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COLORAÇÕES DE CLIQUES, DE BICLIQUES E DE ESTRELAS

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Orientadores: Celina Miraglia Herrera de Figueiredo
Raphael Carlos Santos Machado

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COLORAÇÕES DE CLIQUES, DE BICLIQUES E DE ESTRELAS

Hélio Bomfim de Macêdo Filho

Maio/2014

Orientadores: Celina Miraglia Herrera de Figueiredo
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Programa: Engenharia de Sistemas e Computação

Uma coloração de cliques de um grafo é uma coloração dos vértices em que nenhuma clique é monocromática. Uma k-coloração de cliques é uma coloração de cliques com, no máximo, k cores. Este último problema é Σ_2^P-completo, para todo $k \geq 2$ [Theoret. Comput. Sci. 412 (2011), 3487–3500]. Ao mostrarmos que 2-coloração de cliques para grafos fracamente cordais sem cliques de tamanho 2 é um problema Σ_2^P-completo, fortificamos um resultado da literatura [J. Graph Theory 62 (2009) 139–156] e respondemos a um problema proposto por Kratochvíl e Tuza [J. Algorithms 45 (2002), 40–54]. Também determinamos hierarquias de subclasses de grafos perfeitos aninhadas, de tal forma que a 2-coloração de cliques de cada classe está em uma classe de complexidade distinta, a saber Σ_2^P-completo, $\mathcal{NP}$-completo e $\mathcal{P}$. Por fim, descrevemos um algoritmo de coloração ótima de cliques para grafos livres de corda única com complexidade $O(nm)$.

A coloração de bicliques, a coloração de estrelas, a k-coloração de bicliques e a k-coloração de estrelas são análogos aos problemas anteriores, mas para bicliques e estrelas. Os dois últimos problemas são problemas Σ_2^P-completos, para todo $k \geq 2$ [arXiv 1203.2543 (2012), 1–33]. Mostramos que é co$\mathcal{NP}$-completo verificar se uma dada função que associa uma cor a cada vértice é uma coloração de bicliques (resp. coloração de estrelas). Descrevemos algoritmos de colorações ótimas de bicliques e de estrelas para potências de ciclos e potências de caminhos, todos com complexidade linear. Também descrevemos algoritmos que retornam o menor número de cores entre todas as colorações de bicliques e de estrelas para potências de ciclos e potências de caminhos, todos com complexidade constante. Por fim, descrevemos algoritmos de colorações ótimas de bicliques e de estrelas para grafos livres de corda única com complexidade $O(n^2m)$.

Abstract of Thesis presented to COPPE/UFRJ as a partial fulfillment of the requirements for the degree of Doctor of Science (D.Sc.)

CLIQUE-, BICLIQUE-, AND STAR-COLOURINGS

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A clique-colouring of a graph is a colouring of vertices, such that no clique is monochromatic. A k-clique-colouring is a clique-colouring with at most k colours. The latter is a Σ_2^P-complete problem, for every $k \geq 2$ [Theoret. Comput. Sci. 412 (2011), 3487–3500], and 2-clique-colouring is still a Σ_2^P-complete problem for perfect graphs [J. Graph Theory 62 (2009) 139–156]. We strengthen this result by showing that 2-clique-colouring weakly chordal graphs with no clique of size 2 is also a Σ_2^P-complete problem. This fact answers an open problem posed by Kratochvíl and Tuza 2 [J. Algorithms 45 (2002), 40–54]. We also determine hierarchies of nested subclasses of perfect graphs, whereby 2-clique-colouring of each graph class is in a distinct complexity class, namely Σ_2^P-complete, $\mathcal{NP}$-complete, and $\mathcal{P}$. Finally, we describe an $O(nm)$-time optimal clique-colouring algorithm for unichord-free graphs.

A biclique-colouring (resp. star-colouring) of a graph is a colouring of vertices, such that no biclique (resp. star) is monochromatic. A k-biclique-colouring (resp. k-star-colouring) is a biclique-colouring (resp. star-colouring) with at most k colours. k-biclique-colouring, as well as k-star-colouring, is a Σ_2^P-complete problem, for every $k \geq 2$ [arXiv 1203.2543 (2012), 1–33]. We show that it is co$\mathcal{NP}$-complete to check whether a given function that associates a colour to each vertex is a biclique-colouring (resp. star-colouring). Biclique-chromatic number (resp. star-chromatic number) of a graph G is the least k for which G has a k-biclique-colouring (resp. k-star-colouring). We provide linear-time optimal biclique- and star-colourings algorithms for powers of cycles and powers of paths. We also provide constant-time biclique- and star-chromatic numbers algorithms for powers of cycles and powers of paths. Finally, we describe an $O(n^2m)$-time optimal biclique- and star-colourings algorithms for unichord-free graphs.

Preface

Besides being my DSc thesis, this project also evolved to be a "mini" survey on clique-, biclique-, and star-colourings. The first chapter is devoted to a rather informal discussion (with some specks of formality) about clique-, biclique-, and star-colourings, whilst retaining their intuitive and aesthetic appeal.

Since we are dealing with colourings, even this term being somewhat symbolic, we gave a special attention to provide "real" colours (usually red, blue, and green) in almost all figures throughtout this text. The reader not properly equipped with a colour printer should not be worried, since the colours were chosen to overcome this issue. They provide different scales of gray in a black and white printer.

All chapters were constructed to be mostly independent from each other. Thereby, readers can read the chapters in every order that suits them.

Last, but not least, the reader should notice how beautiful graph theory is. In this text, an unmatched range of techniques are used. The assorted lines of graph theory are addressed in different ways, among which we highlight characterizations, constructive proofs resulting in efficient algorithms, and results of intractability in the three levels of the polynomial hierarchy. All tools follow mathematical nature, such as proofs by contradiction or induction, polynomial reductions between problems, and application of previously established results in the literature, in particular, decomposition theorems, which are vital for solutions that address problems regarding unichord-free graphs. Moreover, as links between graph theory and other branches of mathematics are becoming stronger, we address connections with number theory.

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Chapter 1

Introduction

This chapter is devoted to a mini survey on the thesis subjects.

Let $G = (V, E)$ be a simple graph with $n = |V|$ vertices and $m = |E|$ edges. Reader should be familiar with basic terminology of graph theory and of polynomial hierarchy classes that will be used in this work. All undefined notation follows that of Bondy and Murty [14] (concerning graph theory) and of Papadimitriou [81, Chapter 17] (concerning complexity classes).

We deal with three variations of graph colouring. It is one of the oldest areas of graph theory. Its origin dates back to the renowned four-colour theorem. In 1852, Augustus de Morgan sent a letter to his friend William Hamilton asking if it was possible to colour the regions of every map with four colours, satisfying the constraint that adjacent regions must have different colours. In the early days of this area of research, colouring problem was widely explored in this particular form. The area has grown as a field of mathematics and is now one of the most studied in graph theory. Its interests range from pure theoretical fields to wide application in practical situations [8, 26, 43, 60]. Subsequently, ensued the creation of some variations of graph colouring problem. They can be classified into two major groups: related to local or to global constraints.

