Global and local stability of multi-dimensional Markov chains

Based on joint work with
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Introduction

- We study the evolution of the state of a wireless communication network as a random process (transmissions may be required at random times, they may take random amounts of time, their routing may be random, and how transmitters access the network is random)
- The process may count the number of outstanding transmissions, or the "amount of work"..
- Interested in stability of the system, i.e. ability to cope with the demand

Introduction

- Often the process of interest (number of outstanding transmissions) turns out to be a Markov chain, i.e.

$$P(X_{n+1} \in A | X_n \in B, X_{n-1} \in \cdot, ... X_0 \in \cdot) = P(X_{n+1} \in A | X_n \in B).$$

- For Markov chains we can consider positive recurrence as stability
- Various methods for proving it

General problem

Interested in Markov chains (X_n, Y_n) where

- The first component X_n is predictable, its behaviour is well understood (has a limiting distribution as $n \to \infty$)
- The behaviour of the second component Y_n, assuming that the first component is in its limiting distribution, is understood too
- Can we say something about the limiting distribution of Y_n and the whole process?

Unstable first component: example

Motivating example (S., van de Ven, 2015):

- Time is slotted, all transmission times are equal to 1
- Three transmitters on the line, messages are generated at the first one, and then each message needs to be transmitted from first to second, then from second to third and then from third out of the system. Three transmissions needed for each message
- The first node always has a message to transmit, the second and third have infinite buffers (queues) where they can store waiting messages

Unstable first component: example, continued

- Neighbouring nodes cannot transmit simultaneously
- Each node wakes up first, second or third with probability $1/3$ each. If it has a message to transmit, and if no neighbour is transmitting, the node will start a transmission. Everything is repeated independently in the next time slot
- At any time slot we can have node 1 transmitting alone, node 2 transmitting alone, or nodes 1 and 3 transmitting
- Interested in end-to-end throughput

Nodes 2 and 3

- The state of the system is described by (X_n^2, X_n^3) counting number of messages in nodes 2 and 3
- For node 2,
  $E(arrivals) = \mathbf{P}(arrivals) \geq \mathbf{P}(departure) = E(departures)$, so it is unstable. Can prove that $X_n^2 \to \infty$ a.s.
- For node 3,
  $E(arrivals) = \mathbf{P}(arrivals) < \mathbf{P}(departure) = E(departures)$, so it is stable.
- The vector (X_n^2, X_n^3) cannot be stable but the second component is, in a sense. Can we make this rigorous?

End-to-end throughout

- Assume the state of the system is described by a Markov Chain $Y_n^3 = X_n^3 | X_n^2 = \infty$, which is positive recurrent and hence has a stationary (limiting) distribution
- For limiting probabilities $\pi(y) = \lim_{n\to\infty} \mathrm{P}\left(Y_n^3 = y\right)$ we have

$$\pi_0 = \frac{1}{2}\pi_0 + \frac{2}{3}\pi_1, \quad \pi_1 = \frac{1}{2}\pi_0 + \frac{2}{3}\pi_2$$

  and

$$\pi_i = \frac{1}{3}\pi_{i-1} + \frac{2}{3}\pi_{i+1} \quad \text{for} \quad i \geq 2.$$

- These equations lead to $\pi_0 = \frac{2}{5}$ and $\pi_i = \frac{3}{2^{i+1}}\pi_0$ for $i \geq 1$.
- The throughput is then equal to $\frac{2}{3}(1-\pi_0) = \frac{1}{2}\pi_0 + \frac{1}{3}(1-\pi_0) = \frac{2}{5}$.

