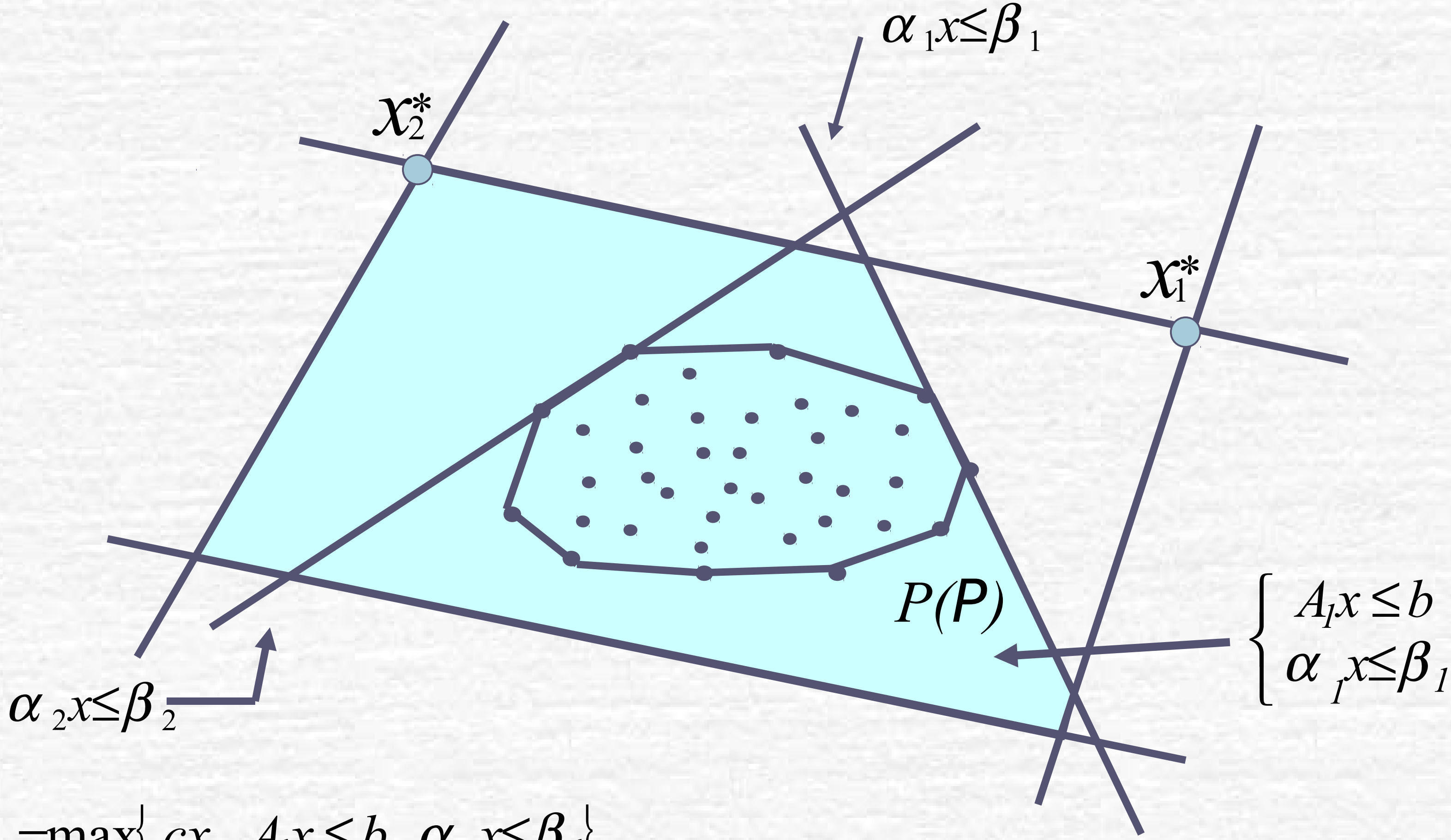
1.3. Cutting plane method



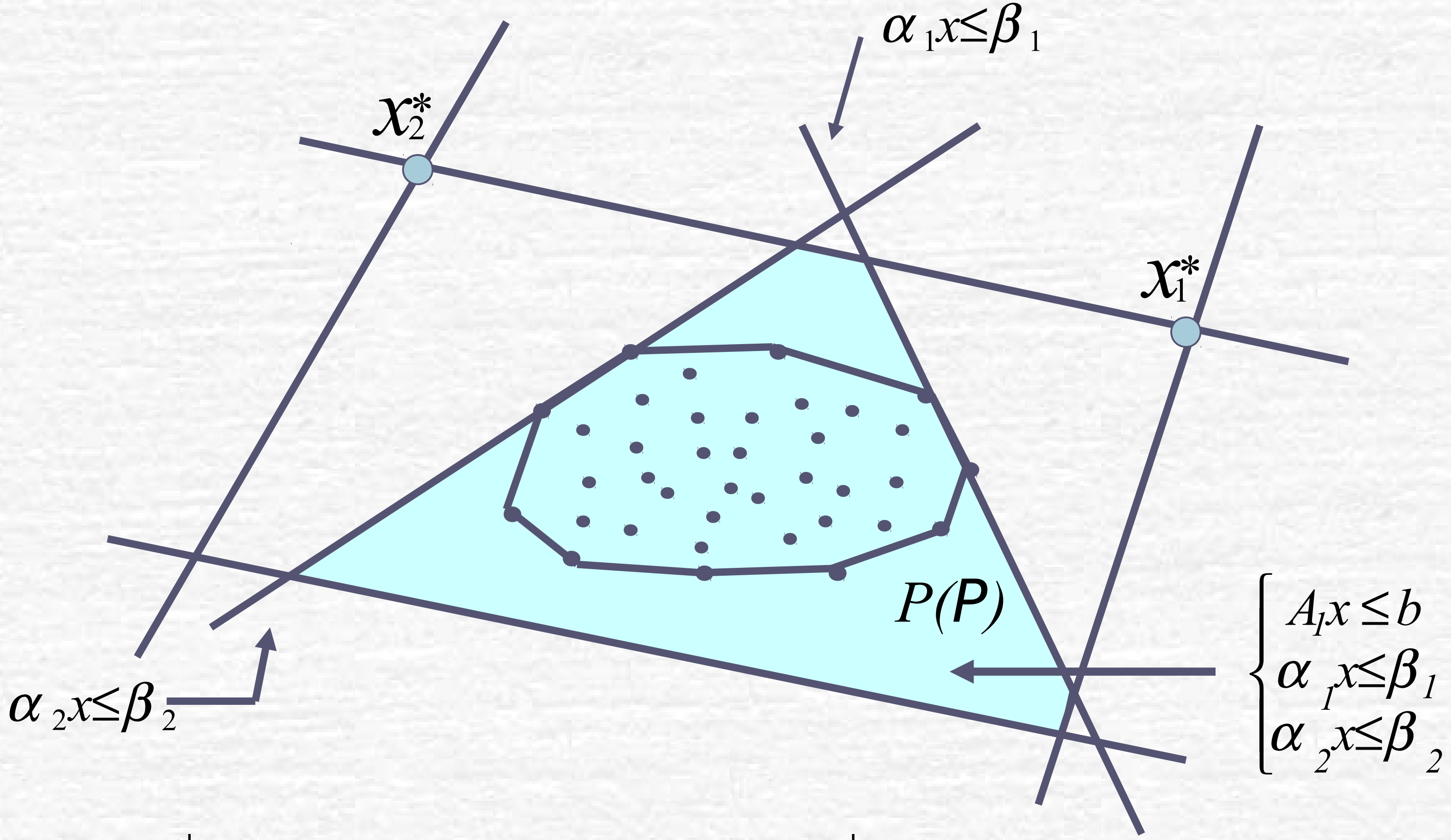


$P_1 = \max\{ cx, A_l x \leq b \}$

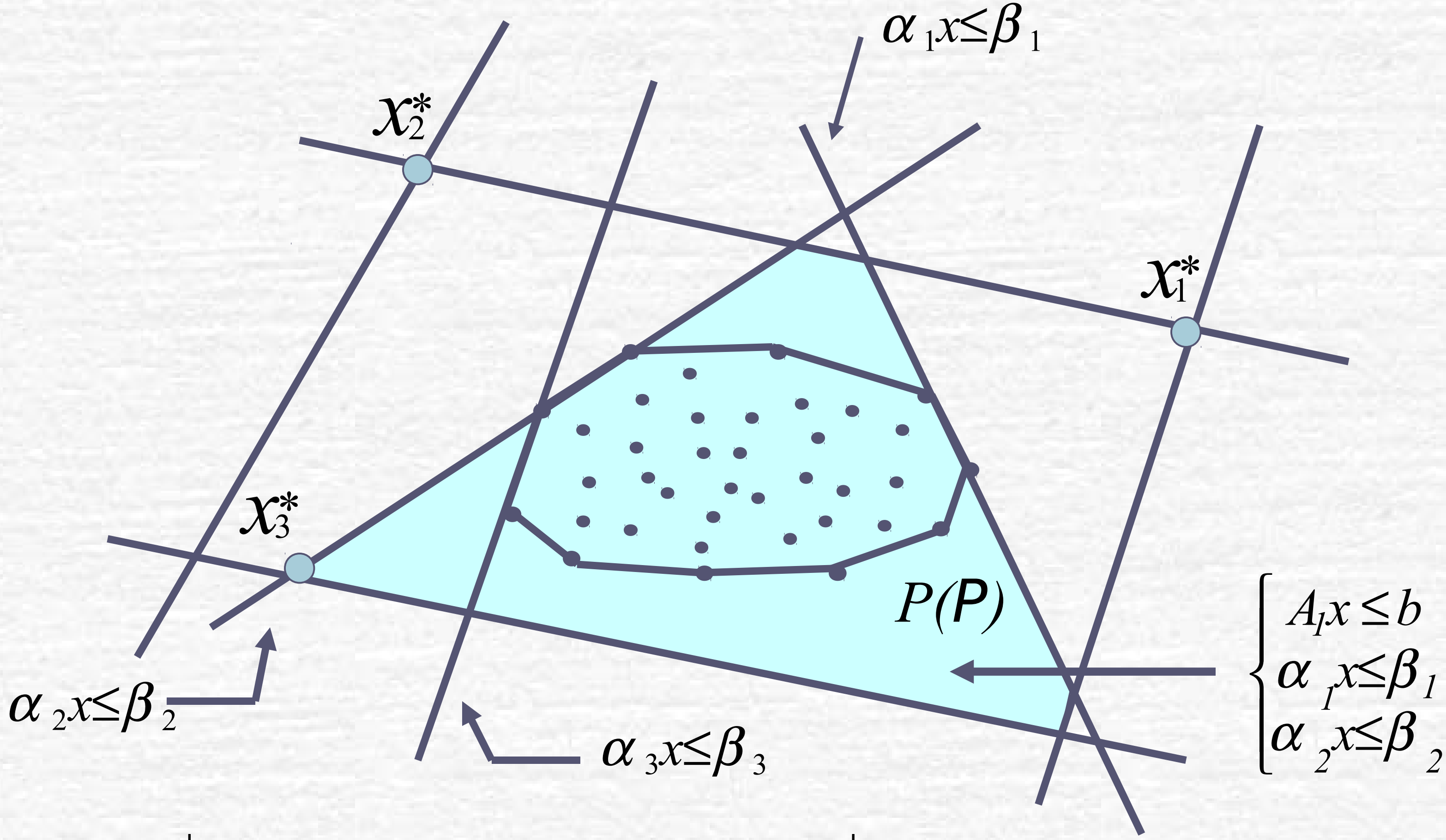
1.3. Cutting plane method



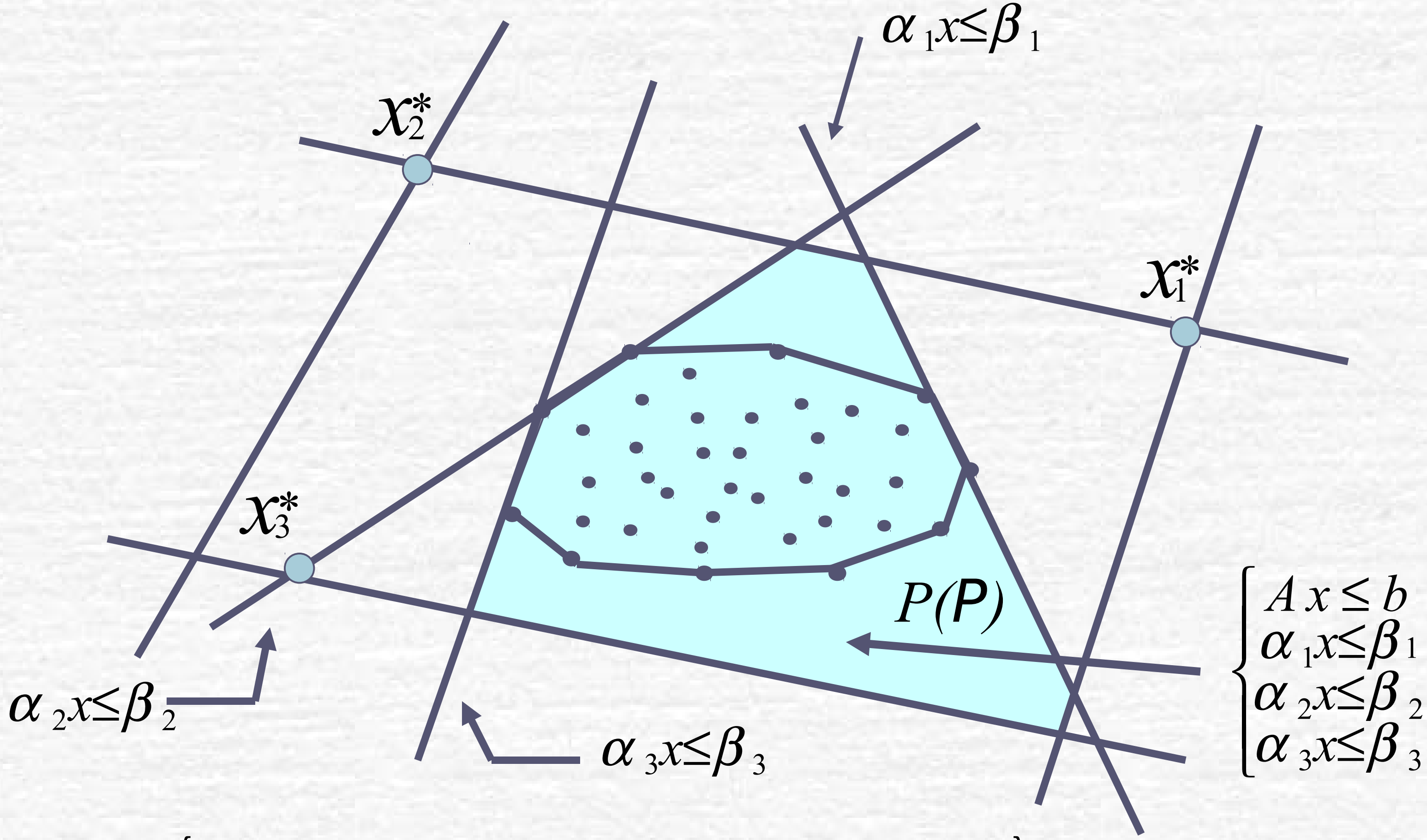
$P_2 = \max\{\, cx\,,\ A_I x \leq b\,,\ \alpha_1 x \leq \beta_1 \}$



$$P_3 = \max\{ cx,\ A_I x \leq b,\ \alpha_1 x \leq \beta_1,\ \alpha_2 x \leq \beta_2 \}$$



$$P_3 = \max\{ cx, A_I x \leq b, \alpha_1 x \leq \beta_1, \alpha_2 x \leq \beta_2 \}$$



