Some remarkable differences between quantum and classical information

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Quantum Computing

Ingredients of the term

Quantum Computing

Ingredients of the term

- Quantum mechanics

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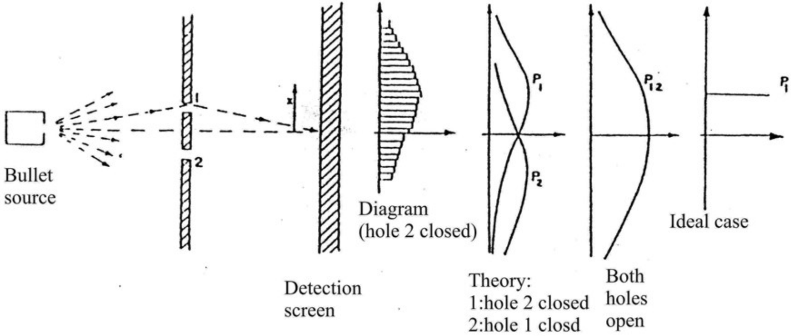
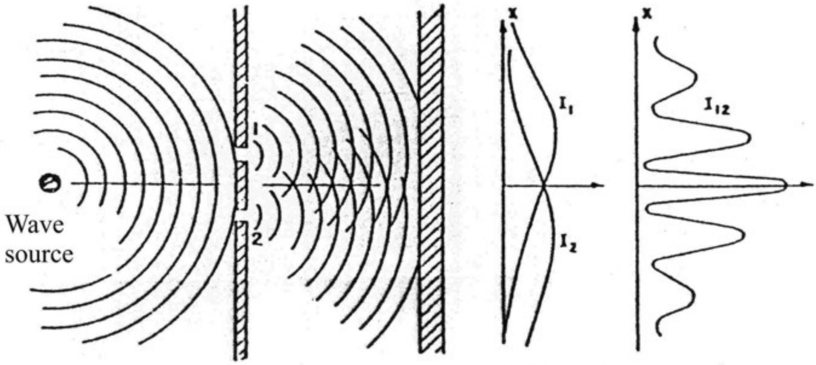
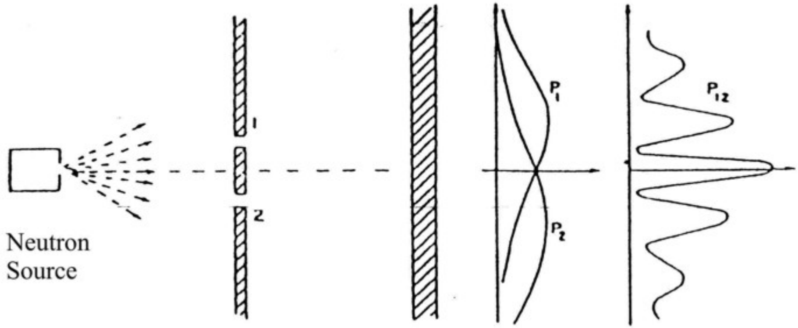
- Quantum mechanics
- Computing

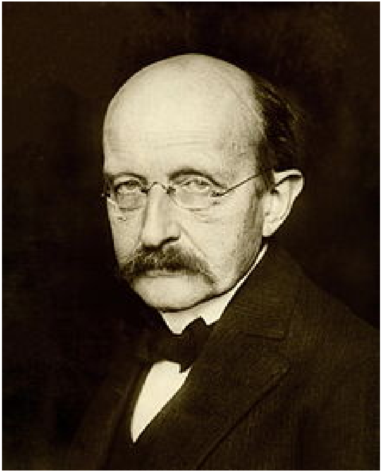
Bullets


Bullet source
1
2
x
Detection screen
Diagram (hole 2 closed)
P_1
P_2
Theory:
1:hole 2 closed
2:hole 1 closd
P_{12}
Both holes open
P_1
Ideal case