We analyze three colouring problems related to global constraints: clique-, biclique- and star-colourings. Clique-colouring consists in assigning colours to the vertices of a graph, in such a way that all non-extensible sets of mutually adjacent vertices have at least two colours. Biclique-colouring consists in assigning colours to the vertices of a graph, in such a way that every non-extensible bipartition with 1) mutually non-adjacent vertices within a partition; and 2) mutually adjacent vertices of distinct partitions; has at least two colours. Finally, star-colouring consists in assigning colours to the vertices of a graph, in such a way that every non-extensible bipartition with 1) mutually non-adjacent vertices within a partition; 2) mutually adjacent vertices of distinct partitions; and 3) a unitary partition; has at least two colours. Since each colouring problem is a generalization of the original one, it also ranges from pure theoretical fields to wide application in practical situations. Potentially, abovementined problems are at least as hard as the original one, as we will see later in this introduction. While the traditional vertex-colouring is classified as $\mathcal{NP}$-complete problem, clique-, biclique-, and star-colouring problems are classified as Σ_2^P-complete problems [73, 95]. For a survey on colouring problems about local constraints, the interested reader may refer to Tuza's [98].

A *clique* of a graph is a maximal set of vertices that induces a complete graph with at least one edge. A *clique-colouring* of a graph G is a function that associates a colour to each vertex such that no (maximal) clique is monochromatic. A *k-clique-colouring* of a graph G is a clique-colouring of G with at most k colours.

Problem 1.1. k-clique-colouring

Input: Graph G

Output: A clique-colouring of G with at most k colours.

Clique is an important classical structure in graphs. Hence, it is natural that clique-colouring problem has been studied for a long time — see, for instance, related works [5, 29, 57, 73]. Besides recent papers regarding aspects of clique-colouring, this subject emerged in the literature in a sparse way. Tuza [97] (reprinted by Hedetniemi and Laskar [50]) solved a very particular case of the following problem posed by Gallai in a private communication.

> Let $\mathcal{G}$ be a given class of graphs, and suppose a graph $G \in \mathcal{G}$ is such that every edge of G is contained in a complete subgraph of order k. Then, there is a vertex set of at most n/k elements that meets all cliques of G.[1]

In particular, Gallai was interested in the class of chordal graphs. Tuza [97] solved the case of $k = 3$ and gave a shorter proof for the case $k = 2$, which was already proved by Aigner and Andreae [2]. After that, Lonc and Rival [64] conjectured that cocomparability graphs have a subset of vertices X, whereby X and its complement $\overline{X}$ are clique-transversals. It is a weak condition which implies that cocomparability graphs are 2-clique-colourable.

We end this very short history of clique-colouring by citing further references, in which clique-colouring have other names: *weak k-colouring* [4], *strong k-division* [51], or simply *colouring of clique-hypergraph* [57].

[1] We recall that a subset of vertices that meets all cliques is the so-called clique-transversal.

Beyond, we additionally cite several graph classes which have been studied in the context of clique-colouring.

- $(2, 1)$-polar graphs with all cliques having size at least 3 are 2-clique-colourable. See Chapter 4.
- A line graph $L(G)$ is 2-clique-colourable if, and only if, $H = L(G)$ has a 2-colouring of its edges without monochromatic triangles and H is not an odd cycle [3].
- Chordal graphs are 2-clique-colourable [82].
- Claw-free graphs of maximum degree at most four, other than odd-hole, are 2-clique-colourable [4].
- Claw-free perfect graphs are 2-clique-colourable [5].
- Claw-free planar graphs are 2-clique-colourable with the exception of odd cycles [92].
- Cocomparability graphs are 3-clique-colourable [35].
- Comparability graphs are 2-clique-colourable [34].
- Generalized split graphs are 3-clique-colourable [4].
- Planar graphs are 3-clique-colourable [75]. Moreover, it can be decided in polynomial-time whether a planar graph is 2-clique-colourable [57].
- Powers of cycles are 3-clique-colourable. Moreover, it can be decided whether a power of a cycle is 2-clique-colourable: if it is not a hole with odd order at least 5 [17].
- The complement of a line graph is 2-clique-colourable with the exception of nine small graphs [3].
- Unichord-free graphs are 3-clique-colourable. Moreover, it can be decided in polynomial-time whether a unichord-free graph is 2-clique-colourable. See Chapter 2.

In the area of clique-colouring, the long-standing open problem is a conjecture of Duffus *et al.* [34]: *the class of perfect graphs is k-clique-colourable for some constant k.* In fact, yet, there is no example of a perfect graph that is not 3-clique-colourable. Bacsó *et al.* [5] exhibited several classes of 3-clique-colourable perfect graphs. Moreover, Défossez [28] exhibited several classes of 3-clique-colourable odd-hole-free graphs.[2] We show that unichord-free perfect graphs are 2-clique-colourable (see Chapter 2). Finally, Bacsó *et al.* [5] proved that almost all perfect graphs are 3-clique-colourable.

Now, we turn our attention to biclique- and star-colourings. A *biclique* of a graph is a maximal set of vertices that induces a complete bipartite graph with at least one edge. A *biclique-colouring* of a graph G is a function that associates a colour to each vertex such that no (maximal) biclique is monochromatic. A *star* of a graph is a biclique with a unitary partition. A *star-colouring* of a graph G is a function that associates a colour to each vertex such that no (maximal) star is monochromatic. A k*-biclique-colouring* of a graph G is a biclique-colouring of G with at most k colours. A k*-star-colouring* of a graph G is a star-colouring of G with at most k colours.

Problem 1.2. k-biclique-colouring

Input: Graph G, Integer k

Output: A biclique-colouring of G with at most k colours.

Problem 1.3. k-star-colouring

Input: Graph G, Integer k

Output: A star-colouring of G with at most k colours.

Research on biclique-colouring has begun in the master thesis of Terlisky [95], under the supervision of Groshaus and Soulignac. On the other hand, research on star-colouring has started in a paper of Groshaus *et al.* [46], which results extend those of Terlisky's master thesis. Star-colouring has appeared naturally from biclique-colouring. Indeed, star and biclique are the most well-known inclusionwise complete bipartite graphs.[3]

Many other problems, very famous in the extremal graph theory literature, initially stated for cliques, have their version for bicliques [10, 56]. For instance, *Ramsey number*[4] [87]. One variation of Ramsey number is a combinatorial game [40] called on-line Ramsey number [9, 59] and it also has a version for bicliques [37]. This problem was further extended and it is, now, called *On-line Ramsey theory* [47]. Although complexity results for complete bipartite subgraph problems are mentioned by Garey and Johnson [42] and the (maximum) biclique problem is shown to be $\mathcal{NP}$-hard by Yannakakis [100], only in the last decade, (maximal) bicliques

[2] Recall that odd-hole-free graphs contain the perfect graphs, since the last ones are precisely the Berge graphs [22].

[3] Note that C_4 is a well-known complete bipartite graph, but it is not inclusionwise.

[4] Ramsey number is the smallest number r such that, in every 2-edge-colouring of the complete graph with r vertices, it is guaranteed to have a clique of size k, which edges have the same colour.

were rediscovered in the context of counting problems [44, 83], enumeration problems [32, 33, 79, 80], and intersection graphs [45].

Beyond, we may additionally cite several graph classes which have been studied in the context of biclique- and star-colourings.

- Split graphs are $(\beta+1)$-biclique-colourable, in which β is the size of the largest set of mutually true twin vertices[5] of the graph. Moreover, it can be decided in polynomial-time whether a {net[6], diamond}-free chordal graph[7] or a threshold graph[8] is β-biclique-colourable (resp. β-star-colourable).

- Unichord-free graphs are $(\beta+1)$-biclique-colourable and $(\beta+1)$-star-colourable. Moreover, it can be decided in polynomial-time whether a unichord graph is β-biclique-colourable (resp. β-star-colourable). See Chapter 2.

