Boundary properties of the satisfiability problems

Vadim Lozin

DIMAP – Center for Discrete Mathematics and its Applications

Mathematics Institute

University of Warwick

Satisfiability

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clauses

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SAT: Determine if there is a truth assignment satisfying each clause

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- 3-SAT where each variable appears (positively or negatively) in at most two clauses is polynomial-time solvable (Tovey)

Complexity of the problem and its restrictions

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Graphs associated with formulas

Given an instance F of the problem, we associate to it a bipartite graph G_F with the vertex set $C \cup X$ and the set of edges connecting each variable $x \in X$ to those clauses in C that contain x (positively or negatively).

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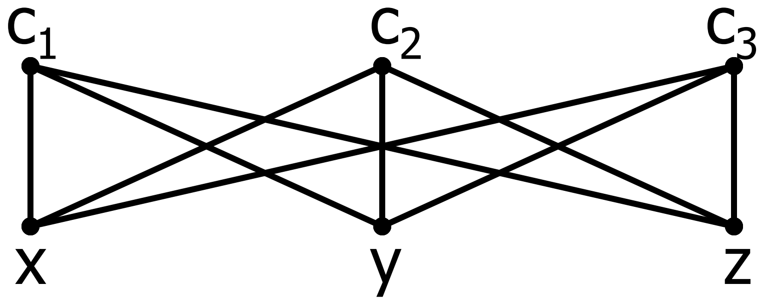

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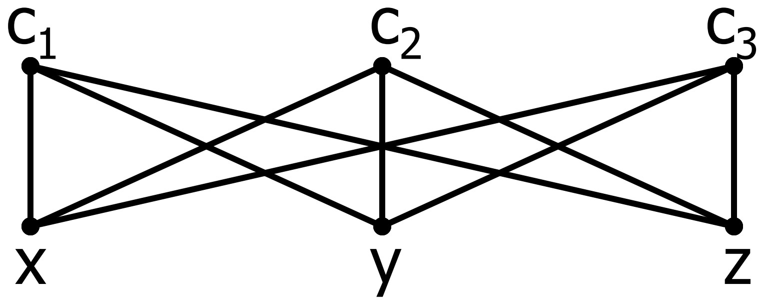

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A formula is planar if its formula graph is planar

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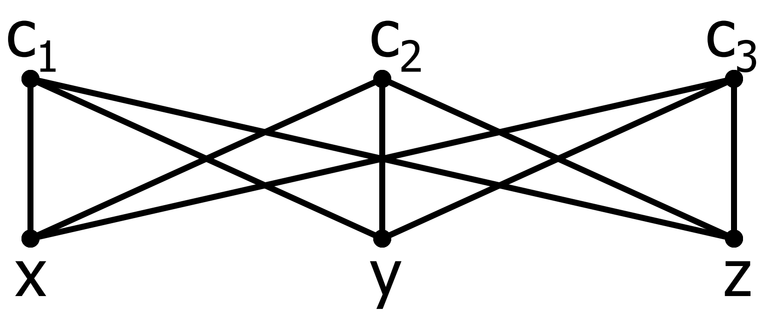

The number of variables in C_i is the degree of C_i,

The number of appearances of x is the degree of x

Satisfiability and graphs

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S. Ordyniak, D. Paulusma and S. Szeider, Satisfiability of Acyclic and almost Acyclic CNF Formulas, Theoretical Computer Science, 481 (2013) 85-99.

proves that satisfiability restricted to instances whose formula graphs are chordal bipartite can be solved in polynomial time.

Hereditary, limit and boundary properties of graphs

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Examples: *bipartite graphs, chordal bipartite graphs, planar graphs, graphs of bounded vertex degree, of bounded tree-width, etc.*

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Theorem. *A class X of graphs is hereditary if and only if X=Free(M) for a set M.*

Hereditary, limit and boundary properties of graphs

The family of hereditary properties contains two important subfamilies: *monotone* (closed under vertex deletions and edge deletions) and *minor-closed* (closed under vertex deletions, edge deletions and edge contractions)

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Let us call any hereditary class of formula graphs with polynomial-time solvable satisfiability problem *good* and all other hereditary classes of formula graphs *bad*.

