Integer Linear Programming Tricks

prof. Mehdi TOLOO, Ph.D.

Department of Systems Engineering,
Faculty of Economics,
VŠB- Technical University of Ostrava,
Czech Republic
Email: mehdi.toloo@vsb.cz
URL: http://homel.vsb.cz/~tol0013/

Contents

1. A variable taking discontinuous values
2. Fixed costs
3. Either-or constraints
4. Conditional constraints
5. Elimination of products of variables
    - 5.1 Two binary variables
    - 5.2 One binary and one continuous variable
    - 5.3 Two continuous variables
6. Applications

1. A variable taking discontinuous values

$$x = 0 \text{ or } l \leq x \leq u$$

<u>Indicator variable</u> method

$$y = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } l \leq x \leq u \end{cases}$$

1. A variable taking discontinuous values

$$y = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } l \le x \le u \end{cases}$$

$$yl \le x \le yu; y \in \{0,1\}$$

$$x \le uy$$
$$x \ge ly$$
$$y \quad \text{binary}$$

Validation?

Validation

x	constraints	y
0	$0 \leq uy$ $0 \geq ly$ $y \in \{0,1\}$	0
$l \leq x \leq u$	$x \leq uy$ $x \geq ly$ $y \in \{0,1\}$	1

2. Fixed cost

Minimize: $$C(x)$$

Subject to:

$$a_i x + \sum_{j \in J} a_{ij} w_j \gtrless b_i \qquad \forall i \in I$$

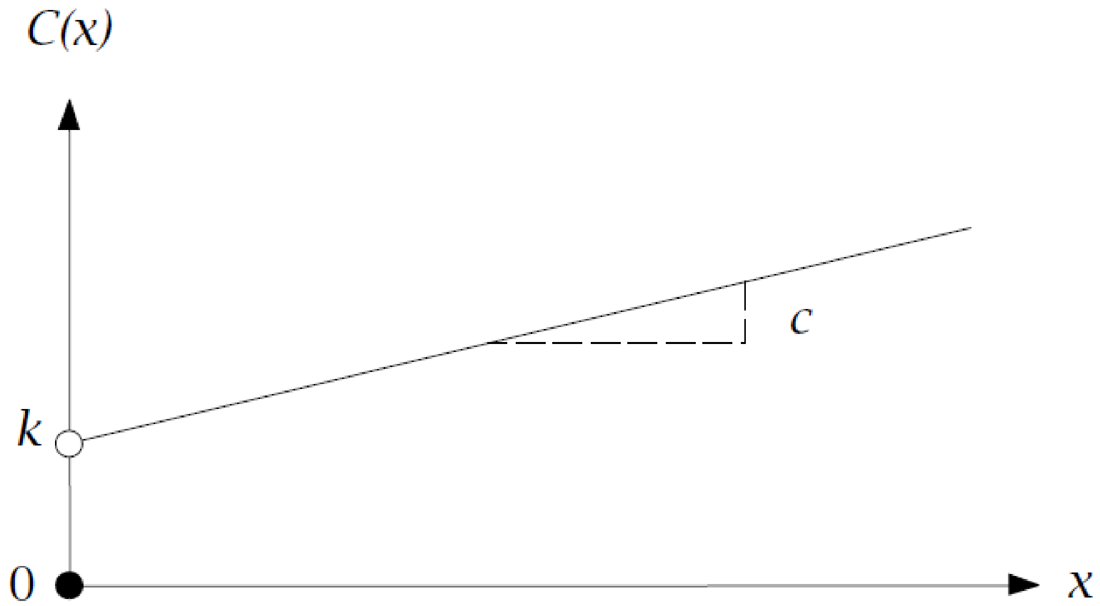
$$x \geq 0$$

$$w_j \geq 0 \qquad \forall j \in J$$

Where:

$$C(x) = \begin{cases} 0 & \text{for } x = 0 \\ k + cx & \text{for } x > 0 \end{cases}$$

2. Fixed cost (illustration)



Modelling Fixed cost

Let $x \leq u$.

Consider the following indicator variable y:

$$y = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

1. $C(x, y) = ky + cx$
2. $x \leq uy$

If $x > 0$ and $x \leq uy$, then $y = 1$.

If $x = 0$, how we have $y = 0$?

The equivalent MBLP model

Minimize: $$ky + cx$$

Subject to:

$$a_i x + \sum_{j \in J} a_{ij} w_j \gtrsim b_i \qquad \forall i \in I$$

$$x \leq uy$$

$$x \geq 0$$

$$w_j \geq 0 \qquad \forall j \in J$$

$$y \quad \text{binary}$$

3. Either-or constraints

Minimize:

$$\sum_{j \in J} c_j x_j$$

Subject to:

$$\sum_{j \in J} a_{1j} x_j \leq b_1 \qquad (1)$$

$$\sum_{j \in J} a_{2j} x_j \leq b_2 \qquad (2)$$

$$x_j \geq 0 \qquad \forall j \in J$$

Where: *at least one of the conditions (1) or (2) must hold*

Modeling either-or constraints

Consider a binary variable y, and sufficiently large upper bounds M_1 and M_2, which are upper bounds on the activity of the constraints. The bounds are chosen such that they are as tight as possible, while still guaranteeing that the left-hand side of constraint i is always smaller than $b_i + M_i$. The constraints can be rewritten as follows:

$$(1) \quad \sum_{j \in J} a_{1j} x_j \leq b_1 + M_1 y$$

$$(2) \quad \sum_{j \in J} a_{2j} x_j \leq b_2 + M_2 (1 - y)$$

Equivalent MBLP model

Minimize:

$$\sum_{j \in J} c_j x_j$$

Subject to:

$$\sum_{j \in J} a_{1j} x_j \leq b_1 + M_1 y$$

$$\sum_{j \in J} a_{2j} x_j \leq b_2 + M_2 (1 - y)$$

$$x_j \geq 0 \qquad \forall j \in J$$

$$y \quad \text{binary}$$

Other forms

- $\begin{cases} \sum_{j\in J} a_{1j}x_j \leq b_1 \\ \text{or} \\ \sum_{j\in J} a_{2j}x_j \geq b_2 \end{cases}$ then $\begin{cases} \sum_{j\in J} a_{1j}x_j \leq b_1 + My \\ \sum_{j\in J} a_{2j}x_j \geq b_2 - M(1-y) \end{cases}$

- $\begin{cases} \sum_{j\in J} a_{1j}x_j \leq b_1 \\ \text{or} \\ \sum_{j\in J} a_{2j}x_j = b_2 \end{cases}$ then $\begin{cases} \sum_{j\in J} a_{1j}x_j \leq b_1 \\ \text{or} \\ \begin{pmatrix} \sum_{j\in J} a_{2j}x_j \leq b_2 \\ \text{and} \\ \sum_{j\in J} a_{2j}x_j \geq b_2 \end{pmatrix} \end{cases}$ hence

$$\begin{cases} \sum_{j\in J} a_{1j}x_j \leq b_1 + My \\ \sum_{j\in J} a_{2j}x_j \leq b_2 + M(1-y) \\ \sum_{j\in J} a_{2j}x_j \geq b_2 - M(1-y) \end{cases}$$

4. Conditional constraints

$$\text{If} \quad (1) \quad \left(\sum_{j\in J} a_{1j}x_j \le b_1\right) \quad \text{is satisfied,}$$

$$\text{then} \quad (2) \quad \left(\sum_{j\in J} a_{2j}x_j \le b_2\right) \quad \text{must also be satisfied.}$$

is equivalent to

$$\left(\sum_{j\in J} a_{1j}x_j > b_1\right) \quad \text{or} \quad \left(\sum_{j\in J} a_{2j}x_j \le b_2\right) \quad \text{must hold.}$$

4. Conditional constraints

$$\sum_{j \in J} a_{1j} x_j \geq b_1 + \epsilon, \quad or \quad \sum_{j \in J} a_{2j} x_j \leq b_2 \quad must\ hold.$$

Indicator variable

$$\sum_{j \in J} a_{1j} x_j \geq b_1 + \epsilon - Ly$$

$$\sum_{j \in J} a_{2j} x_j \leq b_2 + M(1 - y)$$

5. Elimination of products of variables

5.1 Two binary variables

5.2 One binary and one continuous variable

5.3 Two continuous variables

5.1 Two binary variables

Let $b_1, b_2 \in \{0,1\}$ and $z = b_1 . b_2$

$$z = b_1 . b_2 \quad \Leftrightarrow \quad \begin{array}{c} y \leq b_1 \\ y \leq b_2 \\ y \geq b_1 + b_2 - 1 \\ y \in \{0,1\} \end{array}$$

Validation?