- A non-complete power of a path P_n^k is $max(2k+2-n, 2)$-biclique-colourable and $max(2k+2-n, 2)$-star-colourable. See Chapter 3.

- A non-complete power of a cycle is 3-biclique-colourable and 3-star-colourable. Moreover, it can be decided in polynomial-time whether a non-complete power of a cycle is 2-biclique-colourable (resp. 2-star-colourable). See Chapter 3.

An optimization problem related to clique-colouring is the clique-chromatic number of a graph G, which corresponds to the fewest number of colours among all clique-colourings of G. More formally, *clique-chromatic number* of G, denoted by $\kappa(G)$, is the least k for which G has a k-clique-colouring.

Problem 1.4. Clique-chromatic number
Input: Graph G
Output: $\kappa(G)$, which is the least k for which G has a k-clique-colouring.

Regarding clique-colouring and clique-chromatic number, an *optimal clique-colouring* of a graph G corresponds to a $\kappa(G)$-clique-colouring of G, i.e. an optimal clique-colouring of G is a clique-colouring of G with the least number of colours.

[5] Two distinct vertices u and v are *true twins*, if their closed neighbourhood coincide

[6] *Net* is a graph on vertices $\{a_1, \ldots, a_3, b_1, \ldots, b_3\}$ in which $\{a_1, a_2, a_3\}$ is a clique and the only edges between a_i and b_j are a_1b_1, a_2b_2, and a_3b_3.

[7] Diamond-free chordal graphs are precisely block graphs [6].

[8] A threshold graph is the one that can be constructed from an one-vertex graph by repeated applications of the following two operations: 1) addition of a single isolated vertex to the graph, and 2) addition of a single dominating vertex to the graph, i.e. a single vertex that is connected to all other vertices.

> **Problem 1.5.** Optimal clique-colouring
> Input: Graph G
> Output: A $\kappa(G)$-clique-colouring of G.

Following the same line of clique-chromatic number, an optimization problem related to biclique-colouring (resp. star-colouring) is biclique-chromatic number of a graph G (resp. star-chromatic number of a graph G), which corresponds to the fewest number of colours among all biclique-colourings of G (resp. all star-colourings of G). More formally, *biclique-chromatic number* of G (resp. star-chromatic-number of G), denoted by $\kappa_B(G)$ (resp. $\kappa_S(G)$), is the least k for which G has a k-biclique-colouring (resp. k-star-colouring).

> **Problem 1.6.** Biclique-chromatic number
> Input: Graph G
> Output: $\kappa_B(G)$, which is the least k for which G has a k-biclique-colouring.

> **Problem 1.7.** Star-chromatic number
> Input: Graph G
> Output: $\kappa_S(G)$, which is the least k for which G has a k-star-colouring.

Still, regarding biclique-colouring and biclique-chromatic number (resp. star-colouring and star-chromatic number), an *optimal biclique-colouring* of a graph G (resp. *optimal star-colouring* of a graph G) corresponds to a $\kappa_B(G)$-biclique-colouring of G (resp. $\kappa_S(G)$-star-colouring of G), i.e. an optimal biclique-colouring of G (resp. optimal star-colouring of G) is a biclique-colouring of G (resp. star-colouring of G) with the least number of colours.

> **Problem 1.8.** Optimal biclique-colouring
> Input: Graph G
> Output: A $\kappa_B(G)$-biclique-colouring of G.

> **Problem 1.9.** Optimal star-colouring
> Input: Graph G
> Output: A $\kappa_S(G)$-star-colouring of G.

Clique-, biclique-, and star-colourings have hypergraph-colouring versions. Recall that a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is an ordered pair in which V is a set of vertices and $\mathcal{E}$ is a set of hyperedges. A hypergraph colouring consists in assigning colours to the vertices of a graph, in such a way that no hyperedge is monochromatic.

> **Problem 1.10.** Hypergraph-chromatic number
>
> Input: Hypergraph $\mathcal{H}$
>
> Output: $\chi_H(G)$, which is the least number of colours used in the vertices of hypergraph $\mathcal{H}$, such that there is no monochromatic hyperedge.

In order to provide a hypergraph-colouring version of a clique-colouring on a graph G, we use clique-hypergraph of G, defined as follows.

Definition 1.1 (Clique-hypergraph $\mathcal{H}_C(G)$)**.** Let $G = (V, E)$ be a graph. The *clique-hypergraph* of G is $\mathcal{H}_C(G) = (V, \mathcal{E}_C)$, which hyperedge set is $\mathcal{E}_C = \{K \subseteq V \mid K \text{ is a clique of } G, |E(K)| \geq 1\}$.

In order to provide a hypergraph-colouring version of a biclique-colouring on a graph G (resp. star-colouring on a graph G), we use biclique-hypergraph of G (resp. star-hypergraph of G), defined as follows.

Definition 1.2 (Biclique-hypergraph $\mathcal{H}_B(G)$)**.** Let $G = (V, E)$ be a graph. The *biclique-hypergraph* of G is $\mathcal{H}_B(G) = (V, \mathcal{E}_B)$, which hyperedge set is $\mathcal{E}_B = \{K_B \subseteq V \mid K_B] \text{ is a biclique of } G, |E(K_B)| \geq 1\}$.

Definition 1.3 (Star-hypergraph $\mathcal{H}_S(G)$)**.** Let $G = (V, E)$ be a graph. The *star-hypergraph* of G is $\mathcal{H}_S(G) = (V, \mathcal{E}_S)$, which hyperedge set is $\mathcal{E}_S = \{K_S \subseteq V \mid K_S \text{ is a star of } G, |E(K_S)| \geq 1\}$.

A clique-colouring of a graph G is a colouring of its clique-hypergraph $\mathcal{H}_C(G)$. The same is true for biclique- and star-colourings.

Clique-, biclique-, and star-colourings are analogous problems as they refer to hypergraphs' colouring arising from graphs. In particular, hyperedges are the subsets of vertices that are cliques, bicliques, or stars in the original graph.

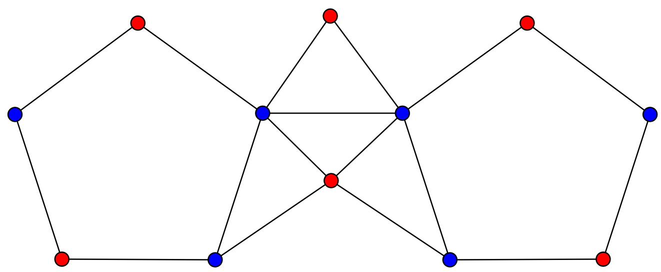
However, Problem 1.10 is more general than clique-, biclique-, and star-hypergraphs colourings. Reader should keep in mind that Problem 1.10 was stated just for didatic reasons. Please, refer to the following examples to reinforce this paragraph.

- A hypergraph with hyperedges $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, and $\{v_1, v_2, v_4\}$ is not a clique-hypergraph, since it would be a K_4 in original graph G, and clique-hypergraph of G would have hyperedge $\{v_1, v_2, v_3, v_4\}$;

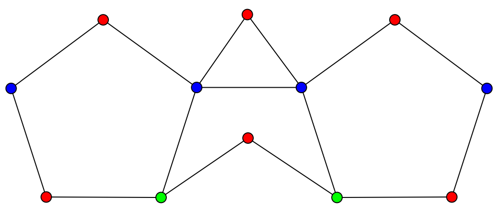
- A hypergraph with hyperedges $\{v_1, v_3\}$, $\{v_1, v_4\}$, $\{v_2, v_3\}$, and $\{v_2, v_4\}$ is not a biclique-hypergraph, since it would be a $K_{2,2}$ in original graph G, and biclique-hypergraph would have a hyperedge $\{v_1, v_2, v_3, v_4\}$.