1
2
Wave
source
x
I1
I2
x
I12

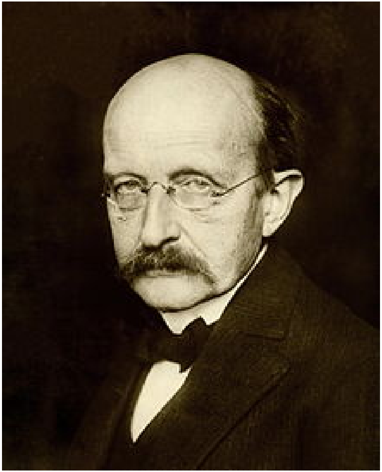
Neutrons





Max Planck (1858–1947)

Quantum Mechanics



Max Planck (1858–1947)
Black body radiation (1900)

Quantum Mechanics

Niels Bohr (1885–1962)

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Hydrogen atom model (1913)

Erwin Schrödinger (1887–1961)

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$\frac{d}{dt}\psi = iH\psi$ (1926)

Louis de Broglie (1892–1987)

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Wawe-particle duality

Mechanics

Newtonian equation of motion

$F = ma$

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Hamiltonian reformulation

$$\frac{d}{dt}x = \frac{\partial}{\partial p}H, \quad \frac{d}{dt}p = -\frac{\partial}{\partial x}H$$

Mechanics

Classical

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Quantum

$\frac{\partial}{\partial t}\psi = -iH\psi$, where ψ is the *wave function*

Wave Function

Max Born's interpretation

$|\psi(x,t)|^2$ is the probability density of the particle position at time t

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$\widehat{\psi}(p) = \mathcal{F}[\psi(x)](p) = \int_{-\infty}^{\infty} \psi(x) e^{-2\pi i x p} \, dx$ is the probability density of the particle momentum.

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Wavefunction ψ gives the full characterization of the system at a fixed time

Finite Quantum Systems

- Nuclear spin
- Photon polarization

Wavefunction ψ defined on a finite set.

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Formally a (pure) state

$\psi = \alpha_1\boldsymbol{\psi}_1 + \alpha_2\boldsymbol{\psi}_2 + \ldots + \alpha_n\boldsymbol{\psi}_n$, where $\{\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n\}$ is an orthonormal basis of n-dimensional complex vector space H_n.

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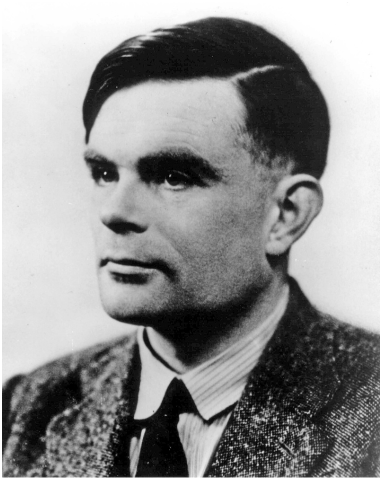
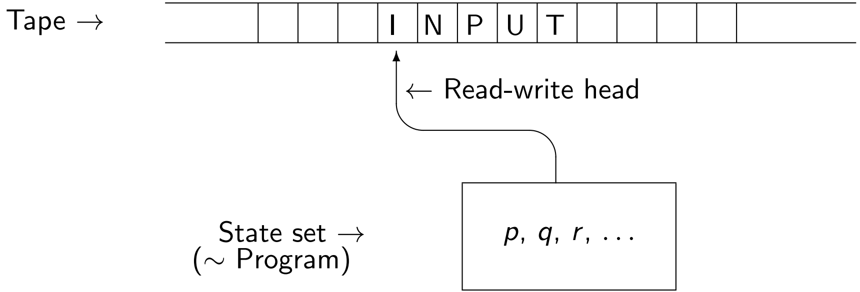
For *mixed states*, representation must be generalized.