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A minimal limit class will be called a *boundary* class.

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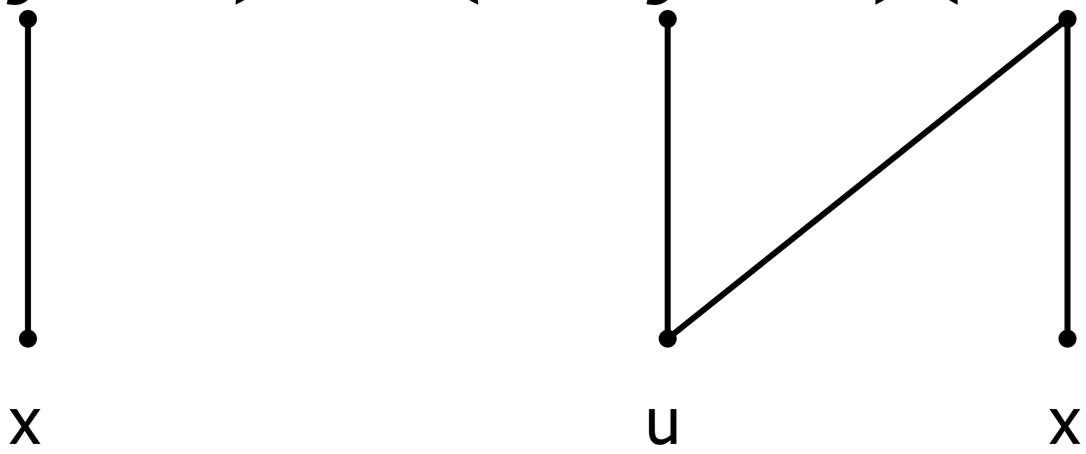

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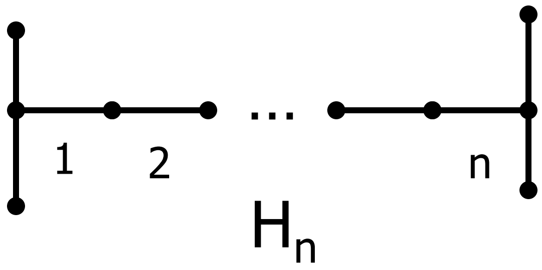

Lemma. *For each fixed k, the satisfiability problem restricted to instances whose formula graphs belong to the class Free($K_{1,4}$,C_3,C_4,…,C_k,H_1,H_2,…H_k) is NP-complete.*

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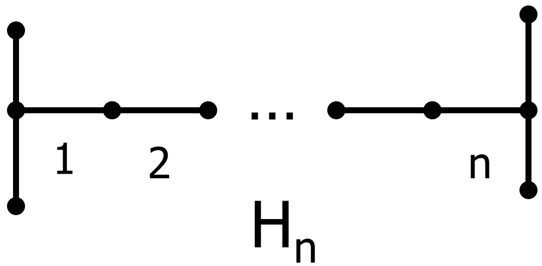

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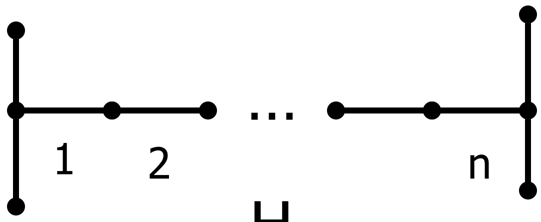

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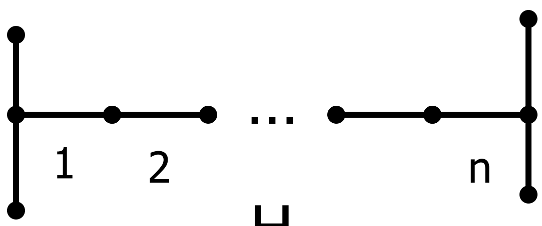