Two binary variables (Validation)

b_1	b_2	$z = b_1 b_2$	constraints	y
0	0	0	$y \leq 0$ $y \leq 0$ $y \geq -1$	0
0	1	0	$y \leq 0$ $y \leq 1$ $y \geq 0$	0
1	0	0	$y \leq 1$ $y \leq 0$ $y \geq 0$	0
1	1	1	$y \leq 1$ $y \leq 1$ $y \geq 1$	1

5.2 One binary and one continuous variable

Let $b \in \{0,1\}$, $0 \leq x \leq U$, and $z = bx$

$$z = bx \quad \Leftrightarrow \quad \begin{array}{c} y \leq Ub \\ y \leq x \\ y \geq x - U(1-b) \\ y \geq 0 \end{array}$$

Validation?

Validation

b	x	$z = bx$	constraints	y
0	$0 \leq x \leq U$	0	$y \leq 0$ $y \leq x$ $y \geq x - U$ $y \geq 0$	0
1	$0 \leq x \leq U$	$0 \leq z(= x) \leq U$	$y \leq U$ $y \leq x$ $y \geq x$ $y \geq 0$	$0 \leq y(= x) \leq U$

5.3 Two continuous variables

Let $x_1, x_2 \in \mathbb{R}$ and $z = x_1 x_2$

Step 1.

$$y_1 = \frac{1}{2}(x_1 + x_2), y_1 \in \mathbb{R}$$

$$y_2 = \frac{1}{2}(x_1 - x_2), y_2 \in \mathbb{R}$$

Step 2. $z = x_1 x_2 = y_1^2 - y_2^2$

Step 3. solve the non-linear problem via *separable programming* (see Taha 2005)

Bounds on x_1 and x_2

If $\mathrm{L}_1 \leq x_1 \leq U_1, L_2 \leq x_2 \leq U_2$

Then

$$\frac{1}{2}(L_1 + L_2) \leq y_1 \leq \frac{1}{2}(M_1 + M_2)$$

$$\frac{1}{2}(L_1 - M_2) \leq y_2 \leq \frac{1}{2}(M_1 - L_2)$$

5.3 Two continuous variables (Special case)

The product $x_1 x_2$ can be replaced by a single variable y whenever:

1. The lower bounds L_1 and L_2 are nonnegative
2. One of the variables, say x_1, is not referenced in any other term except in products of the above form.

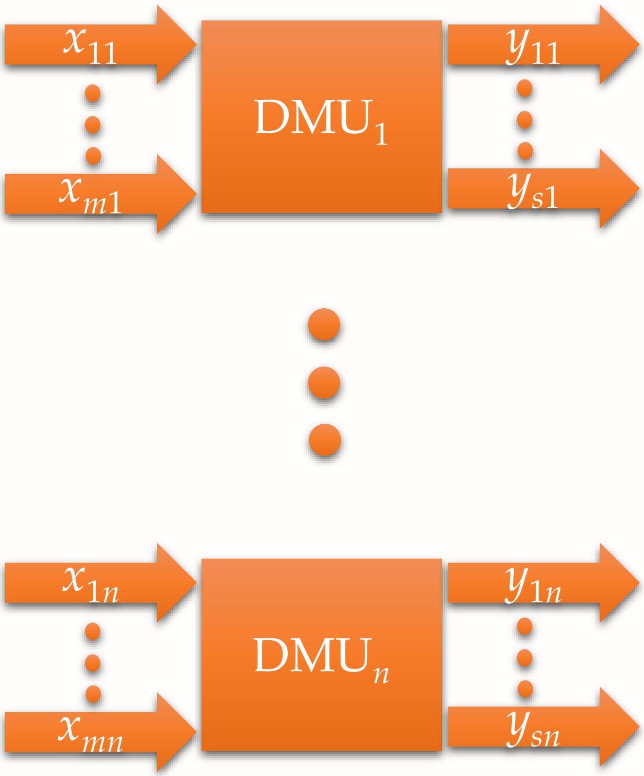
Then substituting for y and adding the following constraint

$$L_1 x_2 \leq y \leq U_1 x_2$$

Applications in DEA

- Flexible measures
- Selective measures
- Best efficient unit

Multiplier form of the CCR model



$$\max \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}$$

s.t.

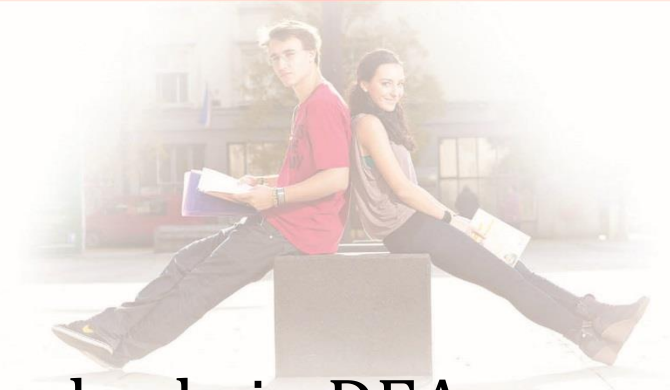
$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1 \qquad j = 1, \dots, n$$

$$v_i \geq 0 \qquad i = 1, \dots, m$$

$$u_r \geq 0 \qquad r = 1, \dots, s$$

Inputs and Outputs in DEA

- Smaller input amounts are preferable.
- Larger output amounts are preferable.
- I/O should reflect an analyst's or a manager's interest in the components that will enter into the relative efficiency evaluations of the DMUs.



Flexible & Selective measures

- Data play an important and critical role in DEA and selecting input and output measures is an essential issue in this method.
- The input versus output status of the chosen performance measures is known.
- In some situations, certain performance measures can play either input or output roles, which are called *flexible measures*.
- *selective measures* deal with *the rule of thumb.*

Mostafa (2009)

Inputs	*Frequency*	*%*	*Outputs*	*Frequency*	*%*
Employees	22	84.62	Loans	12	46.15
Expenses	11	42.31	Number of transactions	8	30.77
Space	5	19.23	Deposits	7	26.92
Terminals	5	19.23	Non- interest income	7	26.92
Capital	5	19.23	Interest income	5	19.23
Deposits	4	15.38	customer response	3	11.54
Assets	4	15.38	Revenues	3	11.54
Number of branches	4	15.38	Profit	3	11.54
Rent	3	11.54	Investment in securities	3	11.54
Number of accounts	3	11.54	Advances	2	7.69
Credit application	2	7.69	Current accounts	2	7.69
ATMs	2	7.69	Error corrections	1	3.85
Location	2	7.69	Liability sales	1	3.85
Costs	2	7.69	Maintenance	1	3.85
Marketing	2	7.69	Marketed balances	1	3.85
Selected financial ratios	1	3.85	Selected financial ratios	1	3.85
Supplier	1	3.85	Insurance commission	1	3.85
Acquired equipment	1	3.85	satisfaction	1	3.85
Funds from customers	1	3.85	ROA	1	3.85
Loanable funds	1	3.85	ROE	1	3.85
Counter transactions	1	3.85			
Net worth	1	3.85			
Borrowings	1	3.85			
Loans	1	3.85			
Size	1	3.85			
Sale FTE	1	3.85			
City employment rate	1	3.85			

Flexible Measures

- Beasley (1990, 1995) firstly faced with a data selection issue in DEA. He found that *research income* measure in the evaluation of research productivity by universities can be considered either as input or output.
- Some other such measures are:
  - *Uptime* measure in evaluating robotics installations (Cook et al., 1992)
  - *Outages* measure in the evaluation of power plants (Cook et al., 1998)
  - *Deposits* measure in the evaluation of bank efficiency (Cook and Zhu 2005)
  - *medical interns* have a similar interpretation in the evaluation of hospital efficiency.

Flexible measures

ELSEVIER

Available online at www.sciencedirect.com
ScienceDirect

European Journal of Operational Research 180 (2007) 692–699

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

www.elsevier.com/locate/ejor

Decision Support

Classifying inputs and outputs in data envelopment analysis

Wade D. Cook [a,*], Joe Zhu [b]

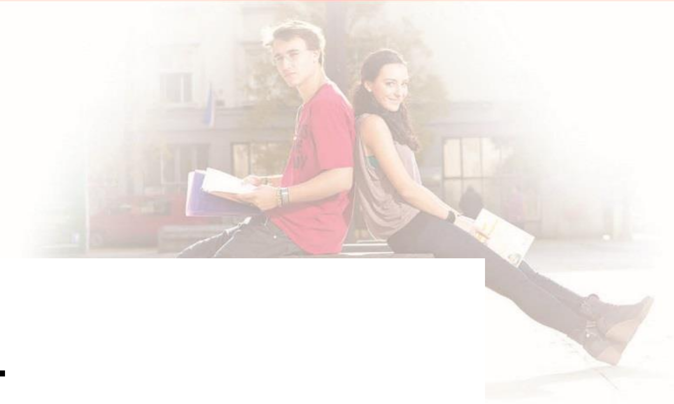
[a] *Department of Management Science, Schulich School of Business, York University, Toronto, Ont., Canada M3J 1P3*
[b] *Department of Management, Worcester Polytechnic Institute, Worcester, MA 01609, USA*

Received 1 December 2003; accepted 29 March 2006
Available online 30 June 2006

Abstract

In conventional data envelopment analysis it is assumed that the input versus output status of each of the chosen performance measures is known. In some situations, however, certain performance measures can play either input or output roles. We refer to these performance measures as flexible measures. This paper presents a modification of the standard constant returns to scale DEA model to accommodate such flexible measures. Both an individual DMU model and an aggregate model are suggested as methodologies for deriving the most appropriate designations for flexible measures. We illustrate the application of these models in two practical problem settings.