$$P_4 = \max\{ cx,\ Ax \leq b,\ \alpha_1 x \leq \beta_1,\ \alpha_2 x \leq \beta_2,\ \alpha_3 x \leq \beta_3 \}$$

1.4. Branch&Cut Method

- Combination of Branch&Bound and Cutting-planes.

- On each node of the tree we solve a linear relaxation of the problem by the cutting-plane method.

1) If an optimal solution is not still found, select a (pending) node of the tree and a fractional variable x_i. Consider two sub-problems by fixing x_i to 1 and x_i to 0 (branching phase).

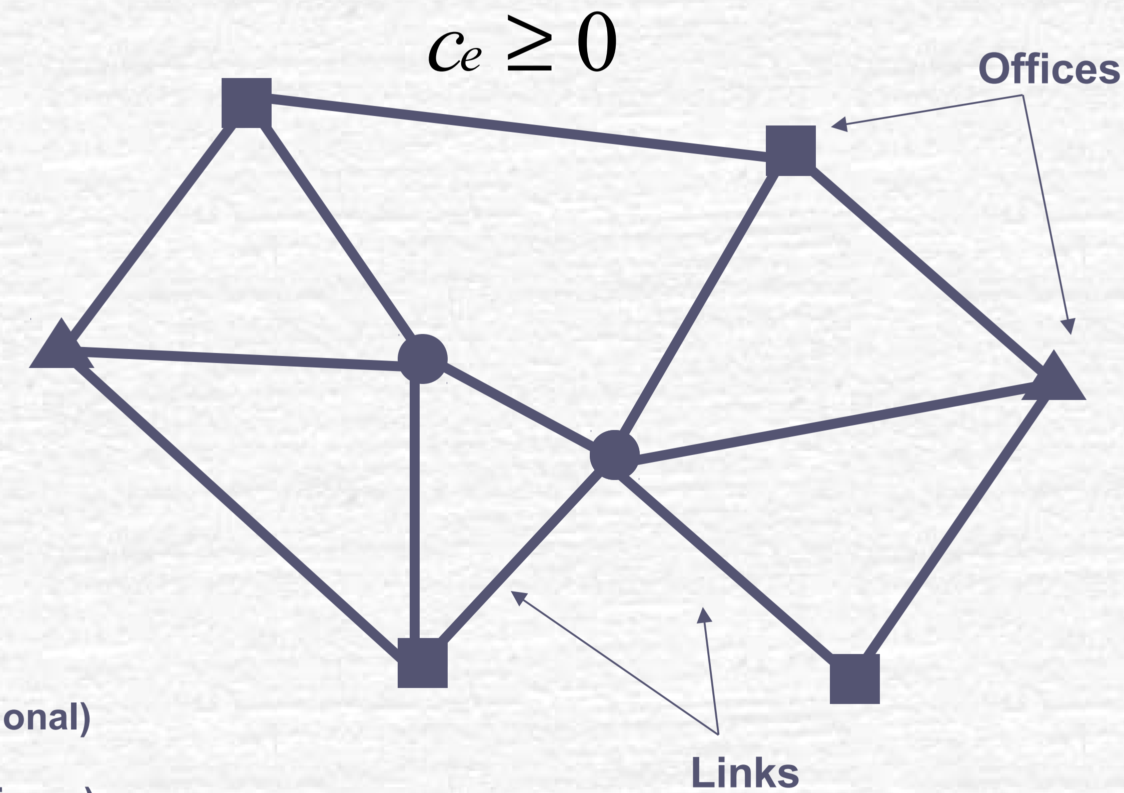
2) Solve each sub-problem by generating new violated constraints (cutting phase).
Go to **1)**.