- A hypergraph with hyperedges $\{v_1, v_3\}$ and $\{v_1, v_2\}$ is not a star-hypergraph, since it would be a $K_{1,2}$ in original graph G, and biclique-hypergraph would have a hyperedge $\{v_1, v_2, v_3\}$.

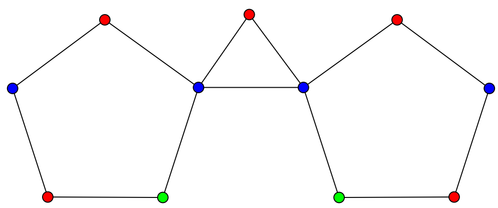
Clique-, biclique-, and star-colourings have some similarities with usual vertex-colouring; in particular, every vertex-colouring is also clique-, biclique-, and star-colourings. Then, clique-, biclique- and star-chromatic numbers are upper bounded by vertex-chromatic number. Optimal vertex- and clique-colourings coincide in the case of K_3-free graphs, while optimal vertex- and biclique-colourings (resp. optimal vertex- and star-colourings) coincide in the (much more restricted) case of $K_{1,2}$-free graphs. Actually, these graphs are simply the disjoint union of complete graphs. Notice that the triangle K_3 is the minimal complete graph that has a proper induced subgraph isomorphic to K_2, while $K_{1,2}$ is the minimal complete bipartite graph that has a proper induced subgraph isomorphic to $K_{1,1} = K_2$. However, there are also essential differences. Most remarkably, clique-, biclique-, and star-colourings of a graph may not determine clique-, biclique-, and star-colourings, respectively, for its subgraphs.[9] Subgraphs may even have a larger clique-, biclique-, and star-chromatic numbers than the original graph. Refer to Figure 1.1 and Figure 1.2.



(a) Graph with clique-chromatic number 2



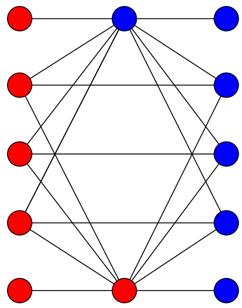
(b) Non-induced subgraph of Figure 1.1a with clique-chromatic number 3



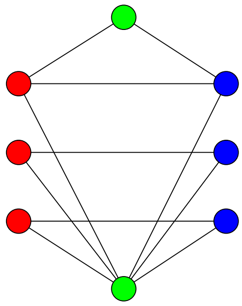
(c) Induced subgraph of Figure 1.1a with clique-chromatic number 3

Figure 1.1: Graph with non-hereditary clique-colouring

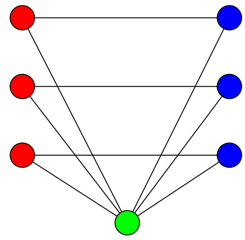
[9] Reader could check that a vertex-colouring of a graph may determine a vertex-colouring of a subgraph, since two adjacent vertices in a subgraph were two adjacent vertices in the original graph. When we are dealing with clique-, biclique-, and star-colourings, the structures where the constraints rely on may not be the same structure in the subgraph, i.e. a clique, a biclique, and a star in a graph, may not be a clique, a biclique, or a star, respectively, in a subgraph.



(a) Graph with biclique- and star-chromatic number 2



(b) Non-induced subgraph of Figure 1.2a with biclique- and star-chromatic number 3



(c) Induced subgraph of Figure 1.2a with biclique- and star-chromatic number 3

Figure 1.2: Graph with non-hereditary biclique-colouring (resp. star-colouring)

On one hand, there is not an inequality involving just clique-chromatic number and biclique-chromatic number (resp. star-chromatic number). The following examples illustrate this situation.

- Let G be a complete graph. Clique-, biclique-, and star-chromatic numbers of G are 2, n, and n, respectively.
- Let G be an odd hole. Clique-, biclique-, and star-chromatic numbers of G are 3, 2, and 2, respectively.

On the other hand, there are powers of cycles in which biclique- and star-chromatifc numbers are 2 and 3, respectively. Hence, biclique-chromatic number is not an upper bound for star-chromatic number. We refer reader to Section 3.4, Page 84 for a detailed discussion.

We consider clique-, biclique-, and star-colouring problems restricted to structured graph classes, obtaining new results for three interesting classes: *unichord-free* graphs[10], powers of paths, and powers of cycles[11]. Class of unichord-free graphs has been investigated in context of colouring problems — namely vertex-colouring [96],

[10]We define unichord-free graphs in the very beginning of Chapter 2.

[11]We define powers of paths and powers of cycles in the very beginning of Chapter 3.

edge-colouring [70] and total-colouring [69]. Regarding clique-colouring problem, we show that every unichord-free graph is 3-clique-colourable, and that 2-clique-colourable unichord-free graphs are precisely those that are perfect. This latter result is interesting because perfect unichord-free graphs are a natural subclass of diamond-free perfect graphs, which attracted much attention in context of clique-colouring. Indeed, clique-colouring diamond-free perfect graphs is an open problem [5, 27]. Regarding biclique- and star-colouring problems, we prove that every unichord-free graph G has its biclique-chromatic number equals to star-chromatic number. Moreover, biclique- and star-chromatic numbers are upper bounded by $\beta + 1$, in which β is the size of the maximum set of mutually true twin vertices of the graph. We remark that a biclique-colouring (resp. star-colouring) assigns distinct colours to all vertices of a set of mutually true twin vertices of a graph. Hence, biclique- and star-chromatic numbers are lower bounded by β. Finally, we show that it can be decided in polynomial-time whether a unichord-free graph is β-biclique-colourable (resp. β-star-colourable). Our proof strongly relies on the decomposition results of Trotignon and Vušković [96].

Clique-, biclique- and star-colouring algorithms of unichord-free graphs follow the same general strategy that is frequently used to obtain vertex-colouring algorithms in classes defined by forbidden subgraphs.[12] A specific structure F is chosen in such a way that either one of the following holds.

1. A graph in the class does not contain F and so belongs to a more restricted subclass for which the problem can be solved; or

2. A graph contains that structure and its presence entails a decomposition into smaller subgraphs in the same class.

In the algorithm of clique-colouring for unichord-free graphs, the chosen structure is a triangle. If a graph is triangle-free, then clique-colouring reduces to vertex-colouring. The existence of a triangle in a unichord-free graph implies a 1-cutset decomposition (to be defined in Chapter 2). Based on an efficient algorithm for vertex-colouring unichord-free graphs [96], the construction of an efficient clique-colouring algorithm is straightforward.

Biclique-colouring algorithm for unichord-free graphs, which is the same for star-colouring, makes a deeper use of the decomposition results of Trotignon and Vušković [96]. As a first step, we apply the strategy of considering the triangle as the chosen structure. As a second step, we consider the square[13] as the chosen structure in the class of triangle-free unichord-free graphs. As a third step, we consider the decomposition of square-free triangle-free unichord-free graphs through a proper

[12] One can easily check that if a graph is F-free, then its subgraphs are also F-free.

[13] A square is a chordless cycle of size 4.

2-cutset decomposition (to be defined in Chapter 2). Then, a biclique-colouring is constructed in the reverse order of the decompositions.