Keywords: DEA; Variable status; Flexible inputs; Flexible outputs; Efficiency



Flexible Measure Cook & Zhu (2007)

$$\max \frac{\sum_{r=1}^{s} u_r y_{ro} + \sum_{l=1}^{L} d_l w_l z_{lo}}{\sum_{i=1}^{m} v_i x_{io} + \sum_{l=1}^{L} (1-d_l) w_l z_{lo}}$$

s. t.

$$\frac{\sum_{r=1}^{s} u_r y_{rj} + \sum_{l=1}^{L} d_l w_l z_{lj}}{\sum_{i=1}^{m} v_i x_{ij} + \sum_{l=1}^{L} (1-d_l) w_l z_{lj}} \leq 1 \qquad j = 1, \dots, n$$

$$v_i \geq 0 \qquad i = 1, \dots, m$$

$$u_r \geq 0 \qquad r = 1, \dots, s$$

$$w_l \geq 0 \qquad l = 1, \dots, L$$

$$d_l \in \{0,1\} \qquad l = 1, \dots, L$$

$d_l^* = 0$ presents an input status
$d_l^* = 1$ presents an output status

There are 2^L various cases (combinations) for L flexible measures.

Charnes & Cooper (1962)'s transformation

$$\max \sum_{r=1}^{s} u_r y_{ro} + \sum_{l=1}^{L} d_l w_l z_{lo}$$

s.t.

$$\sum_{i=1}^{m} v_i x_{io} + \sum_{l=1}^{L} (1 - d_l) w_l z_{lo} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} + \sum_{l=1}^{L} d_l w_l z_{lj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{l=1}^{L} (1 - d_l) w_l z_{lj} \leq 0$$

$$v_i \geq 0 \qquad i = 1, \ldots, m$$

$$u_r \geq 0 \qquad r = 1, \ldots, s$$

$$w_l \geq 0 \qquad l = 1, \ldots, L$$

$$d_l \in \{0,1\} \qquad l = 1, \ldots, L$$

$$0 \leq \delta_l \leq M d_l$$

$$\delta_l \leq w_l \leq \delta_l + M(1 - d_l)$$

Linearization

$$\bar{e}_o^* = \max \sum_{r=1}^{s} u_r y_{ro} + \sum_{l=1}^{L} \delta_l z_{lo}$$

s.t.

$$\sum_{i=1}^{m} v_i x_{io} + \sum_{l=1}^{L} w_l z_{lo} - \sum_{l=1}^{L} \delta_l z_{lo} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} + 2\sum_{l=1}^{L} \delta_l z_{lj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{l=1}^{L} w_l z_{lj} \le 0 \qquad j = 1, \dots, n$$

$$0 \le \delta_l \le M d_l \qquad l = 1, \dots, L$$

$$\delta_l \le w_l \le \delta_l + M(1 - d_l) \qquad l = 1, \dots, L$$

$$v_i \ge 0 \qquad i = 1, \dots, m$$

$$u_r \ge 0 \qquad r = 1, \dots, s$$

$$w_l \ge 0 \qquad l = 1, \dots, L$$

$$d_l \in \{0,1\} \qquad l = 1, \dots, L$$

$d_l^* = 0 \rightarrow \delta_l^* = 0 \rightarrow z_l$ presents an input status

$d_l^* = 1 \rightarrow w_l^* = 0 \rightarrow z_l$ presents an output status

overall input v.s. output status

Let $J_{in}(l) = \{o: d_l^* = 0\}; J_{out}(l) = \{o: d_l^* = 1\}$

1. If $|J_{in}(l)| > |J_{out}(l)|$, then flexible measure l must be selected as input.
2. If $|J_{in}(l)| < |J_{out}(l)|$, then flexible measure l must be selected as output.
3. $|J_{in}(l)| = |J_{out}(l)|$?

Aggregated v.s. Individual

$$\max \frac{\sum_{r=1}^{s} u_r\left(\sum_{j=1}^{n} y_{rj}\right) + \sum_{l=1}^{L} d_l w_l\left(\sum_{j=1}^{n} z_{lj}\right)}{\sum_{i=1}^{m} v_i\left(\sum_{j=1}^{n} x_{ij}\right) + \sum_{l=1}^{L}(1-d_l) w_l\left(\sum_{j=1}^{n} z_{lj}\right)}$$

s. t.

$$\frac{\sum_{r=1}^{s} u_r y_{rj} + \sum_{l=1}^{L} d_l w_l z_{lj}}{\sum_{i=1}^{m} v_i x_{ij} + \sum_{l=1}^{L}(1-d_l) w_l z_{lj}} \leq 1 \qquad j = 1, \dots, n$$

$$v_i \geq 0 \qquad i = 1, \dots, m$$

$$u_r \geq 0 \qquad r = 1, \dots, s$$

$$w_l \geq 0 \qquad l = 1, \dots, L$$

$$d_l \in \{0,1\} \qquad l = 1, \dots, L$$

VŠB TECHNICAL UNIVERSITY OF OSTRAVA FACULTY OF ECONOMICS

Toloo (2009, 2014)

European Journal of Operational Research 198 (2009) 358–360

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

ELSEVIER

Short Communication

On classifying inputs and outputs in DEA: A revised model

Mehdi Toloo *

Department of Mathematics, Islamic Azad University of Central Tehran, Tehran, Iran

ARTICLE INFO

Article history:
Received 22 August 2008
Accepted 27 August 2008
Available online 5 September 2008

Keywords:
Data envelopment analysis
Variable status
Flexible inputs
Flexible outputs
Efficiency
Computational error

ABSTRACT

Cook and Zhu [Cook, W.D., Zhu, J., 2007. Classifying inputs and outputs in data envelopm
European Journal of Operational Research 180, 692–699] introduced a new method to determ
a measure is an input or an output. In practice, however, their method may produce incorre
scores due to a computational problem as result of introducing a large positive number to the
note introduces a revised model that does not need such a large positive number.

European Journal of Operational Research 235 (2014) 810–812

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

ELSEVIER

Short Communication

Notes on classifying inputs and outputs in data envelopment analysis: A comment

CrossMark

Mehdi Toloo *

Department of Business Administration, Faculty of Economics, Technical University of Ostrava, Ostrava, Czech Republic

ARTICLE INFO

Article history:
Received 29 December 2013
Accepted 8 January 2014
Available online 15 January 2014

Keywords:
Data envelopment analysis
Flexible measures
Classifier models

ABSTRACT

Cook and Zhu (2007) introduced an innovative method to deal with flexible measures. Toloo (2009) found a computational problem in their approach and tackled this issue. Amirteimoori and Emrouznejad (2012) claimed that both Cook and Zhu (2007) and Toloo (2009) models overestimate the efficiency. In this response, we prove that their claim is incorrect and there is no overestimate in these approaches.