More examples

- Consider two queues in tandem, with the arrival rate λ and service rates μ_1 and μ_2
- Assume $\lambda > \mu_1$, then the first queue is unstable
- However, if $\mu_2 > \mu_1$, then the second queue should be stable and should behave as a single-server queue with arrival rate μ_1 and service rate μ_2
- Can this be made rigorous? Goodman, Massey (1984) - yes, for a **general Jackson network**
- **Polling systems**
- **Multi-server queues with skill based service** (Adan, Weiss, 2012)

General mathematical model

Assume we have a Markov chain $\{X_n, Y_n\}$ on $\mathcal{Z}_+^2$ such that

- $X_n \to \infty$ a.s.
- Probabilities $\mathbf{P}(Y_1 = j | Y_0 = i, X_0 = k) = p(i,j)$ do not depend on k as long as $k > 0$.
- Probabilities $p(i,j)$ are transition probabilities of a positive recurrent Markov chain (think $\{Y_n | X_n = \infty\}$) with a stationary distribution π

Theorem (Adan, Foss, S., Weiss, (2015))

The conditions above imply that

$$\mathbf{P}(Y_n = j | Y_0 = i, X_0 = k) \to \pi(j)$$

as $n \to \infty$, regardless of i and k and uniformly in j, i.e. π is the limiting distribution for the sequence Y_n.

Remarks

Condition in the result above: probabilities
$\mathbf{P}(Y_1 = j | Y_0 = i, X_0 = k) = p(i,j)$ do not depend on k as long as $k > 0$
This may be *relaxed* to the requirement that probabilities
$\mathbf{P}(Y_1 = j | Y_0 = i, X_0 = k) = p(i,j)$ do not depend on k as long as $k > k^*$.
An interesting question is what happens when probabilities
$\mathbf{P}(Y_1 = j | Y_0 = i, X_0 = k) \to p(i,j)$ as $k \to \infty$.
More applications and examples..?

Model

- M transmitters
- Slotted time
- Each transmission duration equals 1
- High- and low- priority messages (Wi-Max and Wi-Fi)
- At most one high- and at most one low-priority messages transmitted in a time slot
- A transmitter cannot transmit both high- and low-priority messages in a time slot
- Schedule for high-priority messages, no collisions
- Random access for low-priority messages, collided messages return to their origins

High-priority (red) messages assumptions

- At time slot t, ξ_n^t new ones arrive at device n, $\mathbf{E}\xi_n^t = \lambda_R/M$
- $\{\xi_n^t\}$ are i.i.d. in t
- Symmetrical schedule for transmissions: at time slot t, transmitter number $i(t) = ((t-1) \mod M) + 1$ is scheduled to transmit a red message
- If transmitter $i(t)$ has a red message, its transmission is attempted and is successful
- Otherwise no red message is transmitted in the time slot

Red-messages dynamics

Let R_n^t be the number of red messages in the queue of transmitter n at time t. Then

$$\left(R_1^t, .., R_M^t\right)$$

is a Markov chain which is positive recurrent if $\lambda_R < 1$. Moreover,

$$\mathbf{P}(R_{i(t)}^t = 0) \to 1 - \lambda_R, \quad t \to \infty.$$

Low-priority (green) messages assumptions

- At time slot t, η_n^t new ones arrive at device n, $\mathbf{E}\eta_n^t = \lambda_G/M$
- $\{\eta_n^t\}$ are i.i.d. in t
- Transmission attempts are governed by a random-access ALOHA-type protocol: every transmitter not currently transmitting a red message with a non-empty green queue attempts to transmit a green message with (fixed) probability p
- Three possibilities:
  - No transmission attempted
  - Exactly one transmission attempted. It is successful
  - More than one transmission attempts. All of them unsuccessful, messages stay in their queues

Model dynamics

Let G_n^t be the number of green messages in the queue of transmitter n at time t. Then

$$\left(G_1^t, .., G_M^t, R_1^t, .., R_M^t\right)$$

is a Markov chain and we are interested in its long-time behaviour.
Let first $M = 1$. Then a green message transmission will be attempted (and will always be successful) with probability p every time the red queue is empty. We expect that

$$\lambda_G < (1 - \lambda_R)p$$

leads to stability (positive recurrence).