Computability



Alan Turing (1912–1954)

Turing Machine



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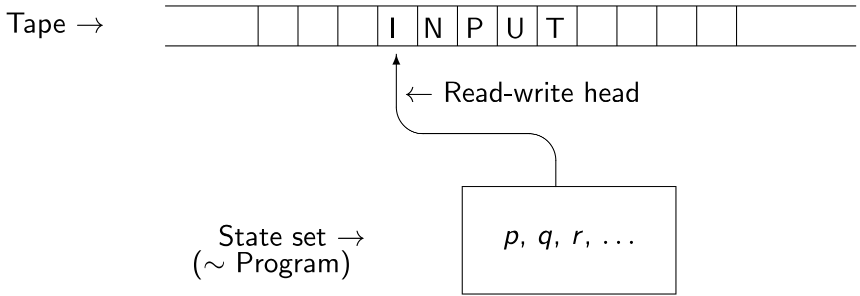

In state p:

- Read a, write b (b depends only on p and a)
- Move read-write head (direction depends only on p and a)
- Go to state q (q depends only on p and a)

Computability

Churc-Turing thesis

Algorithmic computability = Turing Machine computability

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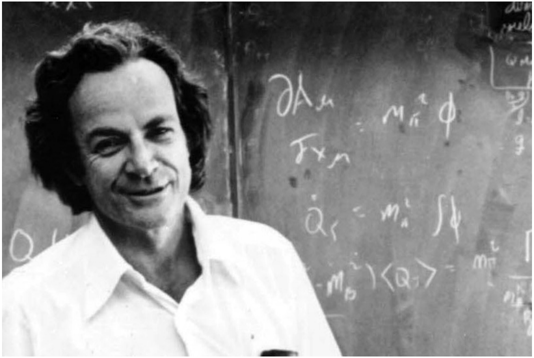
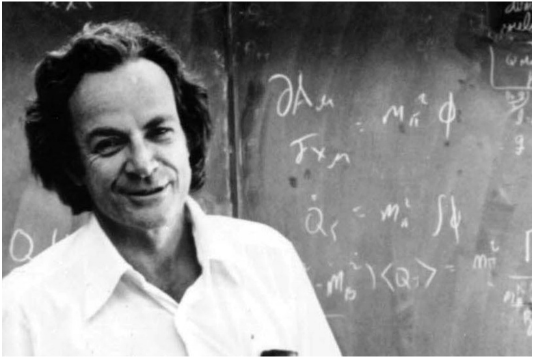
Not provable



Richard Feynman (1918–1988):

Computability



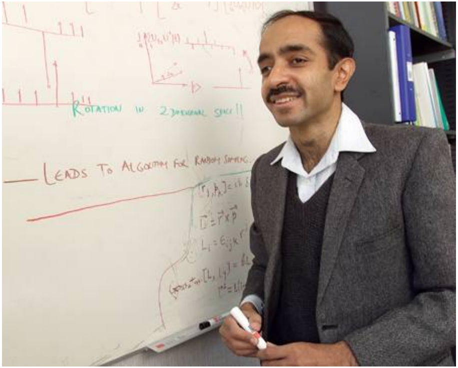
Richard Feynman (1918–1988):
Simulating Physics with Computers (1982)

David Deutsch (1954–): Quantum Turing Machine (1985)

David Deutsch (1954–): Quantum Turing Machine (1985), a "proof" of Church-Turing thesis

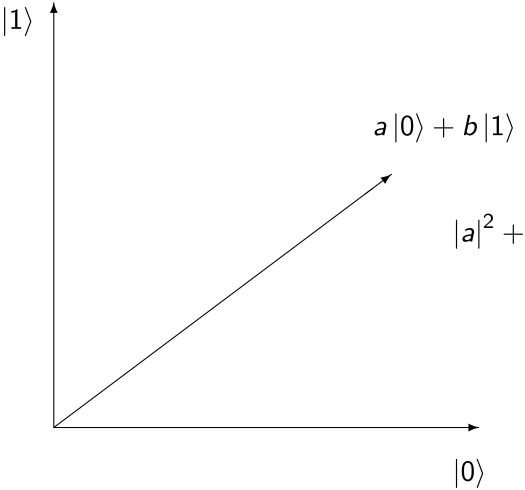
Peter Shor (1959–): Fast Quantum Factoring Algorithm (1994)

Quantum Computing



Lov Grover (1961–): $O(\sqrt{N})$ Quantum Search (1996)

Quantum bit (Qubit)