Toloo (2009)

$$\max \sum_{r=1}^{s} u_r y_{ro} + \sum_{l=1}^{L} \delta_l z_{lo}$$

s.t.

$$\sum_{i=1}^{m} v_i x_{io} + \sum_{l=1}^{L} w_l z_{lo} - \sum_{l=1}^{L} \delta_l w_l z_{lo} = t$$

$$\sum_{r=1}^{s} u_r y_{rj} + 2\sum_{l=1}^{L} \delta_l z_{lj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{l=1}^{L} w_l z_{lj} \le 0 \qquad j = 1, \dots, n$$

$$0 \le \delta_l \le d_l \qquad l = 1, \dots, L$$

$$\delta_l \le w_l \le \delta_l + (1 - d_l) \qquad l = 1, \dots, L$$

$$0 \le v_i \le 1 \qquad i = 1, \dots, m$$

$$0 \le u_r \le 1 \qquad r = 1, \dots, s$$

$$0 \le w_l \le 1 \qquad l = 1, \dots, L$$

$$d_l \in \{0,1\} \qquad l = 1, \dots, L$$

Toloo (2014)

Theorem: $e_o^* = \max\{e_k^*: k = 1, \dots, 2^L\}$

University evaluation (Beasley1990)

- 50 universities
- 2 inputs: General Expenditure (GE) and Equipment Expenditure (EE)
- 3 outputs: Under Graduate Students (UGS), Post Graduate Teaching (PGT), and PG Research (PGR)
- 1 flexible measure: Research Income (RI)

DMU	d^*	$\bar{e}^*$	DMU	d^*	$\bar{e}^*$
University 1	1	1	University 26	0	0.565
University 2	0	0.640	University 27	1	0.855
University 3	0	0.837	University 28	0	1
University 4	1	0.686	University 29	1	0.825
University 5	0	1	University 30	0	0.930
University 6	0	1	University 31	1	0.776
University 7	1	1	University 32	0	0.896
University 8	1	0.812	University 33	1	1
University 9	0	1	University 34	0	1
University 10	1	0.907	University 35	1	1
University 11	0	0.890	University 36	0	0.837
University 12	1	0.709	University 37	1	0.831
University 13	0	0.803	University 38	0	0.833
University 14	0	0.768	University 39	0	0.791
University 15	0	0.704	University 40	1	0.741
University 16	0	0.543	University 41	1	1
University 17	1	0.819	University 42	0	0.847
University 18	1	0.628	University 43	0	0.921
University 19	1	1	University 44	0	1
University 20	0	0.898	University 45	0	0.889
University 21	0	0.700	University 46	0	0.851
University 22	1	0.717	University 47	1	0.688
University 23	0	0.617	University 48	0	0.939
University 24	0	1	University 49	0	1
University 25	1	1	University 50	0	0.842

Cook and Zhu (2007):
20 out of the 50 universities treat the research income measure as an output, i.e., the majority of 30 treat it as an input.

DMU	Efficiency, RI as Input	Efficiency, RI as output	d^*
University 1	1	1	0 or 1
University 2	0.615	0.640	0
University 3	0.837	0.663	0
University 4	0.645	0.686	1
University 5	1	0.893	0
University 6	1	1	0 or 1
University 7	1	1	0 or 1
University 8	0.750	0.812	1
University 9	1	0.658	0
University 10	0.892	0.907	1
University 11	0.890	0.747	0
University 12	0.691	0.709	1
University 13	0.803	0.772	0
University 14	0.768	0.702	0
University 15	0.704	0.688	0
University 16	0.543	0.520	0
University 17	0.536	0.819	1
University 18	0.593	0.628	1
University 19	1	1	0 or 1
University 20	0.858	0.898	0
University 21	0.700	0.669	0
University 22	0.664	0.717	1
University 23	0.617	0.560	0
University 24	0.484	1	0
University 25	0.952	1	1

DMU	Efficiency, RI as Input	Efficiency, RI as output	d^*
University 26	0.425	0.565	0
University 27	0.853	0.855	1
University 28	1	0.809	0
University 29	0.775	0.825	1
University 30	0.831	0.930	0
University 31	0.728	0.776	1
University 32	0.896	0.841	0
University 33	1	1	0 or 1
University 34	1	1	0 or 1
University 35	1	1	0 or 1
University 36	0.837	0.735	0
University 37	0.782	0.831	1
University 38	0.833	0.806	0
University 39	0.791	0.789	0
University 40	0.740	0.741	1
University 41	1	1	0 or 1
University 42	0.847	0.835	0
University 43	0.921	0.643	0
University 44	1	1	0 or 1
University 45	0.883	0.889	0
University 46	0.848	0.851	0
University 47	0.655	0.688	1
University 48	0.939	0.883	0
University 49	1	0.637	0
University 50	0.842	0.835	0

Envelopment form of the CCR model

$$
\begin{aligned}
& \min \theta \\
& \text{s.t.} \\
& \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io} && i = 1, \dots, m \\
& \textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} && r = 1, \dots, s \\
& \lambda_j \ge 0 && j = 1, \dots, n
\end{aligned}
$$

VŠB TECHNICAL UNIVERSITY OF OSTRAVA FACULTY OF ECONOMICS

Flexible measure in envelopment form

$$\begin{cases} \sum_{j=1}^{n} \lambda_j z_{lj} \leq \theta z_{lo} & z_l \text{ as input} \\ \sum_{j=1}^{n} \lambda_j z_{lj} \geq z_{lo} & z_l \text{ as output} \end{cases}$$

$$\sum_{j=1}^{n} \lambda_j z_{lj} \leq \theta z_{lo} + M\bar{d}_l$$

$$\sum_{j=1}^{n} \lambda_j z_{lj} \geq z_{lo} - M(1 - \bar{d}_l)$$

Toloo (2012)

Computers and Mathematics with Applications 63 (2012) 1104–1110

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

ELSEVIER

Alternative solutions for classifying inputs and outputs in data envelopment analysis

Mehdi Toloo

Department of Mathematics, Islamic Azad University, Central Tehran Branch, Tehran, Iran

ARTICLE INFO

Article history:
Received 1 May 2011
Received in revised form 6 December 2011
Accepted 6 December 2011

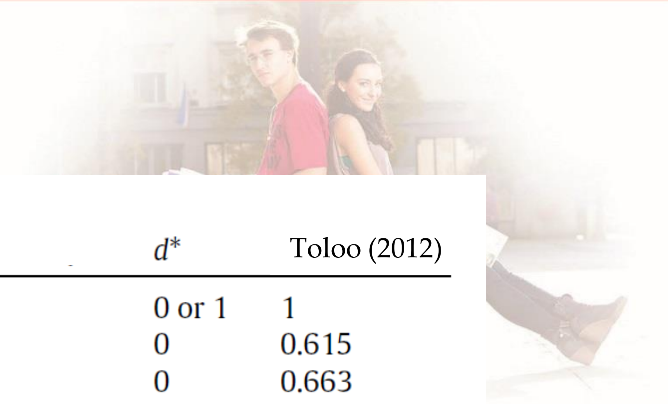
Keywords:
Data envelopment analysis
Flexible measure
Efficiency
Mixed integer linear program

ABSTRACT

In conventional data envelopment analysis (DEA) models, a performance measure whether as an input or output usually has to be known. Nevertheless, in some cases, the type of a performance measure is not clear and some models are introduced to accommodate such flexible measures. In this paper, it is shown that alternative optimal solutions of these models has to be considered to deal with the flexible measures, otherwise incorrect results might occur. Practically, the efficiency scores of a DMU could be equal when the flexible measure is considered either as input or output. These cases are introduced and referred as share cases in this study specifically. It is duplicated that share cases must not be taken into account for classifying inputs and outputs. A new mixed integer linear programming (MILP) model is proposed to overcome the problem of not considering the alternative optimal solutions of classifier models. Finally, the applicability of the proposed model is illustrated by a real data set.

Toloo (2012)

$$
\begin{aligned}
&\underline{e}_o^* = \min \theta \\
&\text{s.t.} \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io} && i = 1, \dots, m \\
&\textstyle\sum_{j=1}^{n} \lambda_j z_{lj} \le \theta z_{lo} + M\bar{d}_l && l = 1, \dots, L \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} && r = 1, \dots, s \\
&\textstyle\sum_{j=1}^{n} \lambda_j z_{lj} \ge z_{lo} - M\left(1 - \bar{d}_l\right) && l = 1, \dots, L \\
&\bar{d}_l \in \{0,1\} && l = 1, \dots, L \\
&\lambda_j \ge 0 && j = 1, \dots, n
\end{aligned}
$$



The comparison of max and min of efficiency scores.