Contents

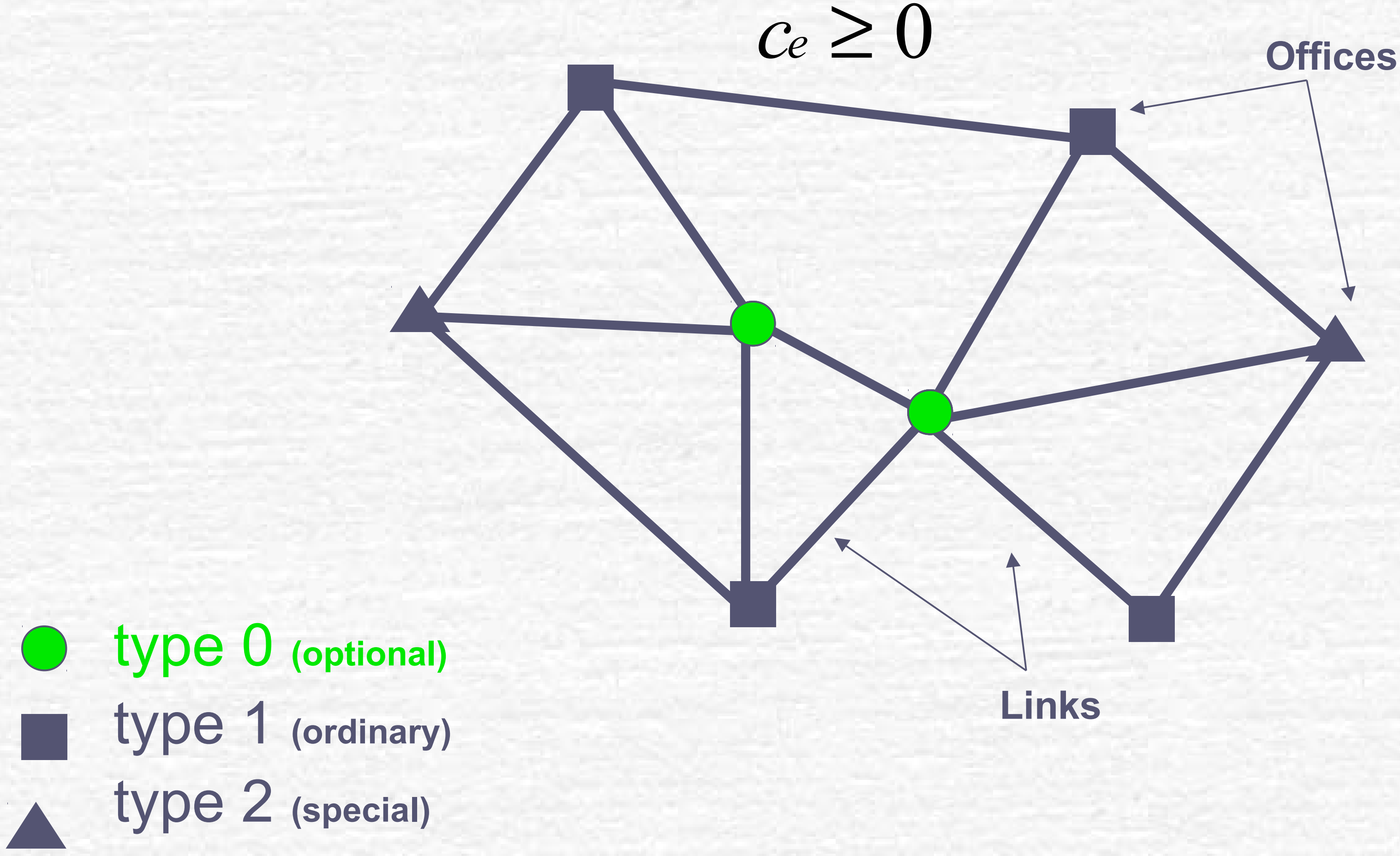
1. Polyhedral techniques

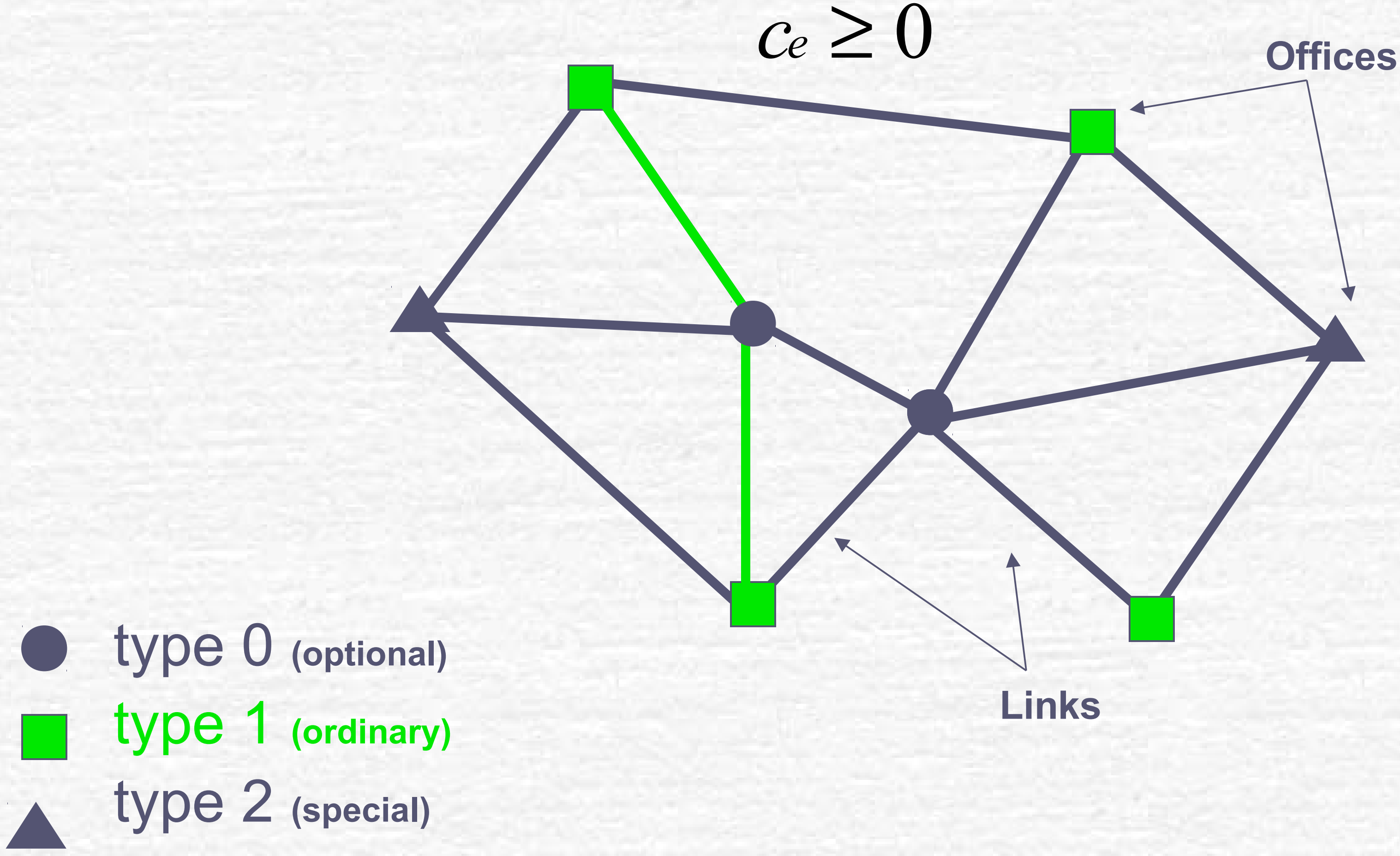
1.1. Introduction
1.2. Polyhedral Approach
1.3. Separation and Optimization
1.4. Cutting plane method
1.5. Branch&Cut