Superposition of states

$|0\rangle$ ja $|1\rangle$

$$|a|^2 + |b|^2 = 1$$

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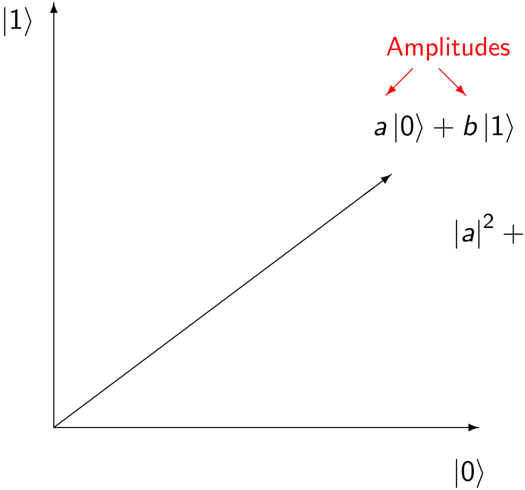

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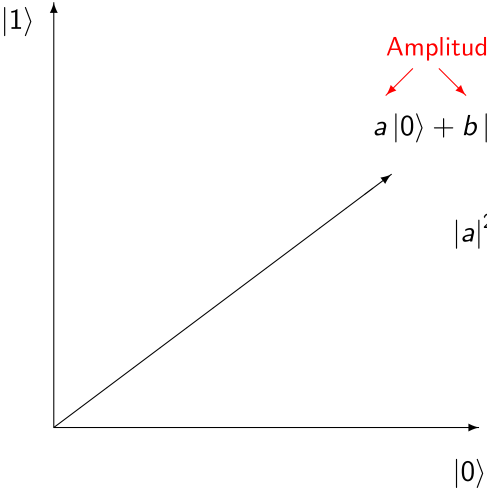

Amplitudes

$$a\,|0\rangle + b\,|1\rangle$$

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Measurement in basis

$\{|0\rangle\,, |1\rangle\}$:

$p(0) = |a|^2$, $p(1) = |b|^2$

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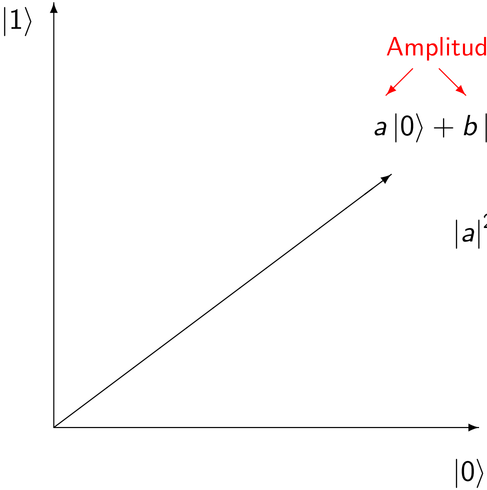

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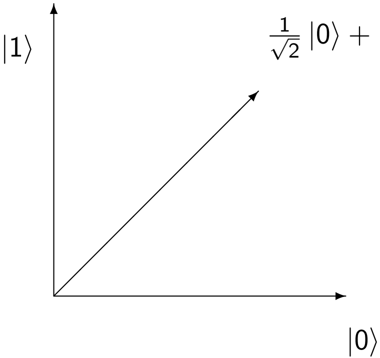
Minimal interpretation

Quantum bit (Qubit)



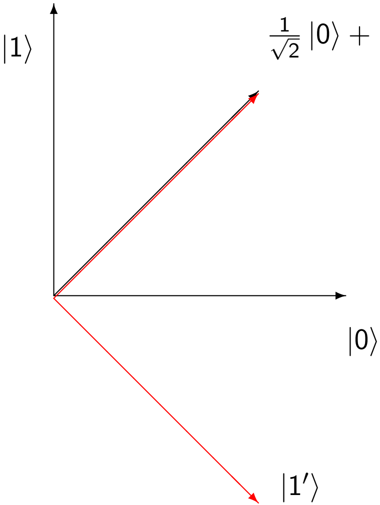
Basis 1:

$\{|0\rangle, |1\rangle\}$

$p(0) = \frac{1}{2}$

Quantum bit (Qubit)

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |0'\rangle$$



Basis 1:

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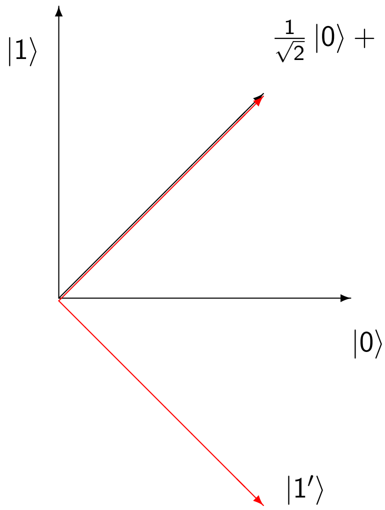
Basis 2:

$$\{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |0'\rangle,$$
$$\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = |1'\rangle\}$$

$$p(0') = 1$$

Quantum bit (Qubit)

$$\frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle \quad = \left|0'\right\rangle$$



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$$p(0') = 1$$

Pure state $\neq$ (generalized) probability distribution

Quantum gate

Example: $W : \mathbb{C}^2 \to \mathbb{C}^2$

$$\begin{aligned} W\left|0\right\rangle &= \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle \\ W\left|1\right\rangle &= \frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle \end{aligned}$$

is *unitary* (Hadamard-Walsh transform)

Interference / Walsh transform once

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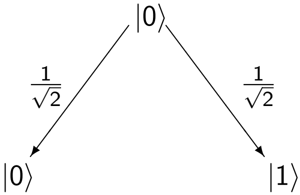

$$|0\rangle$$

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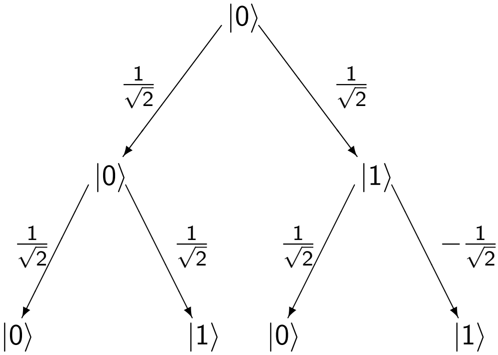
Interference / Walsh transform twice



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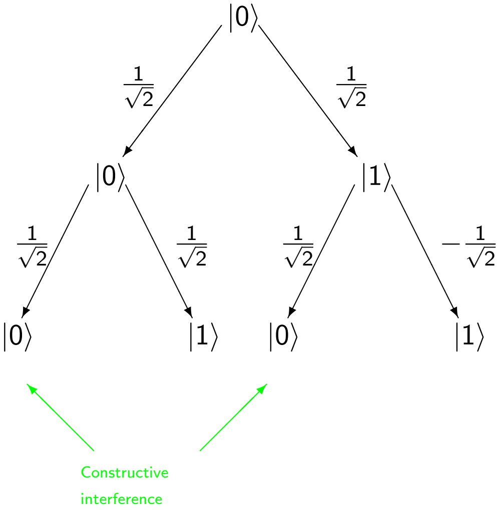

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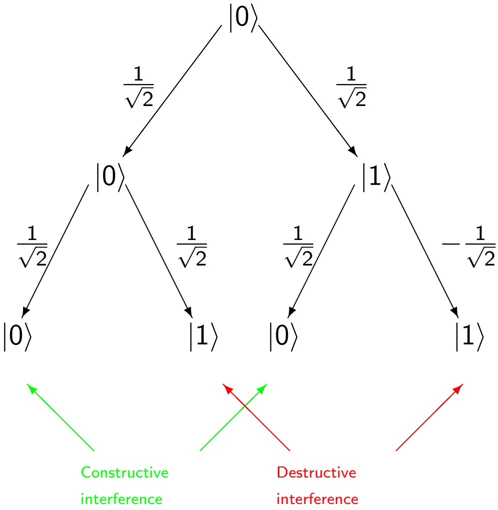

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n quantum bits

- Tensor product notation: $|\boldsymbol{a}\rangle \otimes |\boldsymbol{b}\rangle = |\boldsymbol{a}\rangle\,|\boldsymbol{b}\rangle = |\boldsymbol{a}\,\boldsymbol{b}\rangle$.