DMU	Efficiency, RI as input	Efficiency, RI as output	Cook & Zhu (2207)	d^*	Toloo (2012)
Uni 1	1	1	1	0 or 1	1
Uni 2	0.615	0.640	0.640	0	0.615
Uni 3	0.837	0.663	0.837	0	0.663
Uni 4	0.645	0.686	0.686	1	0.645
Uni 5	1	0.893	1	0	0.893
Uni 6	1	1	1	0 or 1	1
Uni 7	1	1	1	0 or 1	1
Uni 8	0.750	0.812	0.812	1	0.750
Uni 9	1	0.658	1	0	0.658
Uni 10	0.892	0.907	0.907	1	0.892
Uni 11	0.890	0.747	0.890	0	0.747
Uni 12	0.691	0.709	0.709	1	0.691
Uni 13	0.803	0.772	0.803	0	0.772
Uni 14	0.768	0.702	0.768	0	0.702
Uni 15	0.704	0.688	0.704	0	0.688
Uni 16	0.543	0.520	0.543	0	0.520
Uni 17	0.536	0.819	0.819	1	0.536
Uni 18	0.593	0.628	0.628	1	0.593
Uni 19	1	1	1	0 or 1	1
Uni 20	0.858	0.898	0.898	0	0.858
Uni 21	0.700	0.669	0.700	0	0.669
Uni 22	0.664	0.717	0.717	1	0.664
Uni 23	0.617	0.560	0.617	0	0.560
Uni 24	0.484	1	1	0	0.484
Uni 25	0.952	1	1	1	0.952

Technical University of Ostrava, Faculty of Economics
Sokolská třída 33, 702 00 Ostrava 1, Czech Republic
www.ekf.vsb.cz/en

Selective measures

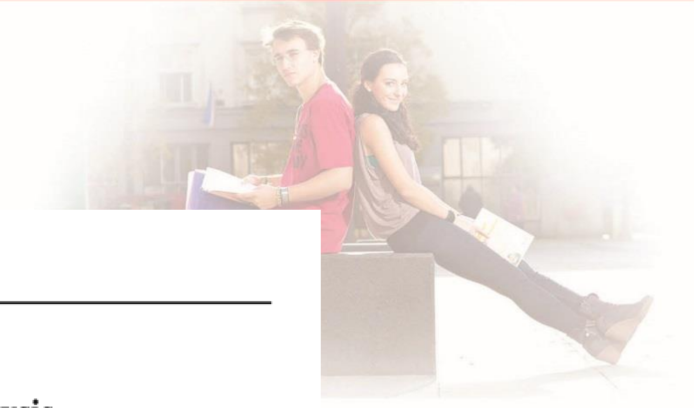
The rule of thumb in DEA:

$$n \geq \max\{3(m+s), m \times s\}$$

How to meet the rule?

illustration

- consider the problem of evaluating 50 branches of a bank with 25 inputs and 30 outputs.
- The total number of measures, i.e. 55, and DMUs do not satisfy the rule of thumb and we subsequently encounter many efficient units.
- To make the problem easier, suppose that the manager pre-selected three inputs, e.g. employees, expenses and space, and three outputs, e.g. loans, profits and deposits.
- With this assumptions, if the manager wants to select 2 out of 22 remaining inputs and 1 out of 27 remaining outputs and also consider all possible combinations of performance measures, then an optimization problem must be solved at most 196350$(= 50 \times \binom{22}{2} \times \binom{27}{1})$ times, which is illogical.



Toloo et al. (2015)

Ann Oper Res (2015) 226:623–642
DOI 10.1007/s10479-014-1714-3

Selective measures in data envelopment analysis

Mehdi Toloo · Mona Barat · Atefeh Masoumzadeh

Published online: 11 September 2014

Abstract Data envelopment analysis (DEA) is a data based mathematical approach, which handles large numbers of variables, constraints, and data. Hence, data play an important and critical role in DEA. Given a set of decision making units (DMUs) and identified inputs and outputs (performance measures), DEA evaluates each DMU in comparison with all DMUs. According to some statistical and empirical rules, a balance between the number of DMUs and the number of performance measures should exist. However, in some situations the number of performance measures is relatively large in comparison with the number of DMUs. These cases lead us to choose some inputs and outputs in a way that produces acceptable results. We refer to these selected inputs and outputs as selective measures. This paper presents an approach toward a large number of inputs and outputs. Individual DMU and aggregate models are recommended and expanded separately for developing the idea of selective measures. The practical aspect of the new approach is illustrated by two real data set applications.

Assumptions

- Let s_1 and s_2 denote subsets of outputs corresponding to fixed-output and selective-output measures, respectively. Similarly, assume that m_1 and m_2 are the parallel subsets of inputs.

Toloo et al. (2015)

$$\max \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}$$

s. t.

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1 \qquad j = 1, \dots, n$$

$$\sum_{r \in s_2} b_r^y + \sum_{i \in m_2} b_i^x \leq \min\left\{\left[\frac{n}{3}\right], 2\sqrt{n}\right\} - (|m_1| + |s_1|)$$

$$\varepsilon b_i^x \leq v_i \leq M b_i^x \qquad i \in m_2$$

$$\varepsilon b_r^y \leq u_r \leq M b_r^y \qquad r \in s_2$$

$$b_i^x, b_r^y \in \{0,1\} \qquad i \in m_2, r \in s_2$$

$$v_i, u_r \geq \varepsilon \qquad i \in m_1, r \in s_1$$

Linearization

$$\max z_{14} = \sum_{r=1}^{s} u_r y_{ro}$$

s. t.

$$\sum_{i=1}^{m} v_i x_{io} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \qquad j = 1, \dots, n$$

$$\sum_{r \in s_2} b_r^y + \sum_{i \in m_2} b_i^x \le \min\left\{\left\lfloor \frac{n}{3} \right\rfloor, 2\sqrt{n}\right\} - (|m_1| + |s_1|)$$

$$\varepsilon b_i^x \le v_i \le M b_i^x \qquad i \in m_2$$

$$\varepsilon b_r^y \le u_r \le M b_r^y \qquad r \in s_2$$

$$b_i^x, b_r^y \in \{0,1\} \qquad i \in m_2, r \in s_2$$

$$v_i, u_r \ge \varepsilon \qquad i \in m_1, r \in s_1$$

Theorem: The selecting model meets the rule of thumb

Special case: $(s_1 = m_1 = \phi)$

$$\max z_{15} = \sum_{r=1}^{s} u_r y_{ro}$$

s. t.

$$\sum_{i=1}^{m} v_i x_{io} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad j = 1, \dots, n$$

$$\sum_{r \in s_2} b_r^y + \sum_{i \in m_2} b_i^x \leq \min\left\{\left[\frac{n}{3}\right], 2\sqrt{n}\right\}$$

$$\varepsilon b_i^x \leq v_i \leq M b_i^x \quad i = 1, \dots, m$$

$$\varepsilon b_r^y \leq u_r \leq M b_r^y \quad r = 1, \dots, s$$

$$\sum_{r=1}^{s} b_r^y \geq 1$$

$$b_i^x, b_r^y \in \{0,1\} \qquad i = 1, \dots, m;\ r = 1, \dots, s$$

Toloo & Tichy(2015)

Measurement 65 (2015) 29–40

Contents lists available at ScienceDirect

Measurement

journal homepage: www.elsevier.com/locate/measurement

ELSEVIER

Two alternative approaches for selecting performance measures in data envelopment analysis

CrossMark

Mehdi Toloo *, Tomáš Tichý

Faculty of Economics, Technical University of Ostrava, Ostrava, Czech Republic

A R T I C L E I N F O

Article history:
Received 10 August 2014
Received in revised form 24 November 2014
Accepted 23 December 2014
Available online 8 January 2015

Keywords:
Decision making
Data envelopment analysis
Multiplier and envelopment forms
The rule of thumb
Selective measures
Banking industry

A B S T R A C T

Data envelopment analysis seeks a frontier to envelop all data with data acting in a critical role in the process and in such a way measures the relative efficiency of each decision making unit in comparison with other units. There is a statistical and empirical rule that if the number of performance measures is high in comparison with the number of units, then a large percentage of the units will be determined as efficient, which is obviously a questionable result. It also implies that the selection of performance measures is very crucial for successful applications. In this paper, we extend both multiplier and envelopment forms of data envelopment analysis models and propose two alternative approaches for selecting performance measures under variable returns to scale. The multiplier form of selecting model leads to the maximum efficiency scores and the maximum discrimination between efficient units is achieved by applying the envelopment form. Also individual unit and aggregate models are formulated separately to develop the idea of selective measures. Finally, in order to illustrate the potential of the proposed approaches a case study using a data from a banking industry in the Czech Republic is utilized.

Envelopment form of selecting model Toloo & Tichy (2015)

$$\max z_{17} = \theta - \varepsilon\left(\sum_{i\in m_1} s_i^x + \sum_{r\in s_1} s_r^y\right) - \varepsilon\left(\sum_{i\in m_2} d_i^x s_i^x + \sum_{r\in s_2} d_r^y s_r^y\right)$$

s.t.

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^x = \theta x_{io} \qquad i \in m_1$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^y = y_{ro} \qquad r \in s_1$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^x = \theta x_{io} + M(1 - d_i^x) \qquad i \in m_2$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^y = y_{ro} - M\left(1 - d_r^y\right) \qquad r \in s_2$$

$$\sum_{i\in m_2} d_i^x = p$$

$$\sum_{r\in s_2} d_r^y = q$$

$$d_i^x, d_r^y \in \{0,1\} \qquad i \in m_2, r \in s_2$$

$$s_i^x, s_r^y \geq 0 \qquad \forall i, \forall r$$

$$\lambda_j \geq 0 \qquad \forall j$$

Linearization

- Let $t_i^x = d_i^x s_i^x$ for $\in m_2$.