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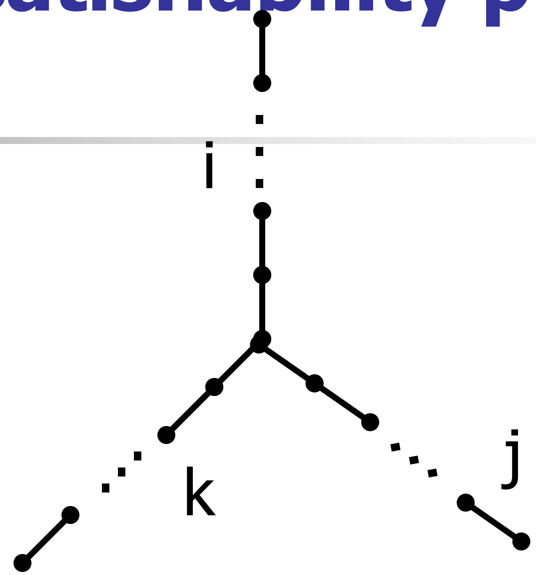

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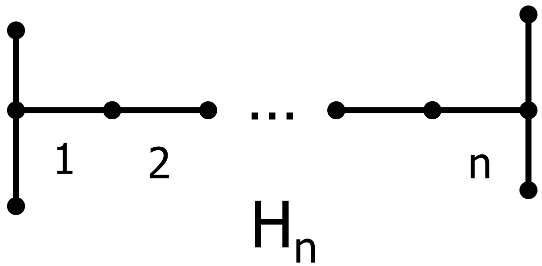

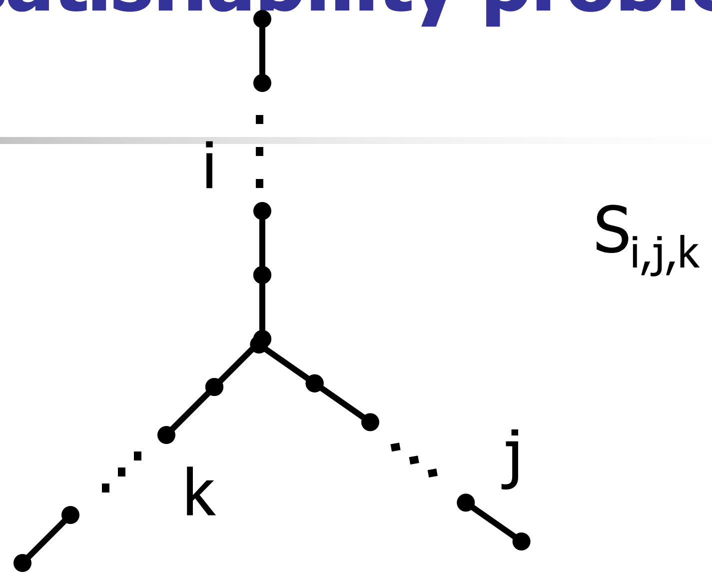

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A limit property of satisfiability problems

Theorem. *The class S is a limit class*

Did you know that

The difference in the speed of clocks at different heights above the earth is now of considerable practical importance, with the advent of very accurate navigation systems based on signals from satellites. If one ignored the predictions of general relativity theory, the position that one calculated would be wrong by several miles!

Stephen Hawking *A brief history of time*

Auxiliary results

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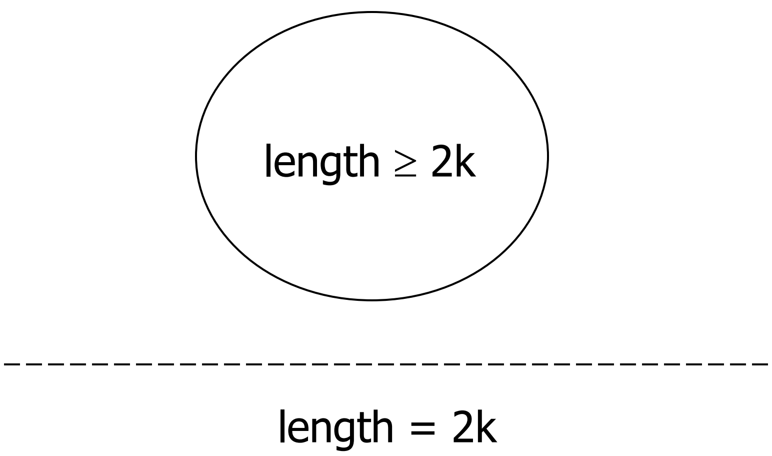