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- For example, $|0\rangle\,|0\rangle = |00\rangle$, $|0\rangle\,|1\rangle = |01\rangle$, etc.

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- General state $\displaystyle\sum_{\boldsymbol{x}\in\{0,1\}^n} c_{\boldsymbol{x}}\,|\boldsymbol{x}\rangle$ (2^n-dimensional Hilbert space), where $\displaystyle\sum_{\boldsymbol{x}\in\{0,1\}^n} |c_{\boldsymbol{x}}|^2 = 1$

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If $U_f\,|\boldsymbol{x}\rangle\,|\mathbf{0}\rangle = |\boldsymbol{x}\rangle\,|f(\boldsymbol{x})\rangle$ can be realized, then

$$U_f \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{x}\in\{0,1\}^n} |\boldsymbol{x}\rangle\,|\mathbf{0}\rangle = \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{x}\in\{0,1\}^n} |\boldsymbol{x}\rangle\,|f(\boldsymbol{x})\rangle$$

(Quantum parallelism)

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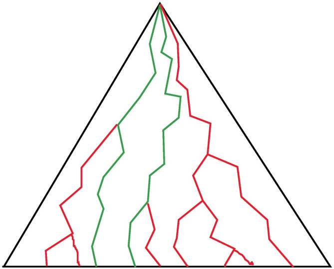
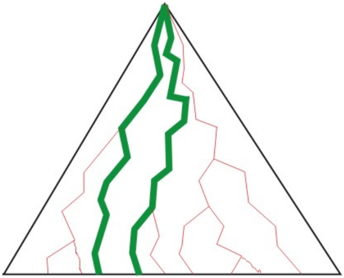
- Direct method offers no advantage over probabilistic guessing!

Nondeterministic Computing

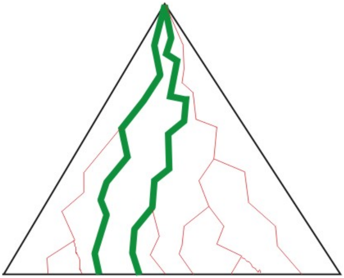


Nondeterministic Computing



- Good computational paths should be supported

Nondeterministic Computing



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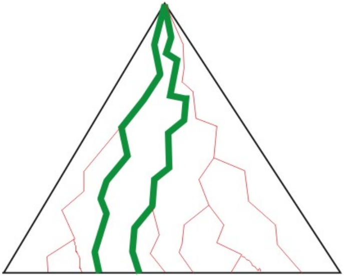

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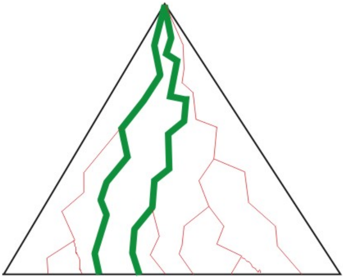

- Good computational paths should be supported
- Seems impossible in classical computing
- Sometimes possible in quantum computing
- The “efficiency” of quantum computing is based on interference

Quantum algorithms

- Interference should favour the good computation paths
- Difficult to control in algorithm design

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Main methods

- Quantum Fourier transform
- Grover iteration
- Adiabatic quantum computing
- Quantum random walks

Quantum Fourier transform

- Discrete Fourier transform on *coefficients* of

$$c_0 |00\dots0\rangle + c_1 |00\dots1\rangle + \dots + c_{2^n-1} |11\dots1\rangle ,$$

- Can be implemented in time Poly(n) (instead of 2^n)
- Exponential advantage for problems with periodic structure
- Main ingredient in Shor's factoring algorithm