We also study biclique- and star-colouring problems on powers of paths and powers of cycles. They have been recently investigated in the context of well studied variations of colouring problems. Restricted to powers of paths: b-chromatic number [36]. Restricted to powers of cycles: chromatic number and choice number [85], total chromatic number [16], clique-chromatic number [17], and b-chromatic number [36]. We remark that total-colouring is an open and difficult problem and it remains unsolved for powers of cycles [16].

As far as possible, we have been neglecting problems in clique-, biclique-, and star-colourings that fit in complexity classes of the polynomial hierarchy above $\mathcal{P}$ (or Σ_0^P), but time has came for action. We give them a special attention in separate sections of this chapter, as follows. Section 1.1 gives an idea about complexity classes. Afterwards, we discuss complexity aspects of clique-colouring in Section 1.2, and of biclique- and star-colourings in Section 1.3. We conclude and make final adjustments in Section 1.4, whereby we deal with complexity aspects of clique-, biclique-, and star-colouring problems, restricted for some graph classes. Moreover, we point out future works.

1.1 One Step Further on Polynomial Hierarchy

Fortnow did a very good survey about polynomial-time hierarchy [39]. Moreover, there is a list of problems known to be complete for the second and higher levels of polynomial-time hierarchy, updated as necessary and presented by Schaefer and Umans [90]. However, as it can be seen in the list, there are relatively few natural complete problems known for these classes. Nevertheless, it is important to notice that those completeness results give one step further than showing the (co)$\mathcal{NP}$-hardness. They remove from consideration the problems that belong to (co)$\mathcal{NP}$, unless the polynomial hierarchy collapses. Another important remark is that if we even know that a problem belongs to some class as Σ_i^P ($i = 3$, for example), we may prove with some clever insight or deeper understanding of the problem that it actually belongs to Σ_2^P or even $\mathcal{NP}$. Nevertheless, showing Σ_3^P-completeness, that is not possible, unless the polynomial-time hierarchy collapses.

1.2 One Step Further on Clique-colouring

We start this section with a motivation of a practical situation of clique-colouring, written by Défossez [29], as follows.[14]

> In a chemical industry, it is necessary to handle dozens of chemical products. Some combinations of products may be catastrofic. In order to prevent human errors from handling these products, possibly generating harmful combinations, an alternative is to distribute the products in different storages, such that no harmful combination is possible. Obviously, one requirement is to minimize the number of storages, in order to save money and space, and to avoid unnecessary efforts in the future, as transportation among storages. Reader is invited to check that cliques are the harmful combinations, each storage represents a class of one colour, and each clique needs at least two colours, i.e. every harmful combination needs to be distributed in at least two different storages.

Suppose that there are four chemical products v_1, v_2, v_3, and v_4 with the following catastrofic combinations: $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, and $\{v_1, v_2, v_4\}$. Clique-hypergraph of this modelling would have hyperedge $\{v_1, v_2, v_3, v_4\}$, since those combinations implies a K_4 in the original graph. Notice that vertices v_1, v_2, v_3 with blue colour and vertex v_4 with red colour is a clique-colouring. Nevertheless, it can cause a catastrofic combination of v_1, v_2, and v_3, since they have the same colour.

Besides the subtle slip, the motivation has intuitive and aesthetic appeal for a possible application of clique-colouring problem.

The rest of this section is build up in order to "climb up" the polynomial-time hierarchy with problems involving clique-colouring. The outline of this section follows.

- It is co$\mathcal{NP}$-complete to check if a given colouring is a proper clique-colouring (consequence of Theorem 1.1);
- It is Σ_2^P-complete to solve the k-clique-colouring problem, for every $k \geq 2$;
- It is Π_3^P-complete to solve the list-colouring version for clique-colouring.

In order to motivate the discussion about the complexity to check whether a colouring of the vertices is a clique-colouring, we give an example which shows that the number of cliques in a graph can grow exponentially with the number of the vertices. Let G be the complement of a perfect matching of size k, i.e. graph G has

[14] This motivation is rather natural and it is very didatic, but a careful reader should notice a subtle slip.

vertices $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$ and edges $u_iu_j, v_iv_j, u_iv_j, 1 \leq i, j \leq k$ and $i \neq j$. Notice that each clique is precisely the union of either u_i or v_i, for each i. Then, we have an exponential number of cliques on the number of vertices: 2^k cliques. Figure 1.3 shows an example of the complement of a perfect matching constructed. In fact, Moon and Moser [76] show that the maximum number of (maximal) cliques in a graph on n vertices is $3^{\frac{n}{3}}$ if $n \equiv 0 \pmod 3$, $4 \cdot 3^{\frac{n}{3}-1}$ if $n \equiv 1 \pmod 3$, and $2 \cdot 3^{\frac{n}{3}}$ if $n \equiv 2 \pmod 3$.

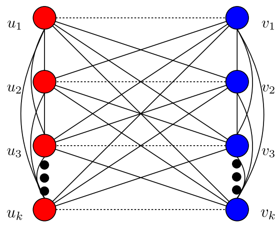


Figure 1.3: Graph with exponential number of cliques

With some clever insight or a deeper understanding of the problem, one could try to give an existential proof of an efficient algorithm (or rather an explicit algorithm) to check whether a given colouring is a clique-colouring. However, it is not possible, as we shall see later. To achieve a result in this direction, we prove the $\mathcal{NP}$-completeness of the following problem: checking whether exists a clique of a graph G contained in a given subset of vertices of G. Indeed, a function that associates a colour to each vertex of a given graph G is a clique-colouring if, and only if, there is **no** clique of G contained in a subset of the vertices of G associated with the same colour. This is the so-called *clique containment* problem. It decides whether exists a clique of a graph G contained in a given subset of vertices of G.

Problem 1.11. Clique containment
Input: Graph $G = (V, E)$ and $T \subseteq V$
Output: Is there a clique K of G such that $K \subseteq T$?

Now, we are able to state the complexity of clique containment problem.

Theorem 1.1 (Bacsó *et al.* [5])**.** *Clique containment problem is $\mathcal{NP}$-complete.*

As a consequence of Theorem 1.1, it is co$\mathcal{NP}$-complete to check whether a colouring of the vertices of a given graph G is a clique-colouring. Défossez [27] shows that clique containment problem is $\mathcal{NP}$-complete, even when restricted to complements of bipartite graphs. As an immediate corollary[15], clique containment problem is $\mathcal{NP}$-complete, even for perfect graphs.

[15] One can easily check that a bipartite graph is a perfect graph. Lovász stated that a complement of a perfect graph is a perfect graph [65], i.e. the complement of a bipartite graph is a perfect graph.

On the other hand, there are cases that it is polynomial to check whether a colouring of the vertices of a graph is a clique-colouring. For instance, assume that the list of cliques is also given as an input of clique-colouring problem, i.e. clique-hypergraph of the graph is the input of the problem. Moreover, there are examples of graphs with some restrictions that it is polynomial to check whether a colouring of the vertices is a clique-colouring. For instance, one can check the following facts.

- Diamond-free graphs have a number of cliques upper bounded by the number of edges. Each edge contained in only one clique.
- Graphs with upper bounded degree have a number of cliques upper bounded by the number of vertices. Every vertex is in, at most, a constant number of cliques.
- Graphs with clique-number upper bounded by $k \in O(1)$ have a number of cliques upper bounded by $O(n^k)$. A trivial upper bound is the sum of every subset of at most k vertices of the graph, i.e. $\sum_{j=1}^{k} \binom{n}{j} = O(n^k)$.[16]

Giving continuity to our climbing, we are now going to state the complexity of solving clique-colouring problem. It is rather natural to classify it as a problem belonging to Σ_2^P, since it is co$\mathcal{NP}$ to check whether a colouring of the vertices of a given graph is a clique-colouring.