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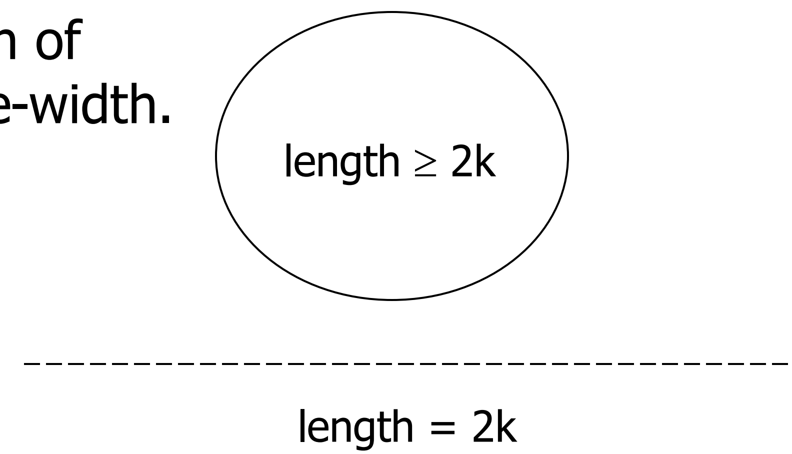

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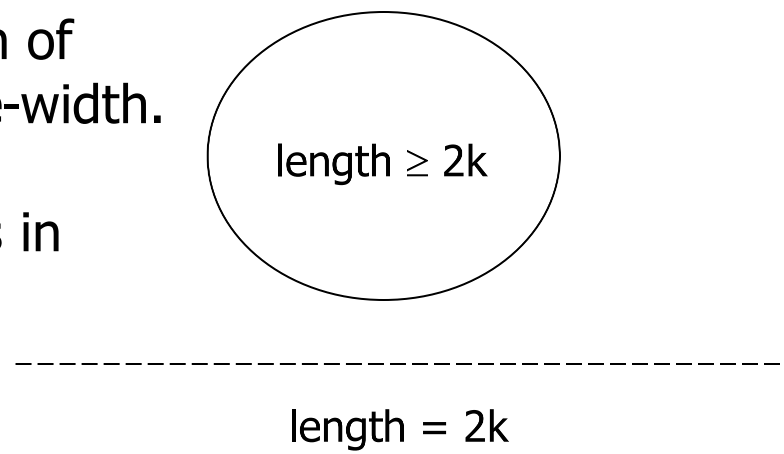
For t>1, deletion of any copy of $S_{k,k,k}$ results in a graph which is of bounded tree-width by the inductive hypothesis.



Minimality criterion

Theorem. *A limit class* $X = Free(M)$ *is minimal if and only if for each graph* $G \in X$ *there is a finite set of graphs* $T \subseteq M$ *such that* $Free(G \cup T)$ *is good.*

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Theorem. *A limit class X=Free(M) is minimal if and only if for each graph G∈X there is a finite set of graphs T⊆M such that Free(G∪T) is good.*

Proof. Assume first X is minimal and suppose by contradiction that ∃ G∈X such that for each finite set T⊆M, the class Free(G∪T) is not good. Let M={F_1,F_2,…}. Then Z_k:=Free(F_1,…,F_k,G) is not good. But then Z=∩Z_k is a limit class and a proper subclass of X, contradicting the minimality of X.