- Impose the following restrictions on the model:

$$0 \le t_i^x \le M d_i^x$$

$$s_i^x - M(1 - d_i^x) \le t_i^x \le s_i^x$$

Equivalent MBLP model

$$\max z_{18} = \theta - \varepsilon\left(\sum_{i\in m_1} s_i^x + \sum_{r\in s_1} s_r^y\right) - \varepsilon\left(\sum_{i\in m_2} t_i^x + \sum_{r\in s_2} t_r^y\right)$$

s. t.

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^x = \theta x_{io} \qquad i \in m_1$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^y = y_{ro} \qquad r \in s_1$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^x = \theta x_{io} + M(1 - d_i^x) \qquad i \in m_2$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^y = y_{ro} - M\left(1 - d_r^y\right) \qquad r \in s_2$$

$$\sum_{i\in m_2} d_i^x = p$$

$$\sum_{r\in s_2} d_r^y = q$$

$$0 \le t_i^x \le M d_i^x \qquad i \in m_2$$

$$s_i^x - M(1 - d_i^x) \le t_i^x \le s_i^x \qquad i \in m_2$$

$$0 \le t_r^y \le M d_r^y \qquad r \in s_2$$

$$s_r^y - M\left(1 - d_r^y\right) \le t_r^y \le s_r^y \qquad r \in s_2$$

$$d_i^x, d_r^y \in \{0,1\} \qquad i \in m_2, r \in s_2$$

$$t_i^x, t_r^y \ge 0 \qquad i \in m_2, r \in s_2$$

$$s_i^x, s_r^y \ge 0 \qquad \forall i, \forall r$$

$$\lambda_j \ge 0 \qquad \forall j$$

Aggregate model

$$
\begin{aligned}
&\max \theta - \varepsilon\left(\textstyle\sum_{i=1}^{m} t_i^x + \sum_{r=1}^{s} t_r^y\right) \\
&\text{s.t.} \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^x = \theta \tilde{x}_i && \forall i \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^y = \tilde{y}_r && \forall r \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^x = \theta \tilde{x}_i + M(1 - d_i^x) && \forall i \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^y = \tilde{y}_r - M\left(1 - d_r^y\right) && \forall r \\
&\textstyle\sum_{i=1}^{m} d_i^x = p \\
&\textstyle\sum_{r=1}^{s} d_r^y = q \\
&0 \le t_i^x \le M d_i^x && \forall i \\
&s_i^x - M(1 - d_i^x) \le t_i^x \le s_i^x && \forall i \\
&0 \le t_r^y \le M d_r^y && \forall r \\
&s_r^y - M\left(1 - d_r^y\right) \le t_r^y \le s_r^y && \forall r \\
&d_i^x, d_r^y \in \{0,1\} && \forall i,\ \forall r \\
&t_i^x, t_r^y \ge 0 && \forall i,\ \forall r \\
&s_i^x, s_r^y \ge 0 && \forall i,\ \forall r \\
&\lambda_j \ge 0 && \forall j
\end{aligned}
$$

where $\tilde{x}_i = \sum_{j=1}^{n} x_{ij}\ (\forall i)$, $\tilde{y}_r = \sum_{j=1}^{n} y_{rj}\ (\forall r)$, $\tilde{z}_l = \sum_{j=1}^{n} z_{lj}\ (\forall l)$.

Table D–3 Iranian bank data set and CCR-efficiency score

	Inputs						*Outputs*						
DMUs	*Employees*	*No. of accounts*	*Assets*	*Space*	*Costs*	*Expenses*	*No. of transactions*	*Deposits*	*Loans*	*Check Card*	*Credit Card*	*OTP*	*Questionable Efficiency*
1	11	1250	1753	97	10020	3137	5214	72149	57537	5105	4839	25	1
2	17	5019	2604	150	11440	4406	5343	89781	51114	8646	8364	24	0.96
3	7	3217	1155	61	8,427	2180	5145	42654	52485	2797	2697	5	1
4	12	1061	1899	105	11816	6477	3249	97812	67298	3373	3096	68	1
5	14	5219	2215	123	12426	3325	6706	77031	43487	8993	8787	58	1
6	14	1389	2357	123	9907	3757	6259	75923	41442	7604	7371	40	1
7	9	7166	1370	79	10365	2714	3652	47763	43262	3608	3497	9	0.68
8	5	1475	829	44	5283	2887	3913	45732	14237	3795	3500	32	1
9	6	1800	985	52	11061	2852	3566	55222	41062	3299	3182	15	0.93
10	6	1689	1023	52	5856	2606	4559	53323	37418	1858	1746	8	1
11	8	1780	1311	70	8745	4442	4441	69734	57883	3030	2882	23	1
12	9	2669	1536	79	7326	1989	5031	49153	47139	4811	4578	31	1
13	8	7175	1367	70	8326	3727	5053	92365	55543	6840	6588	45	1
14	7	2120	1193	61	6525	3473	4762	64235	22347	5382	5188	22	1
15	9	30618	1359	79	8158	3824	6876	89104	45717	7628	7292	105	1
16	7	1464	1111	61	11135	1524	4307	42012	73925	3187	2984	22	1
17	7	8924	1182	68	6920	3573	5331	69360	27246	3743	3524	24	1
18	7	2388	1069	61	5864	2523	4004	51438	26531	4360	4140	17	0.99
19	6	4714	992	52	5039	2398	2342	39948	20223	2688	2574	36	1
20	7	1866	1180	62	8378	3165	4238	154284	43928	4182	4008	18	1

DMUs	Inputs			Outputs			Efficiency
	No. of Employees	Assets	Costs	No. of transactions	Deposits	Loans	
1	11	1753	10020	5214	72149	57537	0.86
2	17	2604	11440	5343	89781	51114	0.70
3	7	1155	8,427	5145	42654	52485	1.00
4	12	1899	11816	3249	97812	67298	0.85
5	14	2215	12426	6706	77031	43487	0.64
6	14	2357	9907	6259	75923	41442	0.75
7	9	1370	10365	3652	47763	43262	0.65
8	5	829	5283	3913	45732	14237	1.00
9	6	985	11061	3566	55222	41062	0.88
10	6	1023	5856	4559	53323	37418	1.00
11	8	1311	8745	4441	69734	57883	0.99
12	9	1536	7326	5031	49153	47139	0.98
13	8	1367	8326	5053	92365	55543	1.00
14	7	1193	6525	4762	64235	22347	0.89
15	9	1359	8158	6876	89104	45717	1.00
16	7	1111	11135	4307	42012	73925	1.00
17	7	1182	6920	5331	69360	27246	0.99
18	7	1069	5864	4004	51438	26531	0.81
19	6	992	5039	2342	39948	20223	0.66
20	7	1180	8378	4238	154284	43928	1.00

directional distance models

$$
\begin{aligned}
& z_{(2)} = \max \beta \\
& \text{s.t.} \\
& \sum_{j \in J} \lambda_j x_{ij} \le x_{io} && i \in I \\
& \sum_{j \in J} \lambda_j y_{rj} \ge (1+\beta) y_{ro} && r \in R \\
& \sum_{j \in J} \lambda_j b_{lj} = (1-\beta) b_{lo} && l \in L \\
& \lambda_j \ge 0 && j \in J \\
& \beta \text{ free in sign}
\end{aligned}
$$

$$
\begin{aligned}
& z_{(3)} = \min \left(\sum_{i \in I} v_i x_{io} - \sum_{r \in R} u_r y_{ro} + \sum_{l \in L} w_l b_{lo} \right) \\
& \text{s.t.} \\
& \sum_{l \in L} w_l b_{lo} + \sum_{r \in R} u_r y_{ro} = 1 \\
& \sum_{i \in I} v_i x_{ij} - \sum_{r \in R} u_r y_{rj} + \sum_{l \in L} w_l b_{lj} \ge 0 && j \in J \\
& v_i, u_r \ge 0 && i \in I, r \in R \\
& w_l \text{ free in sign} && l \in L
\end{aligned}
$$

- The DD models obtain the maximum possible movement from $\text{DMU}_o = (\boldsymbol{x}_o, \boldsymbol{y}_o, \boldsymbol{b}_o)$ for $o \in J$ in the direction $(\boldsymbol{y}_o, -\boldsymbol{b}_o)$

Multi-valued measures

Table 2
Description of inputs and (un)desirable outputs.