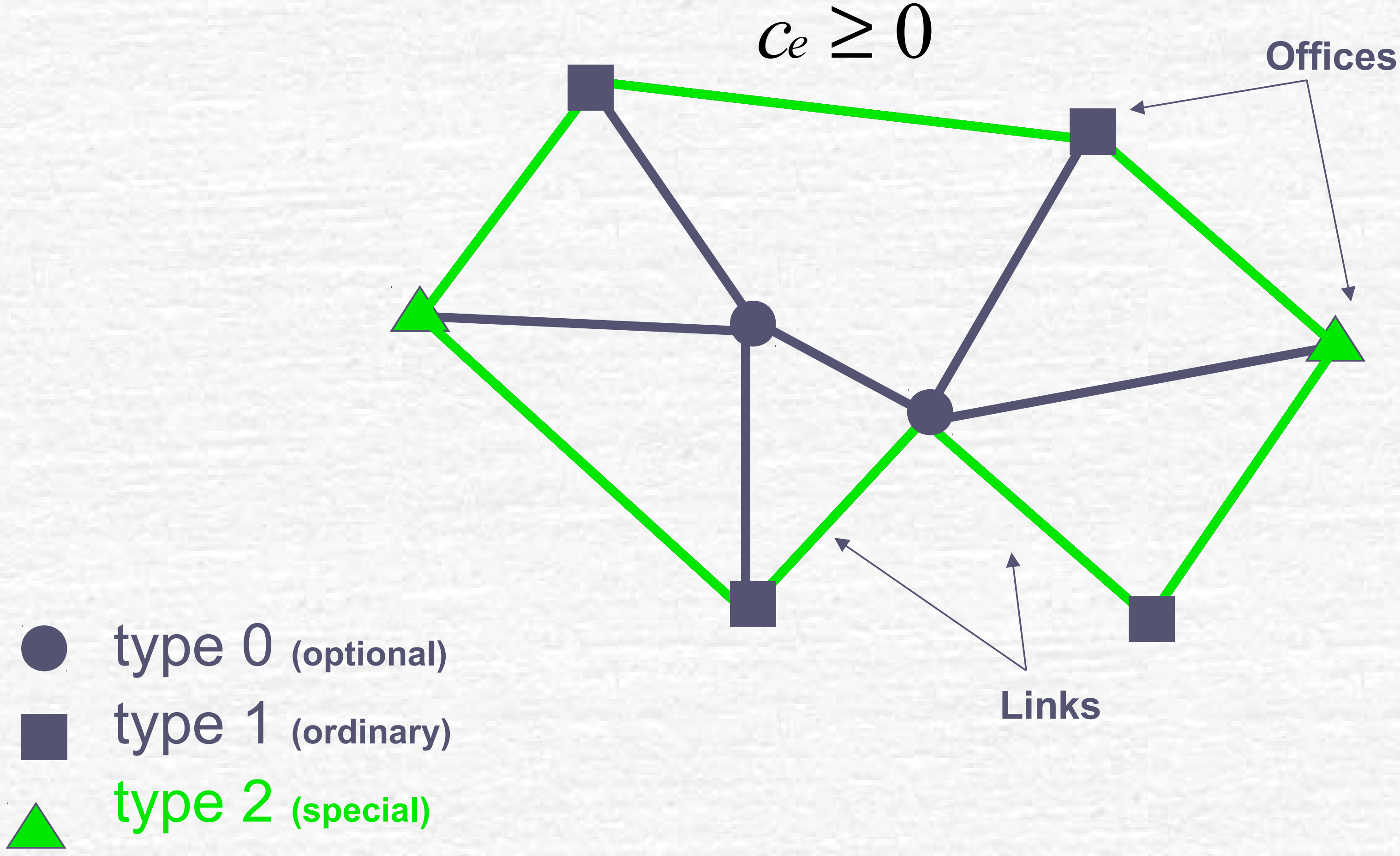
2. Networks survivability



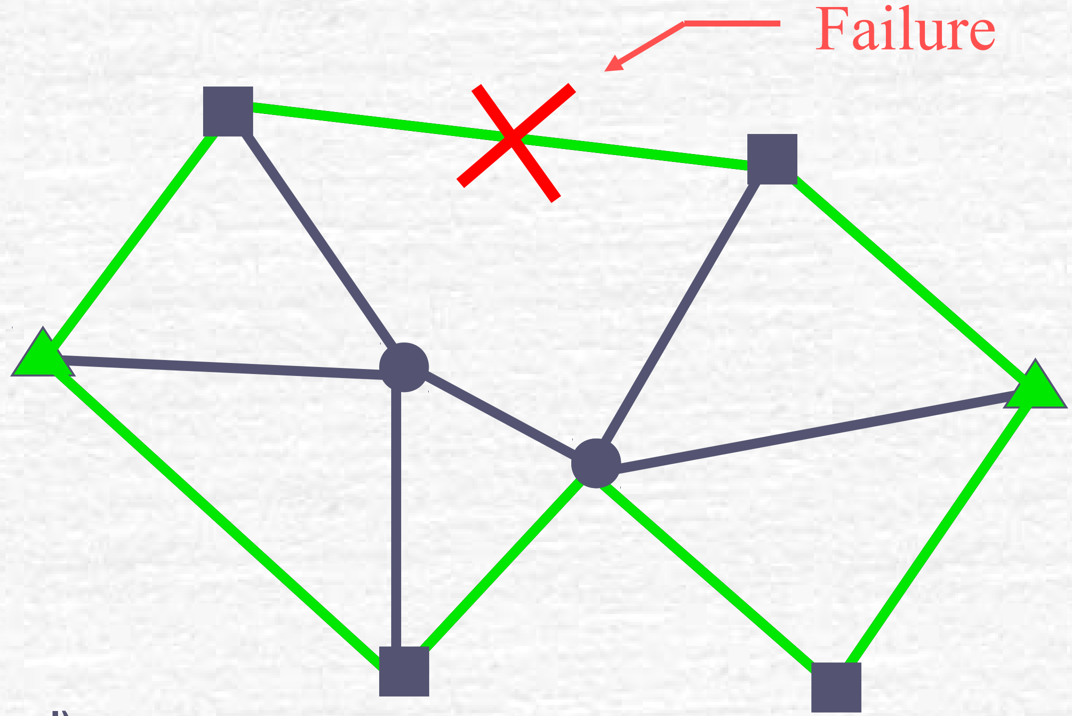
- type 0 (optional)
- type 1 (ordinary)
- type 2 (special)


$c_e \geq 0$
Offices
Links
type 0 (optional)
type 1 (ordinary)
type 2 (special)


$c_e \geq 0$
Offices
Links
type 0 (optional)
type 1 (ordinary)
type 2 (special)



$c_e \geq 0$
Offices
Links
type 0 (optional)
type 1 (ordinary)
type 2 (special)



- type 0 **(optional)**
- type 1 **(ordinary)**
- type 2 **(special)**

Survivability

The ability to restore network service in the event of a catastrophic failure.

Goal

Satisfy some connectivity requirements in the network.

Motivation

Design of optical communication networks.

Contents

2. Network survivability

2.1. A General model

Let $G=(V,E)$ be a graph. If s is a node of G, we associate with s a connectivity type $r(s) \in N$.

If s,t are two nodes, let

$$r(s,t)=min(r(s),r(t))$$

G is said to be survivable if for every pair of nodes s,t, there are at least $r(s,t)$ edge (node)-disjoint paths between s and t.

(Grötschel, Monma, Stoer (1992))

The Survivable Network Design Problem (SNDP)

Given weights on the edges of G, find a minimum weight survivable subgraph of G.

The SNDP is also known as the generalized Steiner tree problem and the multiterminal synthesis problem.

Special cases:

- $r(v)=1$ for every v: the minimum spanning tree problem.

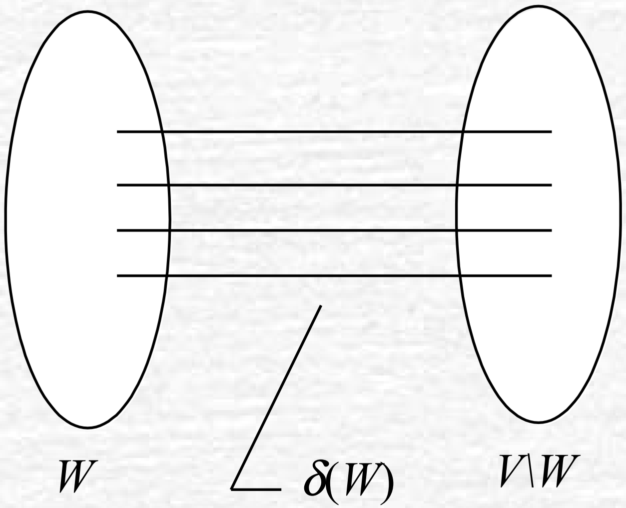
- $r(v)=1$ for two nodes s,t and 0 elsewhere: the shortest path problem between s and t.

- $r(v)\in\{0,1\}$ for every v: the Steiner tree problem.

- $r(v)=k$ for every v (k fixed): the k-edge (k-node) connected subgraph problem .

The SNDP is NP-hard in general.

Formulation of the SNDP (edge case)



If $W \subset V$, $\emptyset \neq W \neq V$, let
$r(W) = \max\{r(s) \mid s \in W\}$
$con(W) = \min\{r(W), r(V \setminus W)\}$

$r(W)$ is the connectivity type of W.

$\delta(W)$ is called a *cut* of G.

$$\sum_{e \in \delta(W)} x(e) = x(\delta(W)) \geq con(W)$$

cut inequalities

The (edge) SNDP is equivalent to the following integer program

$$\min \sum_{e \in E} c(e)x(e)$$

Subject to

$$x(\delta(W)) \geq con(W) \qquad \text{for all } W \subset V,\ \emptyset \neq W \neq V$$

$$0 \leq x(e) \leq 1 \qquad \text{for all } e \in E,$$

$$x(e) \in \{0,1\} \qquad \text{for all } e \in E.$$

Follows from Menger’s theorem (1927).

$$\min \sum_{e \in E} c(e)x(e)$$

Subject to

$x(\delta(W)) \geq con(W)$ for all $W \subset V$, $\emptyset \neq W \neq V$

$0 \leq x(e) \leq 1$ for all $e \in E$,

The linear relaxation can be solved in polynomial time (by the ellipsoid method).

2.2. Polyhedral Results

Let SNDP(G) be the convex hull of the solutions of SNDP, *i.e.*,

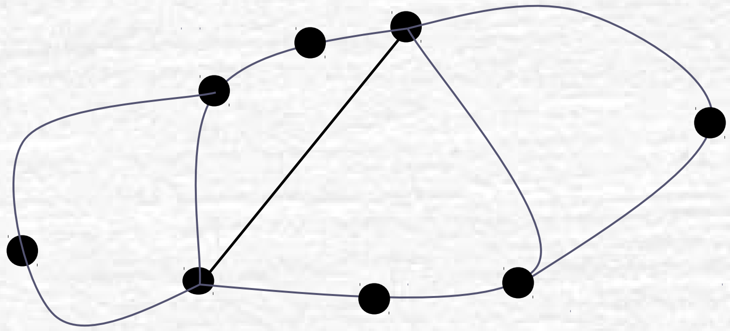
SNDP(G) = $conv\{x \in R^E | \ x$ is a (an integer) solution of SNDP$\}$.

SNDP(G) is called the survivable network design polyhedron.

2.2.1. Restricted graphs

A graph is said to be series-parallel if it can be constructed from an edge by iterative application of the following operations:

1) *Addition of parallel edges*
2) *Subdivision of edges*



Theorem: (Kerivin & M. (2002))
If G is series-parallel and r(v) is even for every v, then SNDP(G) is given by the trivial and the cut inequalities.

Generalizes Cornuéjols, Fonlupt and Naddef (1995), Baïou & M. (1996), Didi-Biha & M. (1999).

Corollary:
If G is series-parallel and r(v) is even for every v, then SNDP can be solved in polynomial time.

General graphs

Low connectivity case: $r(v) \in \{0,1,2\}$

2.2.2. Valid inequalities:

Trivial inequalities:

$$0 \leq x(e) \leq 0 \quad \text{for all } e \in E$$

Cut inequalities:

$$x(\delta(W)) \geq con(W) \qquad \text{for all } W \subset V,\ \emptyset \neq W \neq V$$

Partition inequalities:

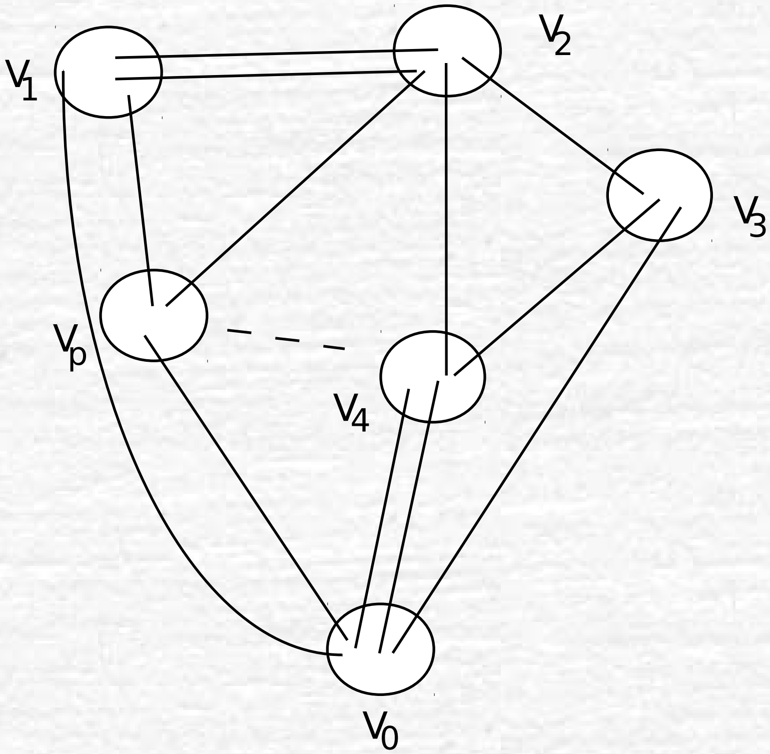
Let $V_1,\ldots,V_p$, $p \geq 2$, be a partition of V such that $con(V_i) \geq 1$ for all V_i. Then the following inequality is valid for SNDP(G).