One could think, in the light of discussion raised in Section 1.1, that it is possible to state that to solve clique-colouring problem and to check whether a colouring of the vertices is a clique-colouring belong to the same level of the hierarchy. However, this is not possible and this is a consequence of what is stated next.

Theorem 1.2 (Marx [73]). *k-clique-colouring problem is Σ_2^p-complete, for every $k \geq 2$.*[17]

Finally, in our last step in this section, we discuss the list-colouring version for clique-colouring. Given a list of k possible colours to each vertex, is there a clique-colouring constrained to choose the colour of a vertex only from the list assigned to that vertex? Moreover, is that possible for every combination of lists of size k?

[16] $\sum_{j=1}^{k} \binom{n}{j} \leq \sum_{j=1}^{k} \left(\frac{en}{j}\right)^j \leq \sum_{j=1}^{k} (en)^j \leq k(en)^k$, in which e is the Euler's number.

[17] Bacsó *et al.* [5] showed that k-clique-colouring problem, for every $k \geq 2$, is $\mathcal{NP}$-complete when the list of cliques is also given as an input of the problem. Indeed, in this case, it is polynomial to check if a k-colouring of the vertices of a graph is a k-clique-colouring. Moreover, it makes sense that the complexity of solving the problem is in a lower level, since the complexity of checking its answer is also in a lower level.

Problem 1.12. k-clique-choosability

Input: Graph $G = (V, E)$

Output: For every list $L : V \to 2^{\mathbb{N}}$ of k admissible colours for each vertex, is there a clique-colouring ψ of G with $\psi(v) \in L(v)$?

It is rather natural to classify the k-clique-choosability problem as a problem belonging to Π_3^P, since a problem belongs to Π_{i+1}^P (resp. Σ_{i+1}^P) if, and only if, checking its answer is a problem in Σ_i^P (resp. Π_i^P). Recall that checking whether a colouring is a clique-colouring is in co$\mathcal{NP}$. A clique-colouring is a natural certificate to show that a graph has a clique-colouring with given lists. Then, deciding whether a graph is clique-colourable with given lists is in Σ_2^P and deciding whether a graph is *not* clique-colourable with given lists is in Π_2^P. Finally, an uncolourable list assignment is a natural certificate to show that a graph is *not* k-clique-choosable. Then, deciding whether a graph is *not* k-clique-choosable is in Σ_3^P, and deciding whether a graph is k-clique-choosable is in Π_3^P.

One could think, in the light of the discussion raised in Section 1.1, that it is possible to state that to solve k-clique-choosability problem and to check an answer of k-clique-choosability belong to the same hierarchical level. However, again, this is not possible and it is a consequence of what is stated next.

Theorem 1.3 (Marx [73])**.** *k-clique-choosability is Π_3^P-complete, for every $k \geq 2$.*

1.3 One Step Further on Biclique-colouring and Star-colouring

Reader is invited to check that bicliques in a C_4-free graph are precisely the stars of the graph. Most of this section provides complexity results for biclique-colouring in C_4-free graphs. As a consequence, complexity results for problems involving star-colouring are obtained. Hence, our focus in this section is mainly on biclique-colouring C_4-free graphs. Consequently, on obtaining the complexity results for star-colouring as corollary.

We start this section with a motivation of a "practical" situation of biclique-colouring, inspired by the novel of Puzo [86]: dons are in an unsustainable infighting and want to organize a large convention to decide the future of local mafia. This convention will be on more than one room to avoid every conflict of two distinct groups of gangsters. Such conflict may occur only if two groups feel confident to start a fireshooting in a room. Moreover, these groups are formed in such a way that each gangster is enemy of every other gangster in the other group and there are not two gangsters, which are enemies, belonging to the same group. In addition to these requirements, reliance to generate a conflict is fulfilled when every gangster in the situation of conflict is in the same room, i.e. there is not any other gangster in the conference as a whole, which can be added to some of these two groups keeping the conflict situation. One requirement of the conference is to minimize the number of rooms used for the conference, so it is possible to save money and space, and to avoid unnecessary efforts as, for example, scheduling more talks than necessary. Reader is invited to see that bicliques are the conflict situations, each room represents a class of one colour, and each biclique needs at least two colours, i.e. every possible combination of gangsters that could generate a conflict situation needs to be distributed in at least two different rooms.

The rest of this section is also built up in order to "climb up" the polynomial-time hierarchy with problems involving biclique- and star-colourings. The outline of this section follows.

- It is co$\mathcal{NP}$-complete to check whether a given colouring is a biclique-colouring (resp. star-colouring);
- It is Σ_2^P-complete to solve the k-biclique-colouring problem (resp. k-star-colouring), for every $k \geq 2$;
- It is Π_3^P-complete to solve the list-colouring version for biclique-colouring (resp. star-colouring).

In order to motivate the discussion around the complexity to check whether a colouring of the vertices is a biclique-colouring, we give an example which shows that the number of bicliques in a graph can grow exponentially with the number of the vertices. Let G be the windmill graph, obtained as follows: k copies of a K_3 with a unique common vertex. We have two types of bicliques.

- Those that contain the unique common vertex. We have 2^k bicliques in this case, each one having two vertices in every K_3.
- Those that do not contain the unique common vertex. Bicliques are pair of vertices. We have k bicliques in this case, each one contained in a K_3.

Then, we have an exponential number of bicliques on the number of vertices: $2^k + k = O(2^{\frac{n}{2}})$ bicliques. Figure 1.4 shows an example of the windmill graph constructed. In fact, Gaspers *et al.* [44] show that the maximum number of (maximal) bicliques in a graph with n vertices is $\theta(3^{\frac{n}{3}})$.

Notice that the windmill graph is C_4-free. Hence, we also have an exponential number of stars on the number of vertices: 2^k+k stars. To the best of our knowledge, there is not a work that shows the maximum number of (maximal) stars in a graph on n vertices.

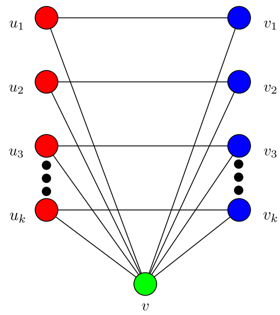


Figure 1.4: Graph with exponential number of bicliques (resp. stars)

With some clever insight or a deeper understanding of the problem, one could try to give an existential proof of an efficient algorithm (or rather an explicit algorithm) to check whether a given colouring is a biclique-colouring.

However, it is not possible, as we shall see. Biclique-colouring problem is a variation of clique-colouring problem. Hence, it is natural to investigate the complexity of biclique-colouring based on the tools that were developed to determine the complexity of clique-colouring. We have already mentioned that it is co$\mathcal{NP}$-complete to check whether a given function that associates a colour to each vertex is a clique-colouring. At that time, Bacsó *et al.* [5] used a reduction from $3DM$. Later, an alternative $\mathcal{NP}$-completeness proof was obtained by a reduction from a variation of $3SAT$, in order to construct the complement of a bipartite graph [5]. Based on this, we provide a corresponding result regarding biclique-colouring problem.