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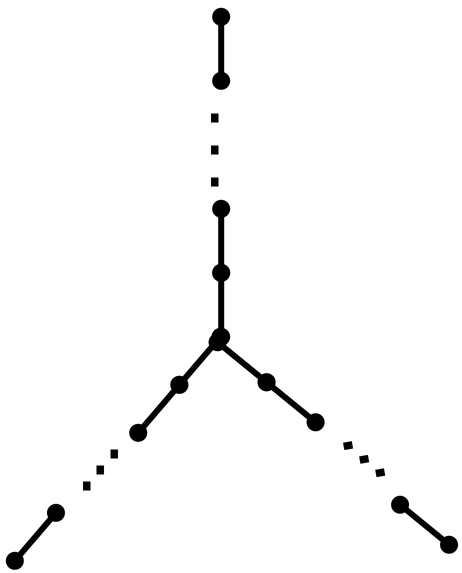
Conversely, assume for each graph G∈X there is a finite set T⊆M such that Free(G∪T) is good. Consider a subclass Z⊂X, a graph G∈X-Z and a finite set T⊆M such that Free(G∪T) is good. Assume Z=∩Z_k for a sequence of bad classes Z_k. But then there must exist an n such that Z_n⊆Free(G∪T) contradicting the assumption.

Theorem. *S=Free($K_{1,4}$,C_3,C_4,…,H_1,H_2,…) is a boundary class*

Theorem. *$S = Free(K_{1,4}, C_3, C_4, \ldots, H_1, H_2, \ldots)$ is a boundary class*

Proof. Let G be a graph in S. W.l.o.g. every connected component of G is of the form $S_{k,k,k}$.



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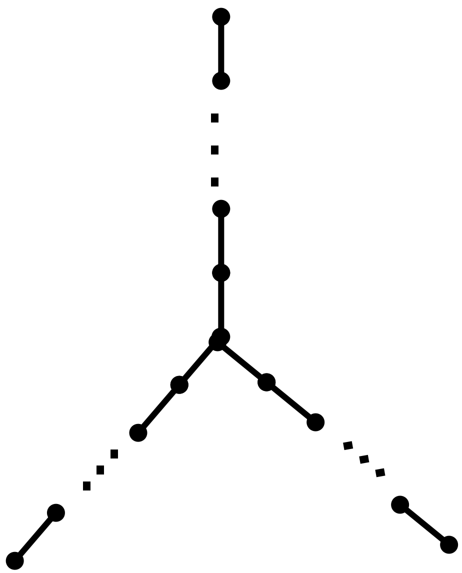

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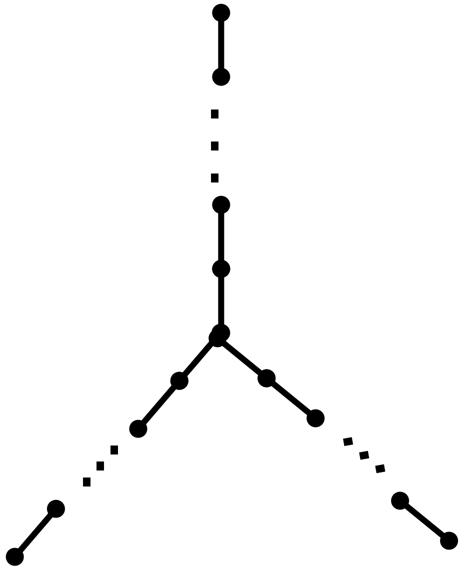

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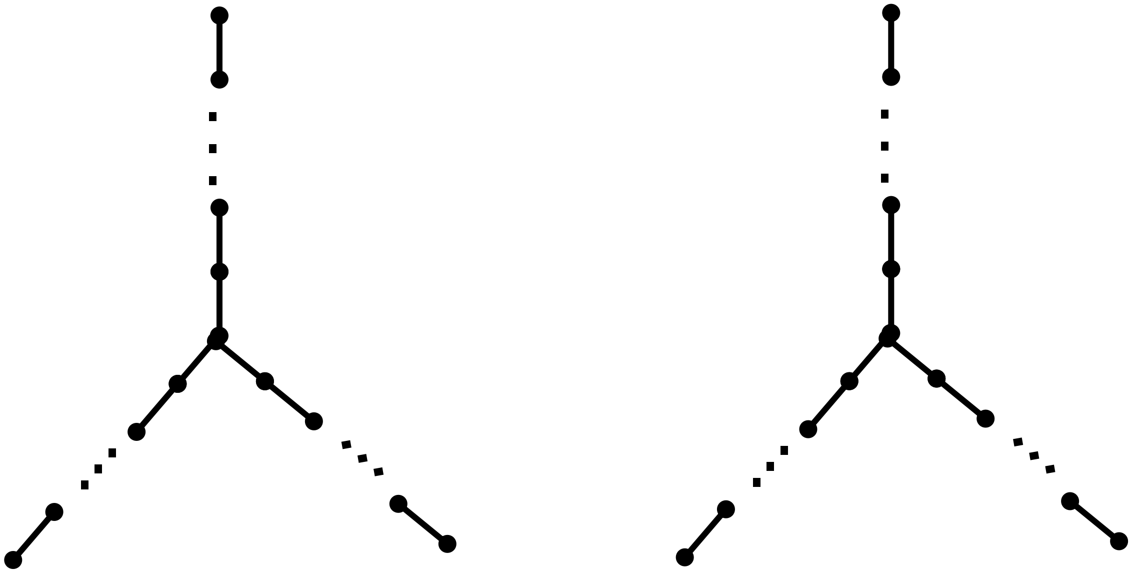