$$x(\delta(V_1,\ldots,V_p)) \geq \begin{cases} p\text{-}1, & \text{if } con(V_i)=1 \text{ for all } V_i \\ p, & \text{if not,} \end{cases}$$

(Grötschel, Monma and Stoer (1992))

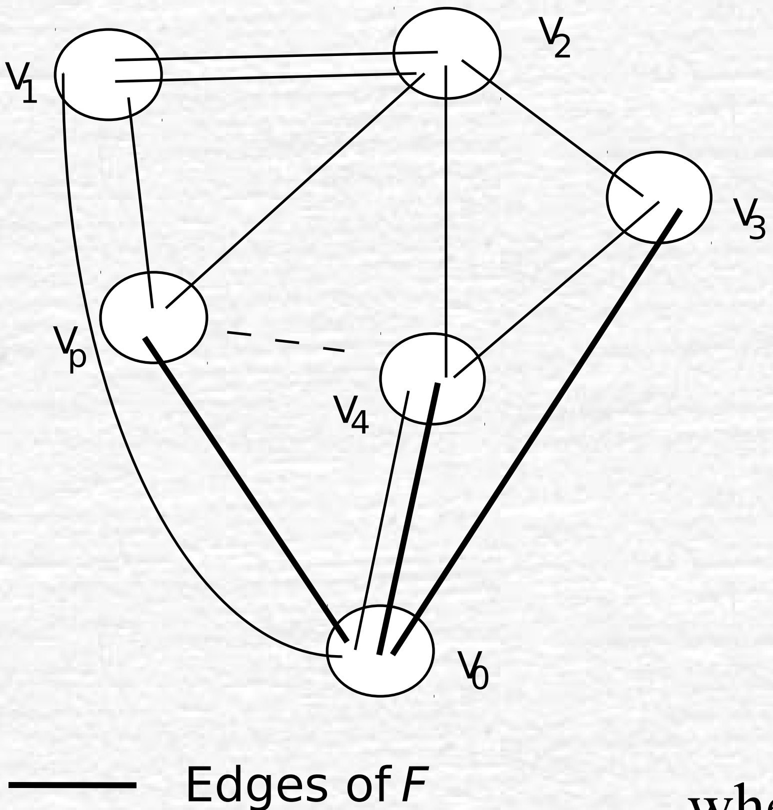
F-partition inequalities:

Let $V_0, V_1, \ldots, V_p$ be a partition of V such that $con(V_i)=2$ for all V_i



***F*-partition inequalities**:

Let $V_0, V_1, \ldots, V_p$ be a partition of V such that $con\,(V_i)=2$ for all V_i



Let F be a set of edges of $\delta(V_0)$ and $|F|$ id odd.

$$x(\delta(V_i)) \geq 2, \qquad i=1,\ldots,p$$

$$-x(e) \geq -1, \qquad e \in F$$

$$x(e) \geq 0, \qquad e \in \delta(V_0) \setminus F$$

$$\Longrightarrow \qquad 2x(\Delta) \geq 2p - |F|,$$

where $\Delta = \delta(V_0, V_1, \ldots, V_p) \setminus F$

Then

$$x(\Delta) \geq p - \frac{|F|-1}{2}$$

is valid for the SNDP(G).

These inequalities are called *F-partition inequalities.* (M. (1994))

Further valid inequalities related to the traveling salesman polytope have been given by Boyd & Hao (1994) for the 2-edge connected subgraph polytope. And general valid inequalities for the SNDP have been introduced by Grötschel, Monma and Stoer (1992) (generalizing the *F*-partition inequalities).

2.3. Separation

Consider the constraints

$$x(\delta(V_1,\ldots,V_p)) \geq p\text{-}1.$$

called *multicut inequalities*.

These arise as valid inequalities in many connectivity problems.

The separation problem for these inequalities reduce to $|E|$ min cut problems Cunningham (1985) .

It can also be reduced to $|V|$ min cut problems Barahona (1992).

Both algorithms provide the *most violated* inequality if there is any.

F-partition inequalities

($r(v)$ =2 for all node v)

Theorem. (Barahona, Baïou & M.) *If F is fixed, then the separation of F-partition inequalities can be solved in polynomial time.*

Let $G'=(V',E')$ be the graph obtained by deleting the edges of F. Hence the F-partition inequalities can be written as

$$x(\delta(V_0,\ldots,V_p)) \geq p-(|F|-1)/2$$

where $(V_0,\ldots,V_p)$ is a partition of V' such that for each edge $uv \in F$, $|\{u,v\} \cap (V_0)|=1$.

There are $2^{|F|}$ possibilities for assigning these nodes.

For each possibility we contract the nodes that must be in V_0 and solve the separation problem for the inequalities.

$$x(\delta(V_0,\ldots,V_p)) \geq p-(|F|-1)/2$$

where $|F|$ is fixed. These are partition inequalities, and hence the separation can be done in polynomial time.

2.4. Critical extreme points
of the 2-edge connected subgraph polytope

(Fonlupt & M. (2006))

We suppose *r(v)*=2 for all *v*.

Consider the linear relaxation of the problem:

$$\min \sum_{e\in E} c(e)x(e)$$

$$x(\delta(W)) \geq 2 \qquad \text{for all } W \subset V,\ \emptyset \neq W \neq V$$

$$0 \leq x(e) \leq 1 \qquad \text{for all } e \in E.$$

2.4. Critical extreme points
of the 2-edge connected subgraph polytope

(Fonlupt & M. (2006))

We suppose $r(v)=2$ for all v.

Consider the linear relaxation of the problem:

$$\min \sum_{e\in E} c(e)x(e)$$

$P(G)$

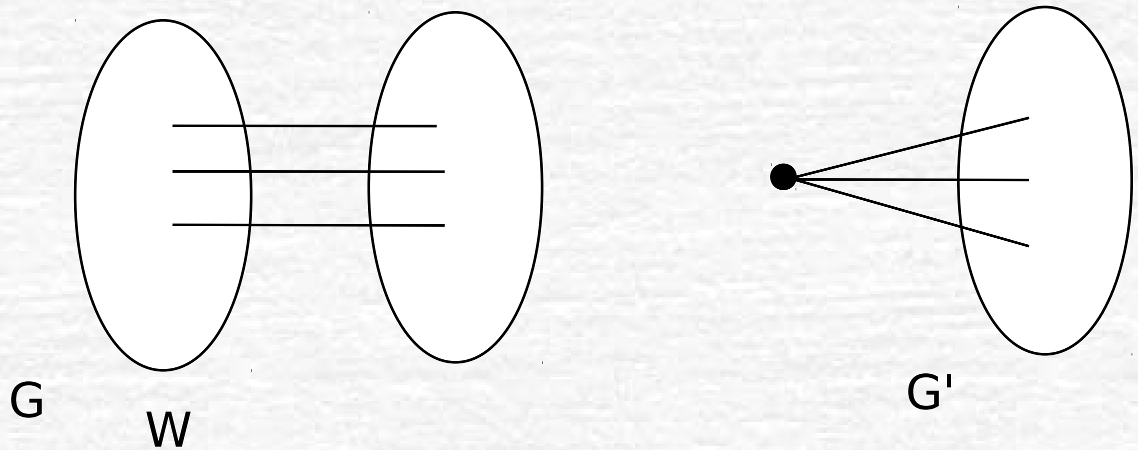
$x(\delta(W))\geq 2$	for all $W\subset V$, $\emptyset \neq W\neq V$
$0\leq x(e)\leq 1$	for all $e \in E$.

Reduction Operations

Let x be a fractional extreme point of $P(G)$.

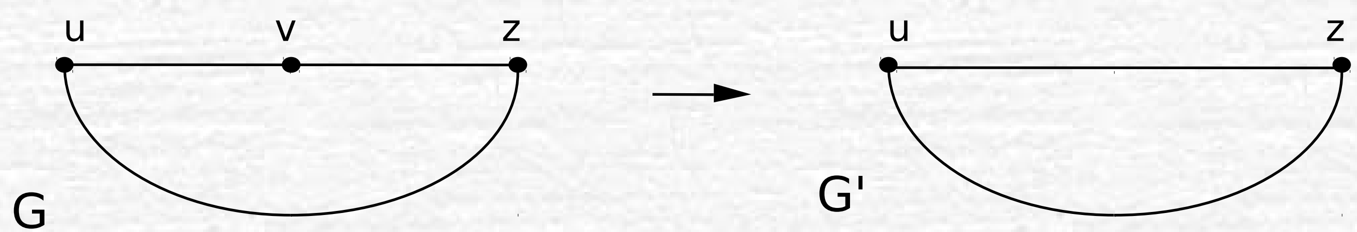
O_1: delete edge e such that $x(e)=0$,

O_2: contract a node set W such that the subgraph induced by W, $G(W)$ is 2-edge connected and $x(e)=1$ for every $e \in E(W)$.



G(W) is 2-edge connected and $x(e)=1$ for every $e \in E(W)$.

O_3: contract an edge having one of its endnodes of degree 2.



<u>Lemma:</u> *Let x be an extreme point of P(G) and x' and G' obtained from x and G by applications of operations O_1, O_2, O_3. Then x' is an extreme point of P(G'). Moreover if x violates a cut, a partition or an F-partition inequality, then x' so does.*

Domination

Let x and y be fractional two extreme points of $P(G)$. Let $F_x=\{e\in E \mid x(e)$ is fractional$\}$ and $F_y=\{e\in E \mid y(e)$ is fractional$\}$.