We show that it is co$\mathcal{NP}$-complete to check whether a given function that associates a colour to each vertex is a biclique-colouring. The co$\mathcal{NP}$-completeness holds even when the input is a $\{C_4, K_4\}$-free graph. Hence, it is also co$\mathcal{NP}$-complete to check whether a given function that associates a colour to each vertex is a star-colouring. To achieve such results, we prove the $\mathcal{NP}$-completeness of a problem similarly as is done to clique cointainment problem: by checking whether there is a biclique of a graph G contained in a given subset of vertices of G. Indeed, a function that associates a colour to each vertex of a given graph G is a biclique-colouring if, and only if, there is **no** biclique of G contained in a subset of the vertices of G associated with the same colour. In contrast to clique containment problem, we named this problem as biclique containment problem. It decides whether there is a biclique of a graph G contained in a given subset of vertices of G.

Problem 1.13. Biclique containment
Input: Graph $G = (V, E)$ and $T_B \subseteq V$
Output: Is there a biclique K_B of G such that $K_B \subseteq T_B$?

In order to show that biclique containment is $\mathcal{NP}$-complete, we use, in Theorem 1.4, a reduction from canonical $\mathcal{NP}$-complete problem known as *3SAT*. Now, we are able to do the proof, as follows.

Theorem 1.4 (Theorem 1, Appendix B)**.** *Biclique containment problem is $\mathcal{NP}$-complete, even if the input graph is $\{K_4, C_4\}$-free.*

Proof. Deciding whether a graph has a biclique in a given subset of vertices is in $\mathcal{NP}$: a biclique is a certificate and verifying this certificate is trivially polynomial.

We prove that biclique containment problem is $\mathcal{NP}$-hard by reducing 3SAT to it. The proof is outlined as follows. For every 3SAT instance ϕ, a graph G is constructed with a subset of vertices denoted by V', such that ϕ is satisfiable if, and only if, there is a biclique B of G such that $B \subseteq V'$.

Let n (resp. m) be the number of variables (resp. clauses) in instance ϕ. We define graph G as follows.

- For each variable x_i, $1 \leq i \leq n$, there are two adjacent vertices x_i and $\overline{x_i}$. Let L be $\{x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$.
- For each clause c_j, $1 \leq j \leq m$, there is a vertex c_j. Moreover, each c_j, $1 \leq j \leq m$, is adjacent to a vertex $l \in \{x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$ if, and only if, the literal corresponding to l is in the clause corresponding to vertex c_j. Let C be $\{c_1, \ldots, c_m\}$.
- There is a universal vertex u adjacent to all x_i, $\overline{x_i}$, $1 \leq i \leq n$, and to all c_j, $1 \leq j \leq m$.

We define the subset of vertices V' as $\{u, x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$. Refer to Figure 1.5 for an example of such construction given a instance $\phi = (x_1 \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge (x_1 \vee x_3 \vee x_5)$.

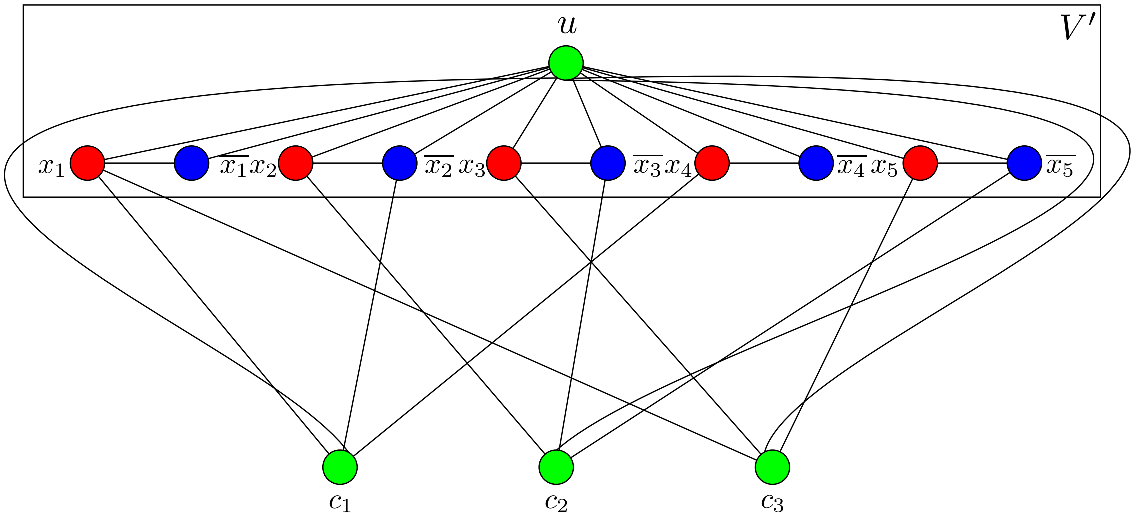


Figure 1.5: Graph construction of Theorem 1.4

We claim that instance ϕ is satisfiable if, and only if, there is a biclique of $G[V']$ that is also a biclique of G.

Each biclique B of $G[V']$ containing vertex u corresponds to a choice of precisely one vertex of $\{x_i, \overline{x_i}\}$, for each $1 \leq i \leq n$. Then, B corresponds to a truth assignment v_B that gives true value to variable x_i if, and only if, the corresponding vertex $x_i \in B$.

Notice that we may assume three properties on 3SAT instance.

- A variable and its negation do not appear in the same clause. Otherwise, every assignment of values (true or false) to such a variable satisfies the clause.
- A variable appears in at least one clause. Otherwise, every assignment of values (true or false) to such a variable is indifferent to instance ϕ.
- Two distinct clauses have at most one literal in common. Else, we can modify the instance as follows. For each clause $(l_i \vee l_j \vee l_k)$, we replace it by $(l_i \vee x'_1) \wedge (\overline{x'_1} \vee l_j \vee x'_2) \wedge (\overline{x'_2} \vee l_k)$ with variables x'_1, x'_2, and x'_3. Clearly, the number of variables and clauses created is upper bounded by 6 times the number of clauses in the original instance. Moreover, the original instance is satisfiable if, and only if, the new instance is satisfiable.

We consider bicliques of $G[V']$ according to two cases.

1. Biclique B does not contain vertex u. Then, B is precisely formed by a pair of vertices, say x_i and $\overline{x_i}$, in which $1 \leq i \leq n$. Now, our assumption says that there is a c_j adjacent to one precise vertex in $\{x_i, \overline{x_i}\}$ which implies that B is not a biclique of G.
2. Biclique B contains vertex u. Then, B is precisely formed by vertex u and one vertex of $\{x_i, \overline{x_i}\}$, for each $1 \leq i \leq n$. B is a biclique of G if, and only if, for each $1 \leq j \leq m$, there is a vertex $l \in L \cap B$ such that c_j is adjacent to l, which, in turn, occurs if, and only if, the truth assignment v_B satisfies ϕ. Therefore, B is a biclique of G if, and only if, v_B satisfies ϕ.

Now, we still have to prove that G is $\{K_4, C_4\}$-free.

For the sake of contradiction, suppose that there is a K_4 in G, say K. There are no two distinct vertices of C in K, since C is an independent set. There are no three distinct vertices of L in K, since there is a non-edge between two of these three vertices. Hence, K precisely contains vertex u, one vertex of C, and two vertices of L. Since K is a complete set, the two vertices in $L \cap K$ are adjacent and the vertex of $C \cap K$ is adjacent to both vertices of $L \cap K$. This contradicts our assumption that a variable and its negation do not appear in the same clause.

Again, for the sake of contradiction, suppose there is a C_4 in G, say H. The universal vertex u cannot belong to H. Since C is an independent set, H contains at most two vertices of C. Now, if H contains two vertices of C, then the other two vertices of H must be two literals, which contradicts our assumption that two distinct clauses have at most one literal in common. Since L induces a matching, H is not contained in L. Therefore, H contains one vertex of C and three vertices of L, which, by the construction of G, gives the final contradiction. $\square$

Then, given a $\{C_4, K_4\}$-free graph, it is co$\mathcal{NP}$-complete to check if a colouring of the vertices of G is a biclique-colouring.