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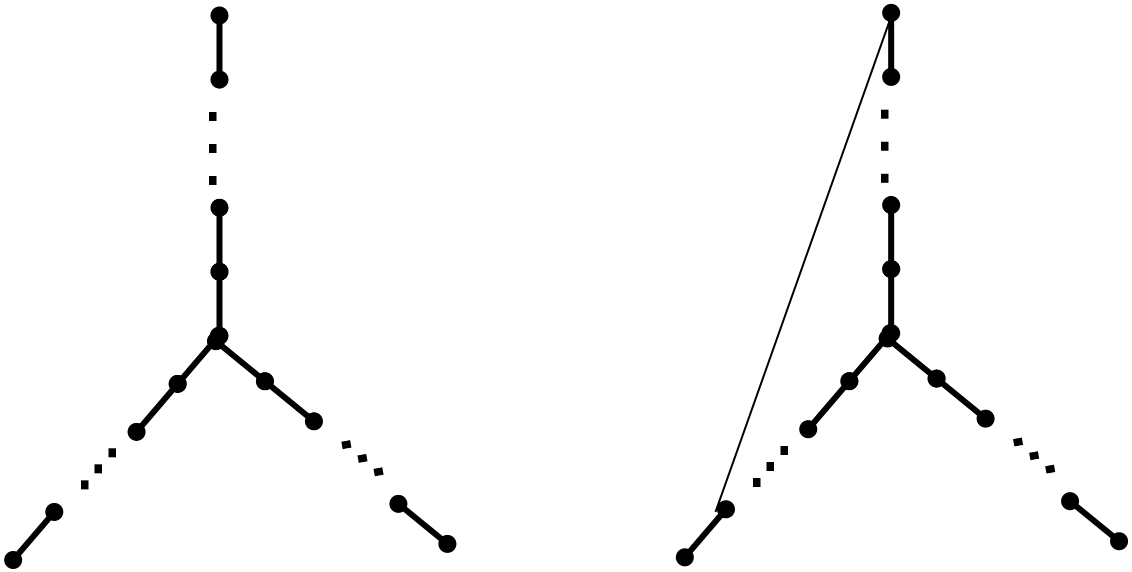