Item		Title	Measurement
Inputs	x_1	Average number of usual weekly hours of work (15–24 years)	Hour
	x_2	Average number of usual weekly hours of work (15–64 years)	Hour
	x_3	Average number of usual weekly hours of work (15 years or over)	Hour
	x_4	Average number of usual weekly hours of work (25–64 years)	Hour
	x_5	Gross fixed capital formation	Million euro
Desirable	y_1	Production (GDP at current market prices)	Million euro
Outputs	y_2	Production (GVA at basic prices)	Million euro
Undesirable	b_1	Unemployment (15 years or over)	Thousand
Outputs	b_2	Unemployment (12 months and more)	Thousand

Contents lists available at ScienceDirect

Omega

journal homepage: www.elsevier.com/locate/omega

Multi-valued measures in DEA in the presence of undesirable outputs☆

Mehdi Toloo*, Jana Hančlová

Department of Systems Engineering, VŠB-Technical University of Ostrava, Sokolská třída 33, 702 00 Ostrava 1, Ostrava, Czechia

A R T I C L E   I N F O

Article history:
Received 5 September 2018
Accepted 17 January 2019
Available online xxx

Keywords:
DEA (Data envelopment analysis)
Directional output distance function
Undesirable outputs
Selecting models
Optimistic and pessimistic approaches
NUTS 2 (Nomenclature des unités territoriales statistiques)

A B S T R A C T

Data envelopment analysis (DEA) evaluates the relative efficiency of a set of comparable decision making units (DMUs) with multiple performance measures (inputs and outputs). Classical DEA models rely on the assumption that each DMU can improve its performance by increasing its current output level and decreasing its current input levels. However, undesirable outputs (like wastes and pollutants) may often be produced together with desirable outputs in final products which have to be minimized. On the other hands, in some real-world situations, we may encounter some specific performance measures with more than one value which are measured by various standards. In this study, we referee such measures as *multi-valued* measures which only one of their values should be selected. For instance, unemployment rate is a multi-valued measure in economic applications since there are several definitions or standards to measure it. As a result, selecting a suitable value for a multi-valued measure is a challenging issue and is crucial for successful application of DEA. The aim of this study is to accommodate multi-valued measures in the presence of undesirable outputs. In doing so, we formulate two individual and summative

Envelopment form of selecting model

$$z_{(4)} = \max \beta$$

s.t.

$$\sum_{j \in J} \lambda_j x_{ij} \leq x_{io} \qquad i \in I^F$$

$$\sum_{j \in J} \lambda_j y_{rj} \geq (1+\beta) y_{ro} \qquad r \in R^F$$

$$\sum_{j \in J} \lambda_j b_{lj} = (1-\beta) b_{lo} \qquad l \in L^F$$

$$\lambda_j \geq 0 \qquad j \in J$$

Multiplier form of selecting model

$$z_{(7)} = \min \left(\sum_{i \in I} v_i x_{io} - \sum_{r \in R} u_r y_{ro} + \sum_{l \in L} w_l b_{lo} \right)$$

s.t.

$$\sum_{l \in L} w_l b_{lo} + \sum_{r \in R} u_r y_{ro} = 1$$

$$\sum_{i \in I} v_i x_{ij} - \sum_{r \in R} u_r y_{rj} + \sum_{l \in L} w_l b_{lj} \geq 0 \qquad j \in J$$

$$0 \leq u_r \leq M d_r^y \qquad r \in R_q^S, \; q = 1, \ldots, Q$$

$$-M d_l^b \leq w_l \leq M d_l^b \qquad l \in L_p^S, \; p = 1, \ldots, P$$

$$v_i, \; u_r \geq 0 \qquad i \in I^F, r \in R^F$$

$$w_l \text{ free} \qquad l \in L^F$$

Application
Toloo & Hanclova (2019)

Table 1
Number of investigated NUTS2 regions in selected countries.

Country	Frequency	Country	Frequency	Country	Frequency	Country	Frequency
AT	5	FI	3	LT	1	RO	8
BE	11	FR	21	LU	1	SE	8
BG	6	HR	2	LV	1	SI	2
CZ	8	HU	7	MT	1	SK	4
DK	5	IE	2	NL	12	UK	27
EL	12	IT	21	PL	15		
						Total	**183**

Table 2
Description of inputs anc

Item	Measurement
Inputs	Hour
	Hour
	Hour
	Hour
	Million euro
Desirable	Million euro
Outputs	Million euro
Undesirable	Thousand
Outputs	Thousand

Table 4
Frequency of selected measures.

Measures	Envelopment selecting model (4)	Multiplier selecting model (7)
x_1	42	**146**
x_2	20	7
x_3	3	13
x_4	**118**	17
y_1	**111**	71
y_2	72	**112**
b_1	52	64
b_2	**131**	**119**

Amin (2009)

Computers & Industrial Engineering 56 (2009) 1701–1702

Contents lists available at ScienceDirect

ELSEVIER

Computers & Industrial Engineering

journal homepage: www.elsevier.com/locate/caie

Short Communication

Comments on finding the most efficient DMUs in DEA: An improved integrated model

Gholam R. Amin *

Department of Computer Engineering, Postgraduate Engineering Centre, Islamic Azad University of South, Tehran Branch, Tehran, Iran

A R T I C L E   I N F O

Article history:
Received 28 December 2007
Received in revised form 21 July 2008
Accepted 22 July 2008
Available online 30 July 2008

Keywords:
Data envelopment analysis
Integrated DEA model
Common set of weights
Most efficient DMU

A B S T R A C T

This paper improves the integrated DEA model proposed for finding the most efficient DMUs introduced by Amin and Toloo, [Amin, Gholam R., Toloo, M. (2007). Finding the most efficient DMUs in DEA: An improved integrated model. *Computers & Industrial Engineering, 52*(2), 71–77]. The paper shows the problem of using the integrated DEA model and presents an improved integrated DEA model for determining a single efficient unit. Also the paper indicates the property of the improvements mathematically and a numerical example shows the usefulness and intelligibility of the study.

Best efficient unit (Amin 2009)

$$\min d_{max}$$

s. t.

$$\sum_{i=1}^{m} v_i x_{ij} \leq 1 \qquad j = 1, \dots, n$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + d_j = 0 \qquad j = 1, \dots, n$$

$$d_{max} - d_j \geq 0 \qquad j = 1, \dots, n$$

$$\sum_{j=1}^{n} \theta_j = n - 1$$

$$\theta_j - \boxed{d_j \beta_j} = 0 \qquad j = 1, \dots, n$$

$$\theta_j \in \{0,1\} \qquad j = 1, \dots, n$$

$$\beta_j \geq 1 \qquad j = 1, \dots, n$$

$$d_j \geq 0 \qquad j = 1, \dots, n$$

$$u_r \geq \varepsilon \qquad r = 1, \dots, s$$

$$v_i \geq \varepsilon \qquad i = 1, \dots, m$$

Applied Mathematical Modelling 36 (2012) 5515–5520

ELSEVIER

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

On finding the most BCC-efficient DMU: A new integrated MIP–DEA model

Mehdi Toloo *

Department of Mathematics, Islamic Azad University, Central Tehran Branch, Tehran, Iran

A R T I C L E I N F O

Article history:
Received 23 December 2010
Received in revised form 20 November 2011
Accepted 27 November 2011
Available online 24 December 2011

Keywords:
Data envelopment analysis
Mixed integer programing model
Most BCC-efficient unit
Common set of weights
Facility layout design

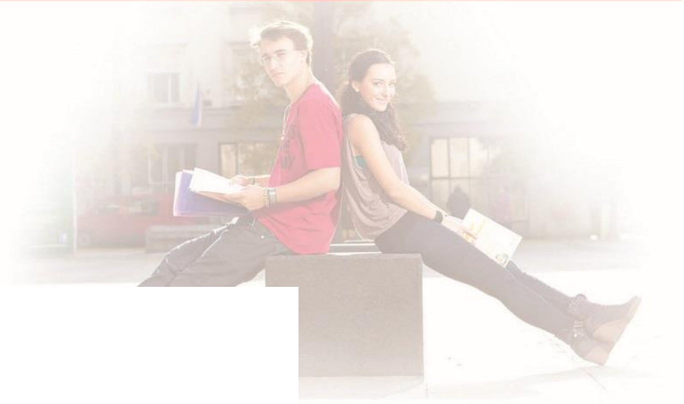
A B S T R A C T

This paper proposes a new integrated mixed integer programing – data envelopment analysis (MIP–DEA) model to improve the integrated DEA model which was introduced by Toloo & Nalchigar [M. Toloo, S. Nalchigar. A new integrated DEA model for finding most BCC-efficient DMU. Appl. Math. Model. 33 (2009) 597–60]. In this study some problems of applying Toloo & Nalchigar's model are addressed. A new integrated MIP–DEA model is then introduced to determine the most BCC-efficient decision making unit (DMU). Moreover, it is mathematically proved that the new model identifies only a single BCC-efficient DMU by a common set of optimal weights. To show applicability of proposed models, a numerical example is used which contains a real data set of nineteen facility layout designs (FLDs).