Grover iteration

- Basic idea: Using k calls of function f, the superposition

$$\frac{1}{\sqrt{2^n}}\Big(\,|00\ldots0\rangle + |00\ldots1\rangle + \ldots + |11\ldots1\rangle\,\Big),$$

  coefficients of such vectors $|\boldsymbol{x}\rangle$ for which $f(\boldsymbol{x}) = 1$ can be increased to $\approx C \cdot \frac{k}{\sqrt{2^n}}$, hence the probability of seeing such an element becomes $\approx |C|^2\,\frac{k^2}{2^n}$.
- Provides a *quadratic* advantage over classical algorithms
- Works on all search problems

EPR Paradox

EPR pair

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

(entangled state), perfect correlation

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(entangled state), perfect correlation; cf. with

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EPR Paradox

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$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

(entangled state), perfect correlation; cf. with

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

(decomposable state).

Compound Systems

Correlation over distance also possible in classical mechanics:

Probability distribution

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Compound Systems

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But

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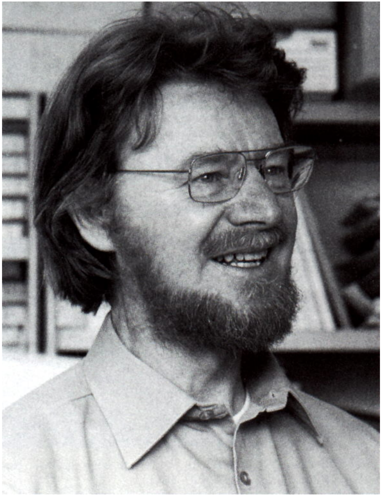
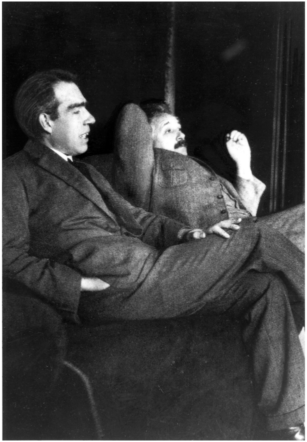
violates a *Bell inequality*.

John Bell



John Steward Bell (1928–1990)

EPR Paradox



Niels Bohr (1885–1962) & Albert Einstein (1879–1955)

Einstein, Podolsky, Rosen: Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

Physical Review 47, 777–780 (1935)

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$\Rightarrow$ Quantum mechanics is an incomplete theory

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Itamar Pitowsky: Quantum Probability – Quantum Logic, Springer (1989)

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Reason: $\mathbb{P}(1 \vee 2) = p_1 + p_2 - p_{12}$ is a probability, too.

Bell Inequalities

Lemma

(p_1, p_2, p_{12}) is an "eligible" probability vector if and only if

$$0 \leq p_{12} \leq p_1, p_2 \leq 1 \quad \textit{and} \quad 0 \leq p_1 + p_2 - p_{12} \leq 1$$

These are Bell inequalities!

Bell Inequalities

In fact,

$$
\begin{aligned}
& (p_1, p_2, p_{12}) \\
= \quad & (1 - p_2 - p_2 + p_{12})(0, 0, 0) \\
+ \quad & (p_2 - p_{12})(0, 1, 0) \\
+ \quad & (p_1 - p_{12})(1, 0, 0) \\
+ \quad & p_{12}(1, 1, 1).
\end{aligned}
$$

However, the representation is not generally unique.

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Probabilities → Expected values

- Let A_1, A_2, B_1, B_2 be ± 1-valued observables.

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- Then $A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 = A_1(B_1 + B_2) + A_2(B_1 - B_2) \in \{-2, 2\}$

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- Hence $|\mathbb{E}(A_1B_1) + \mathbb{E}(A_1B_2) + \mathbb{E}(A_2B_1) - \mathbb{E}(A_2B_2)| \leq 2$

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- Assume Alice and Bob share state $\boldsymbol{x} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

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For these observables,

$$\mathbb{E}(A_1 B_1) + \mathbb{E}(A_1 B_2) + \mathbb{E}(A_2 B_1) - \mathbb{E}(A_2 B_2) = 2\sqrt{2},$$

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Conclusion:

Locality, realism, and quantum mechanics form a contradictory set of assumptions.

From them, you can derive anything.