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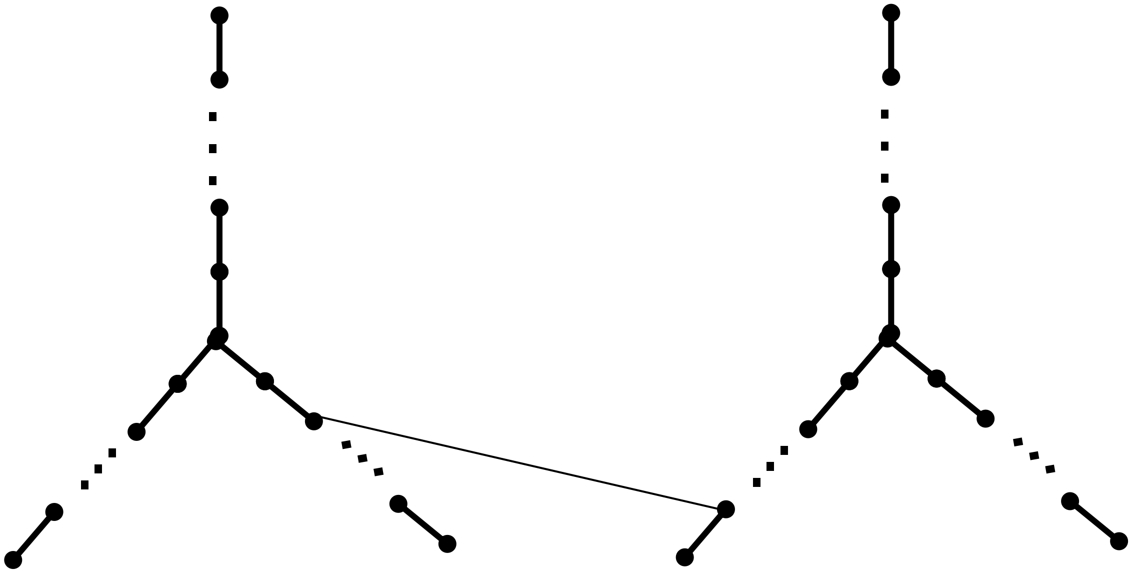

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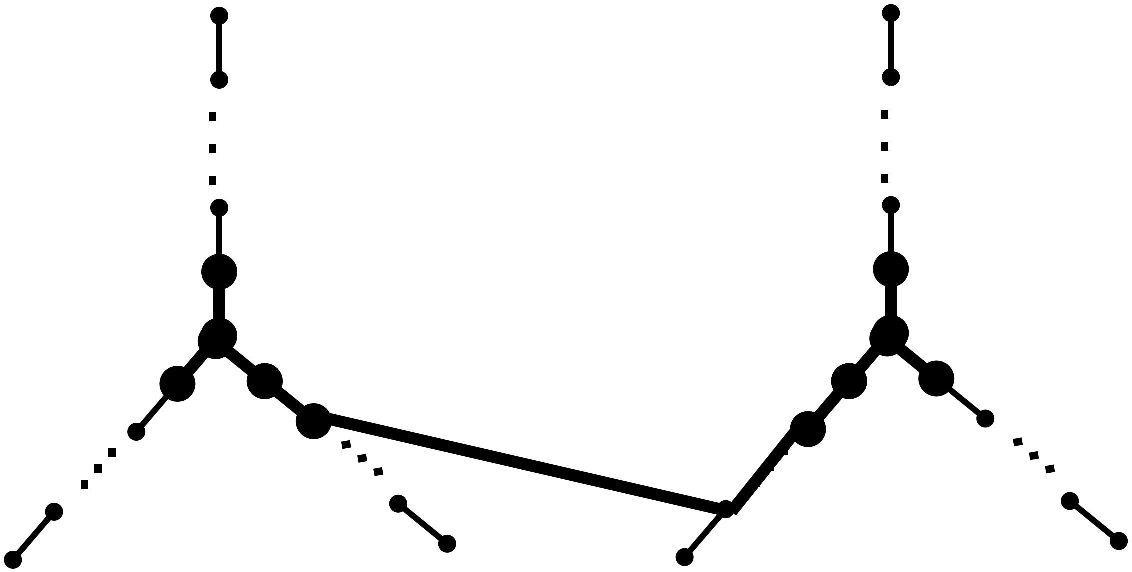

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Free(G,$K_{1,4}$,C_3,…,C_{2k+1},H_1,…,H_{2k+1}) $\subseteq$ Free$_m$(G),

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We will show that $Free(G,K_{1,4},C_3,\ldots,C_{2k+1},H_1,\ldots,H_{2k+1})$ is good. To this end, we will show that graphs in this class do not contain G as a subgraph, not necessarily induced.

$Free(G,K_{1,4},C_3,\ldots,C_{2k+1},H_1,\ldots,H_{2k+1}) \subseteq Free_m(G)$,

Therefore, $Free(G,K_{1,4},C_3,\ldots,C_{2k+1},H_1,\ldots,H_{2k+1})$

is of bounded tree-width and hence is good.

Thank you