Model dynamics

Let now $M \geq 2$. Two cases:

- Transmitter $i(t)$ has a red message to transmit. Then probability of a successful transmission of a green message is $(M-1)p(1-p)^{M-2}$
- Transmitter $i(t)$ does not have a red message to transmit. Then probability of a successful transmission of a green message is $Mp(1-p)^{M-1}$

Stability should be achieved if

$$\lambda_G < \lambda_R(M-1)p(1-p)^{M-2} + (1-\lambda_R)Mp(1-p)^{M-1}.$$

General mathematical model

Let $\{X^t\}$ and $\{Y^t\}$ be random sequences taking values in measurable spaces $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ and $(\mathcal{Y}, \mathcal{B}_\mathcal{Y})$, respectively, and assume that $\{(X^t, Y^t)\}$ is a Markov Chain. Assume also that

- $\{X^t\}$ is a Markov Chain with autonomous dynamics: for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$,
$$\mathbf{P}_{x,y}(X^1 \in \cdot) = \mathbf{P}_x(X^1 \in \cdot).$$
- The X-chain is *Harris ergodic*, so there exists a stationary distribution $\pi = \pi_X$ such that
$$\sup_{B \in \mathcal{B}} \left| \mathbf{P}_x \left(X^t \in B \right) - \pi(B) \right| \to 0$$
as $t \to \infty$, for any $x \in \mathcal{X}$.

Auxiliary chain

Introduce an auxiliary (time-homogeneous) Markov chain $\{\widehat{Y}^t\}$ with transition probabilities

$$\mathbf{P}(\widehat{Y}^{t+1} \in \cdot \,|\, \widehat{Y}^t = y) = \int_{\mathcal{X}} \pi_X(dx) \mathbf{P}(Y^1 \in \cdot \mid X^1 = x, Y^0 = y).$$

Hypothesis

If $\{\widehat{Y}^t\}$ is positive recurrent, then so is (X^t, Y^t).

Theorem

Natural (Foster-Lyapunov type conditions) imply positive recurrence of both $\{\widehat{Y}^t\}$ and (X^t, Y^t).

Conditions on Y

For the sequence $\{Y^t\}$ we assume that there exists a non-negative measurable function L_2 such that:

- The expectations of the absolute values of the increments of the sequence $\{L_2(Y^t)\}$ are bounded from above by a constant U:

$$\sup_{x\in\mathcal{X},y\in\mathcal{Y}} \mathbf{E}_{x,y} \left| L_2\left(Y^1\right) - L_2\left(Y^0\right) \right| \leq U < \infty.$$

- There exist a non-negative and non-increasing function $h(N)$, $N \geq 0$ such that $h(N) \downarrow 0$ as $N \to \infty$, and a measurable function $f : \mathcal{X} \to (-\infty, \infty)$ such that

$$\int_{\mathcal{X}} f(x)\pi(dx) := -\varepsilon < 0$$

and

$$\mathbf{E}_{x,y}\left(L_2\left(Y^1\right) - L_2(y)\right) \leq f(x) + h(L_2(y))$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

Main result

Remark

Conditions on Y imply positive recurrence of $\widehat{Y}^t$.

Theorem (Foss, S, Turlikov (2012))

Under assumptions for X and Y, the Markov chain $\{(X^t, Y^t)\}$ is positive recurrent.

Corollary

Stability of the communication network under natural conditions

(Some of the) related models

- *Bin-packing problems* (Gamarnik, 2004; Gamarnik and Squillante, 2005)
- *Cat-and-mouse Markov chain* (Litvak and Robert, 2012)
- *Other communication-networks applications* (Borst, Jonckheere, Leskela, 2008; Shah, Shin, 2012)
- Queueing systems, storage processes, etc.....
- Some models in **economics** (Foss, S., Thomas, Worrall, 2015)

Work in progress and research plans

- No knowledge of stationary distribution
- Non-autonomous dynamics, "more" dependence
- Other applications