We say that x dominates y if $F_y\subset F_x$.

Question:

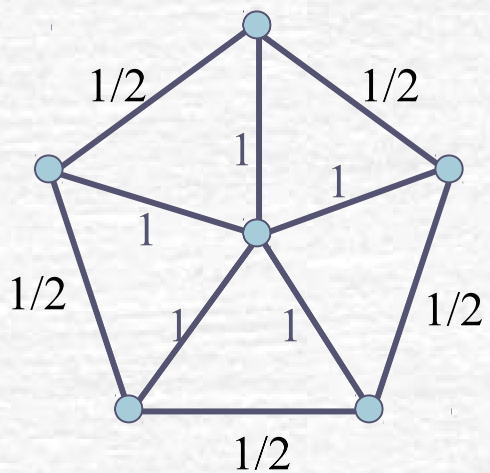
Characterise the minimal fractional extreme points.

Definition : A fractional extreme point *x* of *P(G)* is said to be *critical* if:

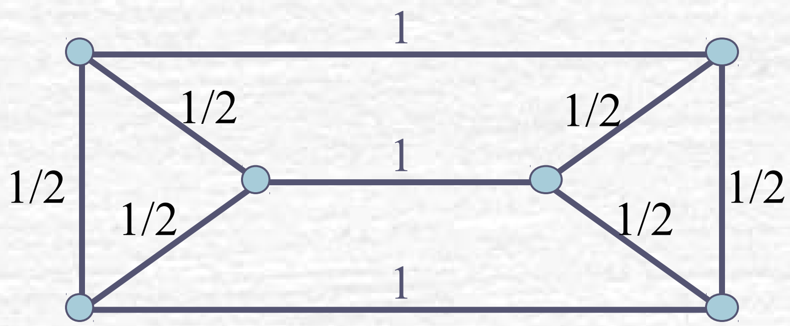
1) none of the operations O_1, O_2, O_3 can be applied for it,

2) it does not dominate any fractional extreme point of *P(G)*.

Example:



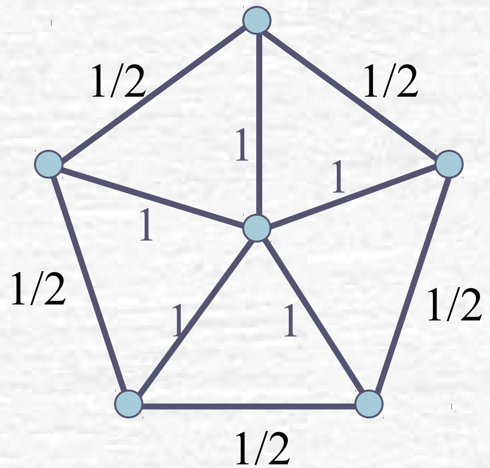
Critical



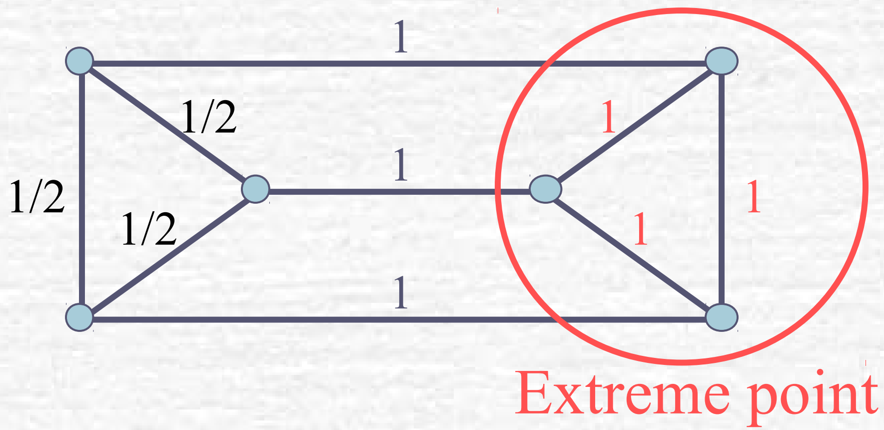
Non-critical

Definition : A fractional extreme point *x* of *P(G)* is said to be *critical* if:
1) none of the operations O_1, O_2, O_3 can be applied for it,
2) it does not domine any fractional extreme point of *P(G)*.

Example:



Critical



Non-critical

2. *Network survivability*

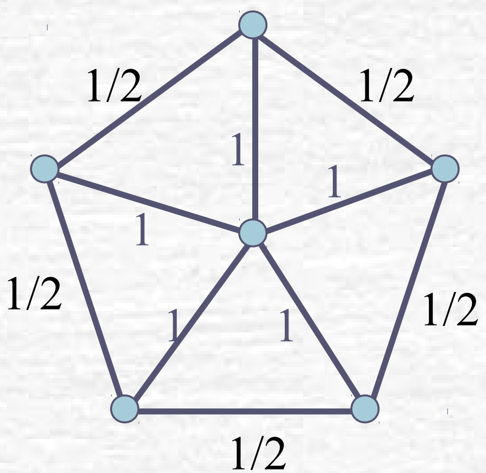
2.4. *Critical extreme points*

<u>Definition :</u> A fractional extreme point *x* of *P(G)* is said to be *critical* if

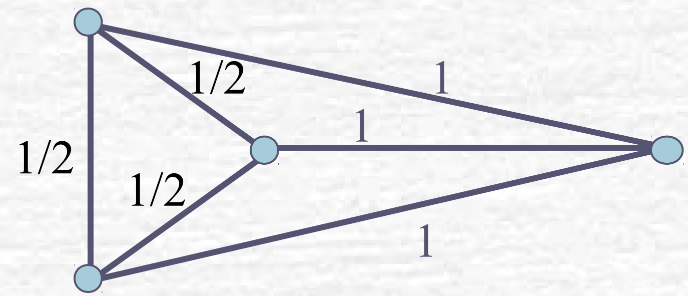
1) none of the operations O_1, O_2, O_3 can be applied for it,

2) it does not domine any fractional extreme point of *P(G)*.

Example:



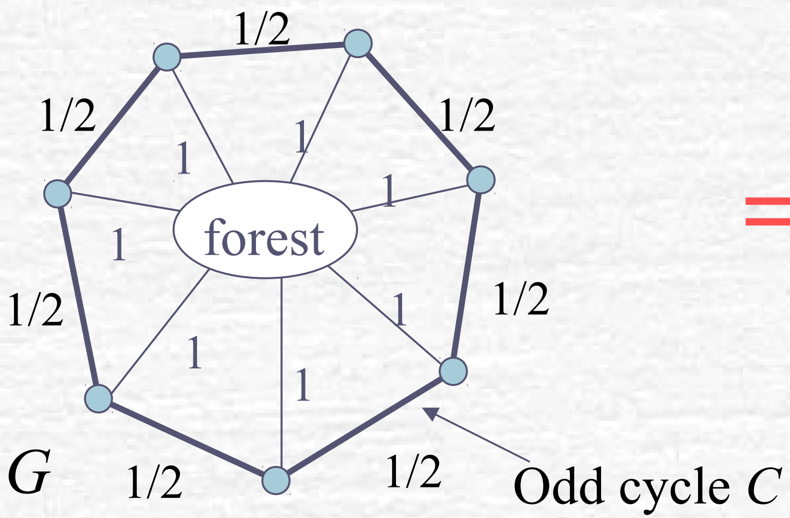
Critical



Critical

Extreme point

Theorem: *An extreme point of P(G) is critical if and only if G and x are of the following form:*



$$\Rightarrow \quad \sum_{e \in C} x(e) \geq \frac{|C|+1}{2}$$

is valid and defines a facet

(it is an *F*-partition inequality)

Theorem: *If x is a critical extreme point of P(G), then x can be separated (in polynomial time) by an F-partition inequality.*

The concept of critical extreme points has been extended (with respect to appropriate reduction operations) to 2-node connected graphs and (1,2)-survivable networks (Kerivin, M., Nocq (2001)), And to *k*-edge connected graphs (Didi Biha & M. (2004)).

2.5. Branch&Cut algorithm

(Kerivin, Nocq, M. (2003))

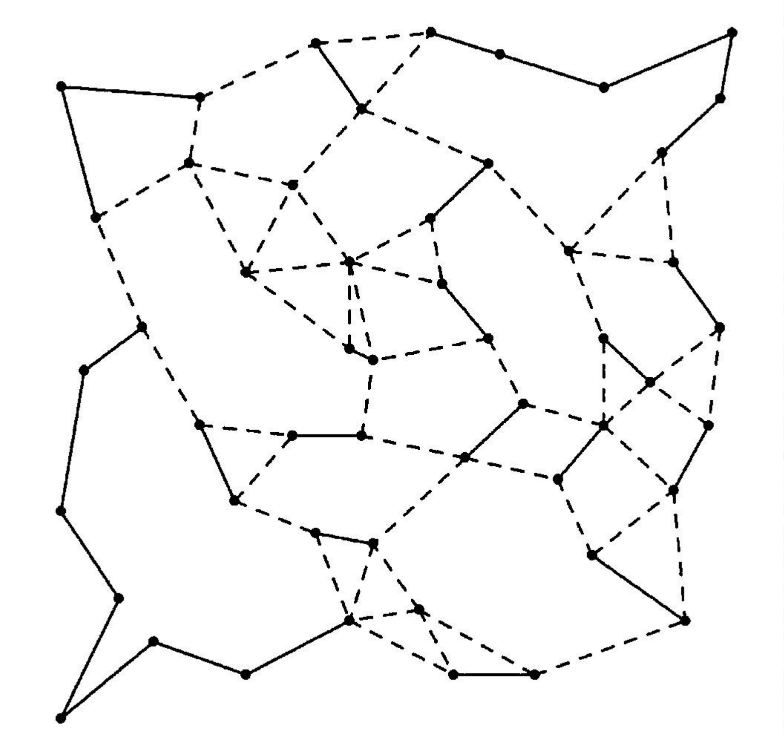
$r(v) \in \{1,2\}$ for all v

Used constraints:

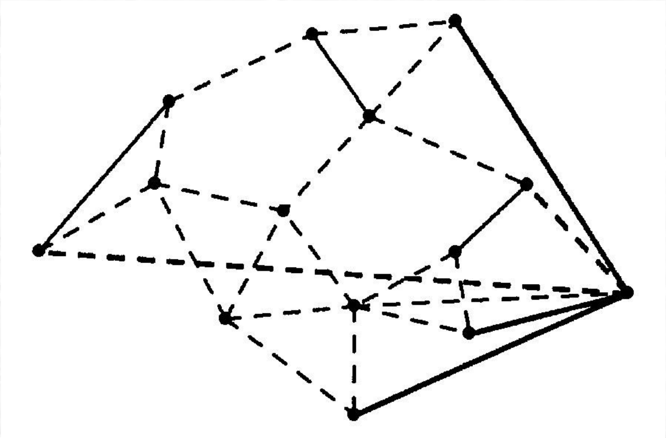
trivial inequalities
cut inequalities
F-partition inequalities
partition inequalities

If x is a fractional extreme point (critical or not), we apply the reduction operations. Let G' and x' be the graph and the solution thus obtained.