Linearization: Toloo (2012)

$$
\begin{aligned}
\min \quad & d_{\max} \\
\text{s.t.} \quad & d_{\max} - d_j \geq 0, \quad j = 1, \ldots, n \\
& \sum_{i=1}^{m} x_{ij} w_i \leq 1, \quad j = 1, \ldots, n \\
& \sum_{r=1}^{s} y_{rj} u_r - \sum_{i=1}^{m} x_{ij} w_i + d_j = 0, \quad j = 1, \ldots, n \\
& \sum_{j=1}^{n} \theta_j = n - 1 \\
& d_j \leq \theta_j \leq M d_j, \quad j = 1, \ldots, n \\
& d_j \geq 0, \theta_j \in \{0, 1\}, \quad j = 1, \ldots, n \\
& w_i \geq \varepsilon^*, \quad i = 1, \ldots, m \\
& u_r \geq \varepsilon^*, \quad r = 1, \ldots, s.
\end{aligned}
$$

Dual-role factors Cook & Zhu (2006)



$$\max \sum_{r=1}^{s} u_r y_{ro} + \sum_{k=1}^{K} \gamma_k w_{ko} - \sum_{k=1}^{K} \delta_k w_{ko}$$

s.t.

$$\sum_{i=1}^{m} v_i x_{io} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} + \sum_{k=1}^{K} \gamma_k w_{kj} - \sum_{k=1}^{K} \delta_k w_{kj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad \forall j$$

$$u_r, v_i, \gamma_k, \delta_k \geq 0 \quad \forall r, i, k$$

1. If $\delta_k^* > 0$, then the dual-role factor w_k plays the role of a non-discretionary input.
2. If $\gamma_k^* > 0$, then the dual-role factor w_k plays the role of an output.
3. If $\delta_k^* = \gamma_k^* = 0$, then the dual-role facto w_k is neglected.

Omega 77 (2018) 15–31

Contents lists available at ScienceDirect

Omega

journal homepage: www.elsevier.com/locate/omega

ELSEVIER

Dual-role factors for imprecise data envelopment analysis☆

CrossMark

Mehdi Toloo[a,*], Esmaeil Keshavarz[b], Adel Hatami-Marbini[c]

[a] Department of Systems Engineering, Faculty of Economics, VŠB-Technical University of Ostrava, Sokolska třída 33, 702 00, Ostrava, Czech Republic
[b] Department of Mathematics, Islamic Azad University, Sirjan Branch, Sirjan, Iran
[c] Department of Strategic Management and Marketing, Leicester Business School, De Montfort University, Hugh Aston Building, The Gateway, LE1 9BH Leicester, UK

A R T I C L E I N F O

Article history:
Received 15 October 2016
Accepted 5 May 2017
Available online 15 May 2017

Keywords:
Efficiency evaluation
Imprecise data
Dual-role factors
Fuzzy decision-making
Bank industry

A B S T R A C T

Efficiency analyses are crucial to managerial competency for evaluating the degree to which resources are consumed in the production process of gaining desired services or products. Among the vast available literature on performance analysis, Data Envelopment Analysis (DEA) has become a popular and practical approach for assessing the relative efficiency of Decision-Making Units (DMUs) which employ multiple inputs to produce multiple outputs. However, in addition to inputs and outputs, some situations might include certain factors to simultaneously play the role of both inputs and outputs. Contrary to conventional DEA models which account for precise values for inputs, outputs and dual-role factors, we develop a methodology for quantitatively handling imprecision and uncertainty where a degree of imprecision is not trivial to be ignored in efficiency analysis. In this regard, we first construct a pair of interval DEA models based on the pessimistic and optimistic standpoints to measure the interval efficiencies where

Imprecise Dual-role factors Toloo et al. (2018)

$$e_o^u = \max \sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} \gamma_k w_{ko}^u - \sum_{k=1}^{K} \delta_k w_{ko}^l$$

s.t.

$$\sum_{i=1}^{m} v_i x_{io}^l = 1$$

$$\sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} \gamma_k w_{ko}^u - \sum_{k=1}^{K} \delta_k w_{ko}^l - \sum_{i=1}^{m} v_i x_{io}^l \leq 0$$

$$\sum_{r=1}^{s} u_r y_{rj}^l + \sum_{k=1}^{K} \gamma_k w_{kj}^l - \sum_{k=1}^{K} \delta_k w_{kj}^u - \sum_{i=1}^{m} v_i x_{ij}^u \leq 0 \quad \forall j \neq o$$

$$u_r, v_i, \gamma_k, \delta_k \geq 0 \quad \forall r, i, k$$

Technical University of Ostrava, Faculty of Economics
Sokolská třída 33, 702 00 Ostrava 1, Czech Republic
www.ekf.vsb.cz/en

MBNLP model

$$e_o^u = \max \sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} b_k \gamma_k w_{ko}^u - \sum_{k=1}^{K} d_k \delta_k w_{ko}^l$$

s.t.

$$\sum_{i=1}^{m} v_i x_{io}^l = 1$$

$$\sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} b_k \gamma_k w_{ko}^u - \sum_{k=1}^{K} d_k \delta_k w_{ko}^l - \sum_{i=1}^{m} v_i x_{io}^l \leq 0$$

$$\sum_{r=1}^{s} u_r y_{rj}^l + \sum_{k=1}^{K} b_k \gamma_k w_{kj}^l - \sum_{k=1}^{K} d_k \delta_k w_{kj}^u - \sum_{i=1}^{m} v_i x_{ij}^u \leq 0 \qquad \forall j \neq o$$

$$b_k + d_k \leq 1 \qquad \forall k$$

$$u_r, v_i, \gamma_k, \delta_k \geq 0 \qquad \forall r, i, k$$

$$b_k, d_k \in \{0, 1\} \qquad \forall k$$

Linearization

$$e_o^u = \max \sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} \hat{\gamma}_k w_{ko}^u - \sum_{k=1}^{K} \hat{\delta}_k w_{ko}^l$$

s.t.

$$\sum_{i=1}^{m} v_i x_{io}^l = 1$$

$$\sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} \hat{\gamma}_k w_{ko}^u - \sum_{k=1}^{K} \hat{\delta}_k w_{ko}^l - \sum_{i=1}^{m} v_i x_{io}^l \le 0$$

$$\sum_{r=1}^{s} u_r y_{rj}^l + \sum_{k=1}^{K} \hat{\gamma}_k w_{kj}^l - \sum_{k=1}^{K} \hat{\delta}_k w_{kj}^u - \sum_{i=1}^{m} v_i x_{ij}^u \le 0 \quad \forall j \ne o$$

$$0 \le \hat{\gamma}_k \le M b_k \quad \forall k$$

$$0 \le \hat{\delta}_k \le M d_k \quad \forall k$$

$$b_k + d_k \le 1 \quad \forall k$$

$$u_r, v_i \ge 0 \quad \forall r, i$$

$$b_k, d_k \in \{0, 1\} \quad \forall k$$

$$\bar{e}_o^l = \min \theta$$

s.t.

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x_{ij}^l + \lambda_o x_{io}^u \leq \theta x_{io}^u \qquad \forall i$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_{rj}^u + \lambda_o y_{ro}^l \geq y_{ro}^l \qquad \forall r$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j w_{kj}^l + \lambda_o w_{ko}^u \leq w_{ko}^u + M(1-d_k) \qquad \forall k$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j w_{kj}^u + \lambda_o w_{ko}^l \geq w_{ko}^l - M(1-b_k) \qquad \forall k$$

$$b_k + d_k \leq 1 \qquad \forall k$$

$$\lambda_j \geq 0 \qquad \forall j$$

$$b_k, d_k \in \{0, 1\} \qquad \forall k$$

Selected References

1. Amin, G. R. (2009). Comments on finding the most efficient DMUs in DEA: An improved integrated model. *Computers & Industrial Engineering*, *56*(4), 1701–1702.
2. Cook, W. D., Zhu, J. (2007). Classifying inputs and outputs in data envelopment analysis. European Journal of Operational Research, 180(2), 692–699.
3. Cook, W. D., Green, R. H., Zhu, J. (2006). Dual-role factors in data envelopment analysis. *IIE Transactions*, *38*(2), 105–115.
4. Toloo, M. (2009). On classifying inputs and outputs in DEA: A revised model. *European Journal of Operational Research*, *198*(1), 358–360.
5. Toloo, M. (2014). Notes on classifying inputs and outputs in data envelopment analysis: a comment. *European Journal of Operational Research*, *235*, 810–812.
6. Toloo, M., Keshavarz, E., Hatami-Marbini, A. (2018). Dual-role factors for imprecise data envelopment analysis. Omega, 77, 15–31.
7. Toloo, M., Hančlová, J. (2019). Multi-valued measures in DEA in the presence of undesirable outputs. *Omega*, (in press).
8. Williams, H. P. (2013). Model building in mathematical programming. Wiley.

Thank you for attention