If a cut, a partition or an F-partition constraint is violated by x' for G', then it can be lifted to a constraint of the same type violated by x for G.



G 51 nodes



G' 14 nodes

F

$x(\delta(V_1,\ldots,V_p)\backslash F) \geq 11$

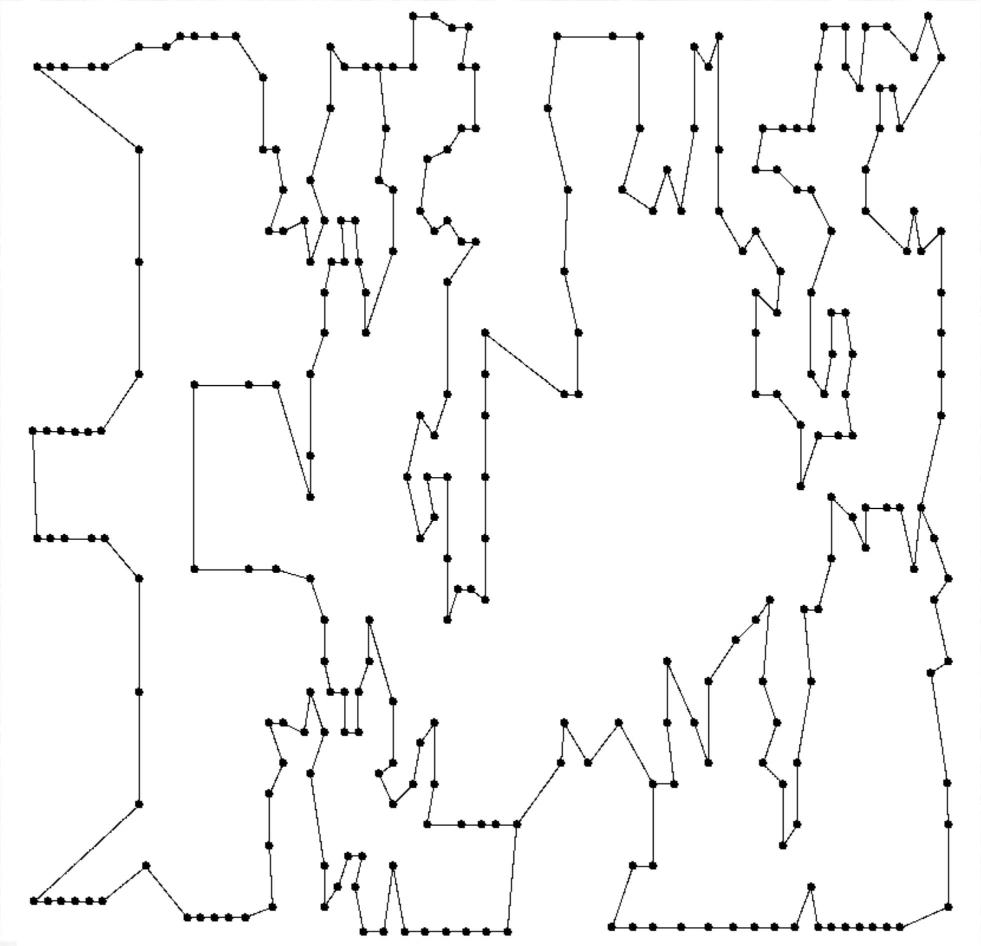
This contraint cuts the extreme point of *G'* and that of *G*.

#nodes 299
(type 2)

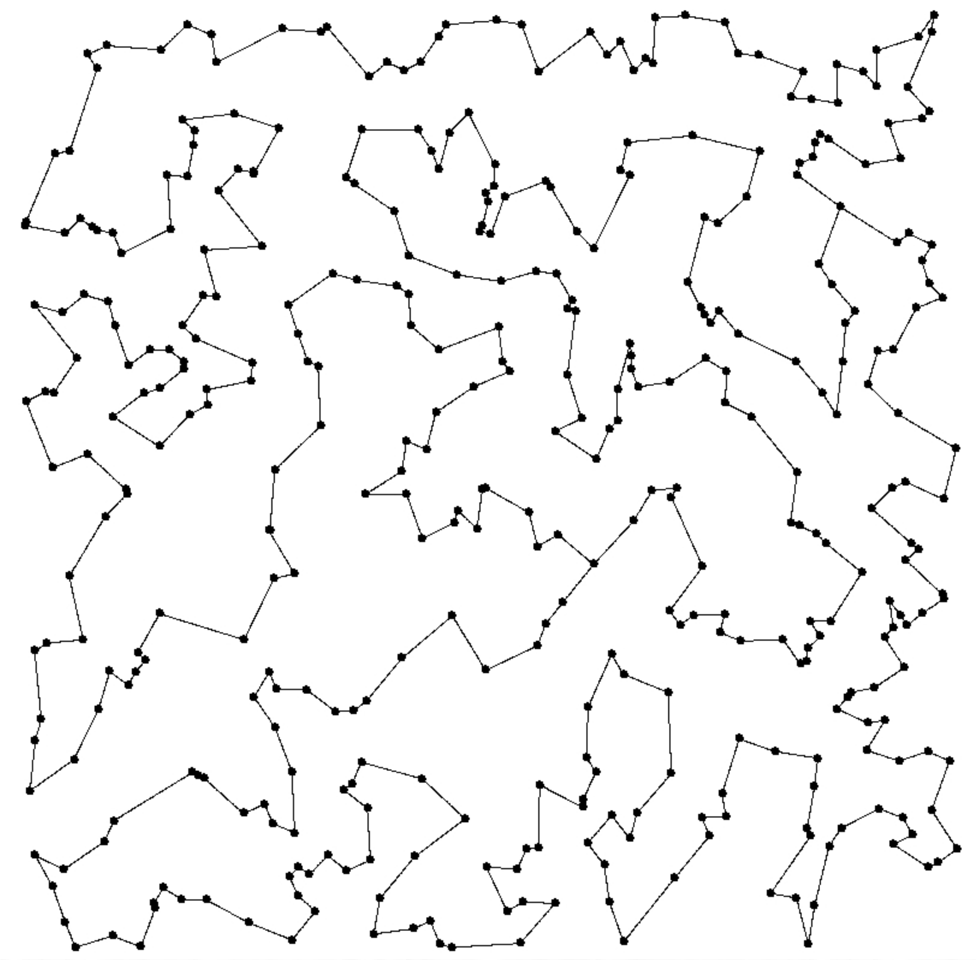
#variables 44551

#constraints 357

CPU Time 142 sec



#nodes (type 2) 2-node connected	400
#variables	79400
#constraints	1369
CPU Time	152 min



#nodes 48

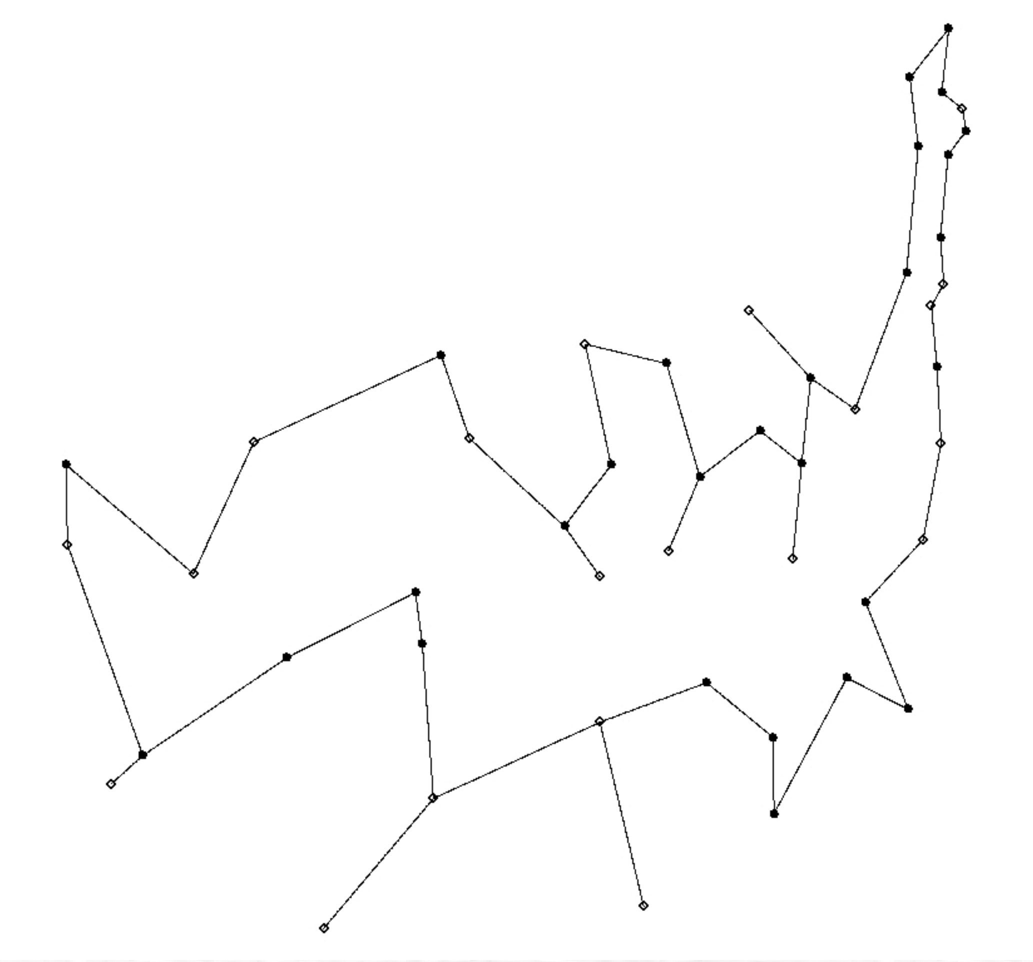
#type 1 20

#type 2 28

#variables 1 128

#constraints 428

CpuTime 202 sec



2.6. Survivable networks with length constraints

Motivation: to have effective routing cost

Local rerouting:

Each edge must belong to a bounded cycle (ring).
SONET/SDH networks

End-to-end rerouting:

the paths between the terminals should not exceed a certain length (a certain number of hops) (hop-constrained paths).
ATM networks, INTERNET

The minimum hop-constrained path problem

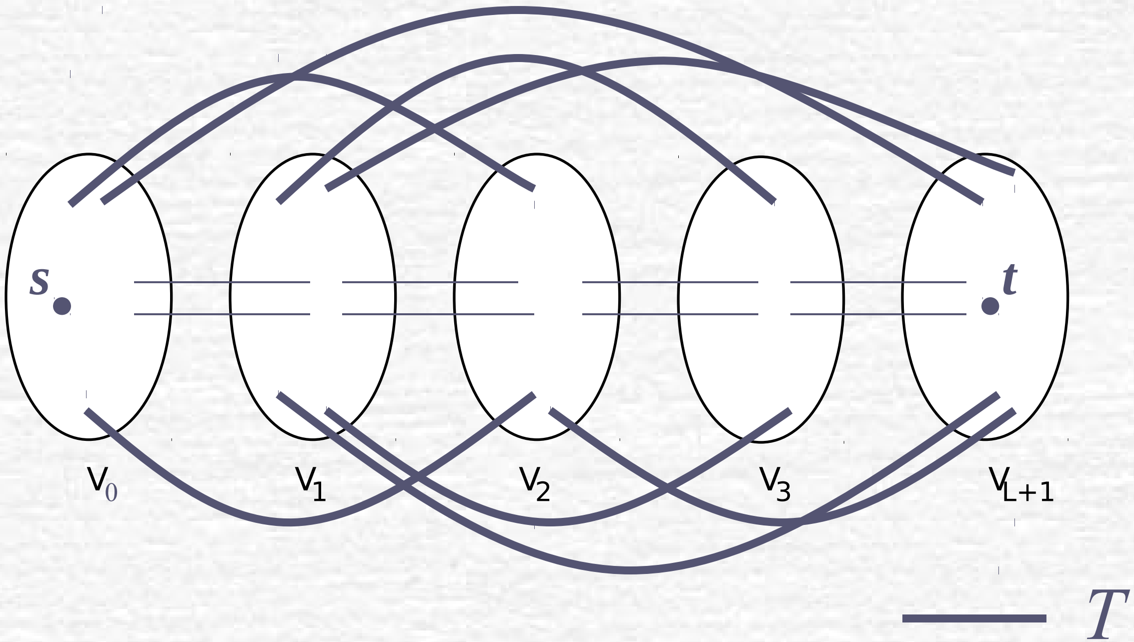
Determine a minimum path between two given nodes s and t, of length no more than L (L fixed).

Dahl & Gouveia (2001)

Formulation in the natural space of variables
Valid inequalities
Description of the associated polytope when L=2,3.

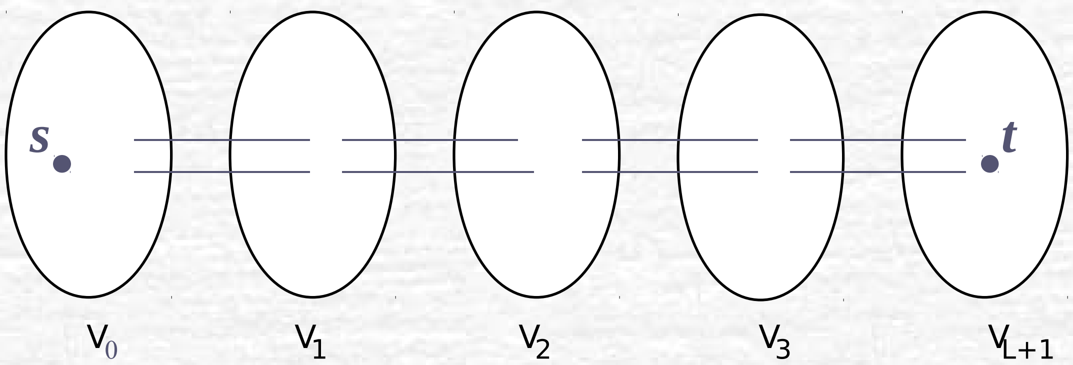
The L-path cut inequalities (Dahl (1999))

Let $V_0, V_1, \ldots, V_{L+1}$ be a partition of V such that $s \in V_0$ and $t \in V_{L+1}$.



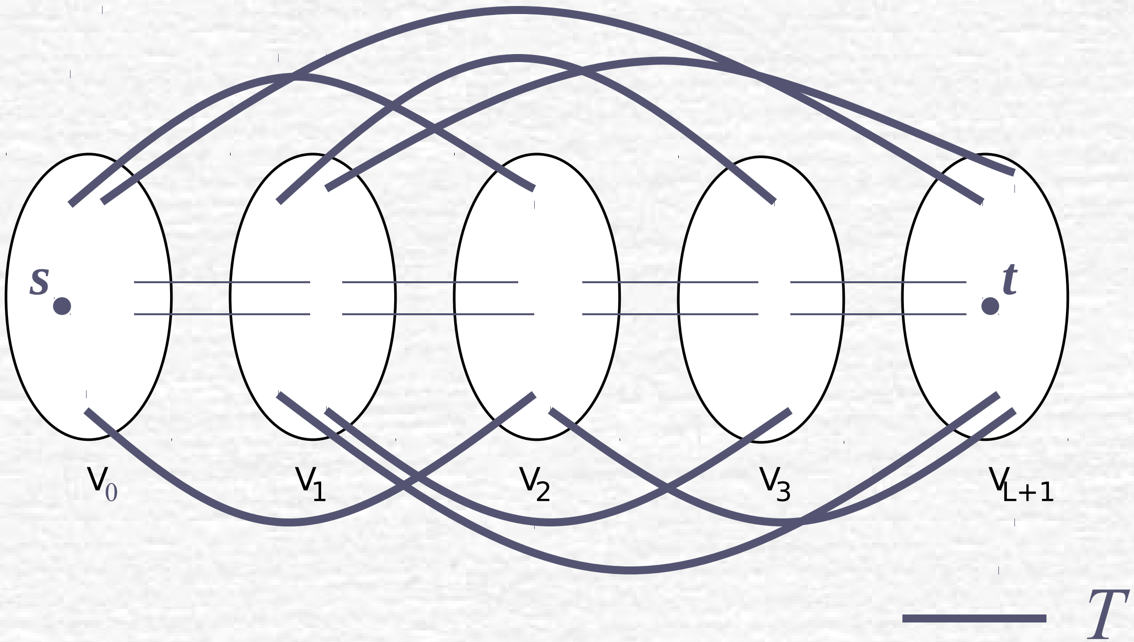
The L-path cut inequalities (Dahl (1999))

Let $V_0, V_1, \ldots, V_{L+1}$ be a partition of V such that $s \in V_0$ and $t \in V_{L+1}$.



The L-path cut inequalities (Dahl (1999))

Let $V_0, V_1, \ldots, V_{L+1}$ be a partition of V such that $s \in V_0$ and $t \in V_{L+1}$.



$x(T) \geq 1$

(L-path cut inequalities)

If at least *K* paths are required between *s* and *t, then*

$$x(T) \geq K$$

is valid for the corresponding polytope.

The separation problem for the *L*-path cut inequalities can be solved in polynomial time, if $L \leq 3$.

Fortz, M., McCormick, Pesneau (2003)

The hop-constrained network design problem (HCNDP):

Given a graph with weights on the edges, a set of terminal-pairs (origines-destinations), two intgers K, L, find a minimum weight subgraph such that between each pair of terminals there are at least K paths of length no more than L.

Formulation for $L \leq 3$

Theorem: (Huygens, M., Pesneau (2004))
For $L\leq 3$, the problem is equivalent to the integer program:

$$\min \sum_{e \in E} c(e)x(e)$$

$$
\begin{array}{ll}
x(\delta(W)) \geq K & \textit{for all st-cut } \delta(W) \\
x(T) \geq K & \textit{for all L-path cut } T \\
0 \leq x(e) \leq 1 & \textit{for all } e \in E, \\
x(e) \in \{0,1\} & \textit{for all } e \in E.
\end{array}
$$

The linear relaxation of the program, when $L\leq 3$, can be solved in polynomial time.

K=2, $L \leq 3$, and only one pair of terminals (s,t)

Théorem: (Huygens, M., Pesneau (2004))

The associated polytope is given by the inequalities

$x(T) \geq 2$	*for all L-path cut T,*
$x(\delta(W)) \geq 2$	*for all st-cut $\delta(W)$,*
$0 \leq x(e) \leq 1$	*for all $e \in E$.*

$\Rightarrow$ A polynomial time algorithm for the problem (when K=2, L=2,3) and only one pair of terminals.

Generalized for $L \leq 3$ and K arbitrary

Bendali, Diarrassouba, M. Mailfert (2010)

Conclusion

- Each combinatorial optimization problem needs a specific polyhedral investigation.

- A deep knowledge of the associated polyhedron is necessary for and efficient Branch&Cut algorithm.

- A Branch-and-Cut algorithm can be combined with a column generation technique (if the number of variables is big).

-The polyhedral approches are the most powerful techniques for solving hard combinatorial optimization to optimality.

Conclusion

- The Survivable network design problems are difficult to solve (even special cases).

- The problems with length constraints remain the most complicated SNDP. A better knowledge of their facial structure would be usefull to establish efficient cutting plane techniques.

- Develop usefull cutting plane and column generation techniques for the more general model with length constraints, capacity assignment and routing.