Corollary 1.5 (Corollary 2, Appendix B)**.** *Let G be a $\{C_4, K_4\}$-free graph. It is co$\mathcal{NP}$-complete to check if a colouring of the vertices of G is a biclique-colouring.*

Since the constructed graph of Corollary 1.5 is C_4-free and bicliques in a C_4-free graph G are precisely stars of G, we can restate as follows below.

Corollary 1.6 (Corollary 13, Appendix B)**.** *Let G be a $\{C_4, K_4\}$-free graph. It is co$\mathcal{NP}$-complete to check if a colouring of the vertices of G is a star-colouring.*

An anonymous referee of a submitted paper for publication gave us an overview for an alternative proof of Theorem 1.4, as follows.

> "Groshaus *et al.* [46] show how to transform a QSAT2 instance $\varphi(x, y)$ into a graph $G(x, y)$ in such a way that $(\exists x)(\forall y)\varphi(x, y)$ is satisfiable if, and only if, $G(x, y)$ has a biclique-colouring with 2 colours. In the particular case in which φ has no x-literals, we obtain that $(\forall y)\varphi(y)$ is a tautology if, and only if, $G(y)$ has a biclique-colouring with 2 colours. The main observation is that $G(y)$ has a unique 2-colouring c, also obtainable in polynomial-time, that satisfies some connection constraints required for it to be a biclique-colouring. Thus, $(\forall y)\varphi(y)$ is a tautology if, and only if, c is a biclique-colouring of $G(y)$. Since the tautology problem is co$\mathcal{NP}$-complete and both $G(y)$ and c are obtainable in polynomial-time, it follows that biclique-colouring verification is co$\mathcal{NP}$-hard. It should also be noted that graph $G(y)$ is $\{C_4, K_4\}$-free."

On one hand, Groshaus *et al.* [46] do not mention this fact explicitly. Then, it was still open the question to determine the complexity of checking whether a colouring of the vertices of a graph is a biclique-colouring. On the other hand, we believe that our proof that reduces from 3-SAT is much simpler and we explicitly describe the complexity of the problem.

Giving continuity to our climbing, we are now going to state the complexity of solving biclique- and star-colouring problems. It is rather natural to classify those problems as belonging to Σ_2^P, since it is co$\mathcal{NP}$ to check whether a colouring of the vertices of a given graph is a clique-colouring.

One could think, in the light of the discussion raised in Section 1.1, that it is possible to state that to solve star-colouring problem and to check whether a colouring of the vertices is a star-colouring belong to the same hierarchical level. However, this is not possible and it is a consequence of what is stated next.

Theorem 1.7 (Groshaus *et al.* [46]). *k-star-colouring problem is Σ_2^P-complete, for every $k \geq 2$, even if the input graph is $\{K_{k+2}, C_4\}$-free.*

Since the constructed graph of Theorem 1.7 is C_4-free and bicliques in a C_4-free graph G are precisely stars of G, we can restate as follows below.

Corollary 1.8 (Groshaus *et al.* [46]). *k-biclique-colouring problem is Σ_2^P-complete, for every $k \geq 2$.*

Reader can get a big picture of this chapter results at Page 24.

Finally, in our last step in this section, we discuss the list-colouring version for biclique-colouring (resp. star-colouring). Given a list of k possible colours to each vertex, is there a biclique-colouring (resp. star-colouring) constrained to choose the colour of a vertex only from the list assigned to that vertex? Moreover, is it possible to do that for every combination of lists of size k?

> **Problem 1.14.** k-biclique-choosability
>
> Input: Graph $G = (V, E)$
>
> Output: For every list $L : V \to 2^{\mathbb{N}}$ of k admissible colours for each vertex, is there a biclique-colouring ψ of G with $\psi(v) \in L(v)$?

> **Problem 1.15.** k-star-choosability
>
> Input: Graph $G = (V, E)$
>
> Output: For every list $L : V \to 2^{\mathbb{N}}$ of k admissible colours for each vertex, is there a star-colouring ψ of G with $\psi(v) \in L(v)$?

It is rather natural to classify both problems as belonging to Π_3^P, since a problem belongs to Π_{i+1}^P (resp. Σ_{i+1}^P) if, and only if, checking its answer is a problem in Σ_i^P (resp. Π_i^P). Recall that checking whether a colouring is a biclique-colouring (resp. star-colouring) is in co$\mathcal{NP}$. A biclique-colouring (resp. star-colouring) is a natural certificate to show that a graph has a biclique-colouring (resp. star-colouring) with given lists. Then, deciding whether a graph is biclique-colourable (resp. star-colourable) with given lists is in Σ_2^P and deciding whether a graph is *not* biclique-colourable (resp. star-colourable) with given lists is in Π_2^P. Finally, an uncolourable list assignment is a natural certificate to show that a graph is *not* k-biclique-choosable (resp. k-star-choosable). Then, deciding whether a graph is *not* k-biclique-choosable (resp. k-star-choosable) is in Σ_3^P, and deciding whether a graph is k-biclique-choosable (resp. k-star-choosable) is in Π_3^P.

One could think, in the light of the discussion raised in Section 1.1, that it is possible to state that to solve k-biclique-choosability problem (resp. k-star-choosability problem) and to check an answer of k-biclique-choosability (resp. k-star-choosability) belong to the same hierarchical level. However, again, this is not possible and it is a consequence of what is stated next.

Theorem 1.9 (Groshaus *et al.* [46]). *k-star-choosability is Π_3^P-complete, for every $k \geq 2$, even if the input graph is $\{K_{k+2}, C_4\}$-free.*

Since the constructed graph of Theorem 1.9 is C_4-free and bicliques in a C_4-free graph G are precisely stars of G, we can restate as follows below.

Corollary 1.10 (Groshaus *et al.* [46]). *k-biclique-choosability is Π_3^P-complete, for every $k \geq 2$.*

1.4 Final Introductory Remarks

Concluding our trip surrounding complexities aspects of clique-, biclique- and star-colourings, we trace a really brief discussion constraining our input for some graph classes.

On one hand, k-clique-colouring problem is $\mathcal{NP}$-complete, when restricted to the following graph classes.

- $(2, 1)$-polar graphs, for $k = 2$. See Chapter 4.
- $(3, 1)$-polar graphs, for $k = 2$. See Chapter 4.
- $(3, 1)$-polar graphs with all cliques having size at least 3, for $k = 2$. See Chapter 4.
- K_4-free perfect graphs, for $k = 2$ [57].
- $\{K_4, \textit{diamond}\}$-free perfect graphs, for $k = 2$ [29].

On the other hand, k-clique-colouring problem is Σ_2^P-complete, when restricted to the following graph classes.

- Graphs with maximum degree 3, for $k = 2$ [5].
- Perfect graphs, for $k = 2$, which implies for odd-hole-free graphs as well [27].
- Weakly chordal graphs, for $k = 2$. See Chapter 4.

- Weakly chordal graphs with all cliques having size at least 3, for $k = 2$. See Chapter 4. We remark that this result is the cherry on the cake of this thesis, since it was an open problem left by Kratochvíl and Tuza to determine the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3 [57].

On one hand, k-biclique and k-star-colouring problems are $\mathcal{NP}$-complete, when restricted to the following graph classes.

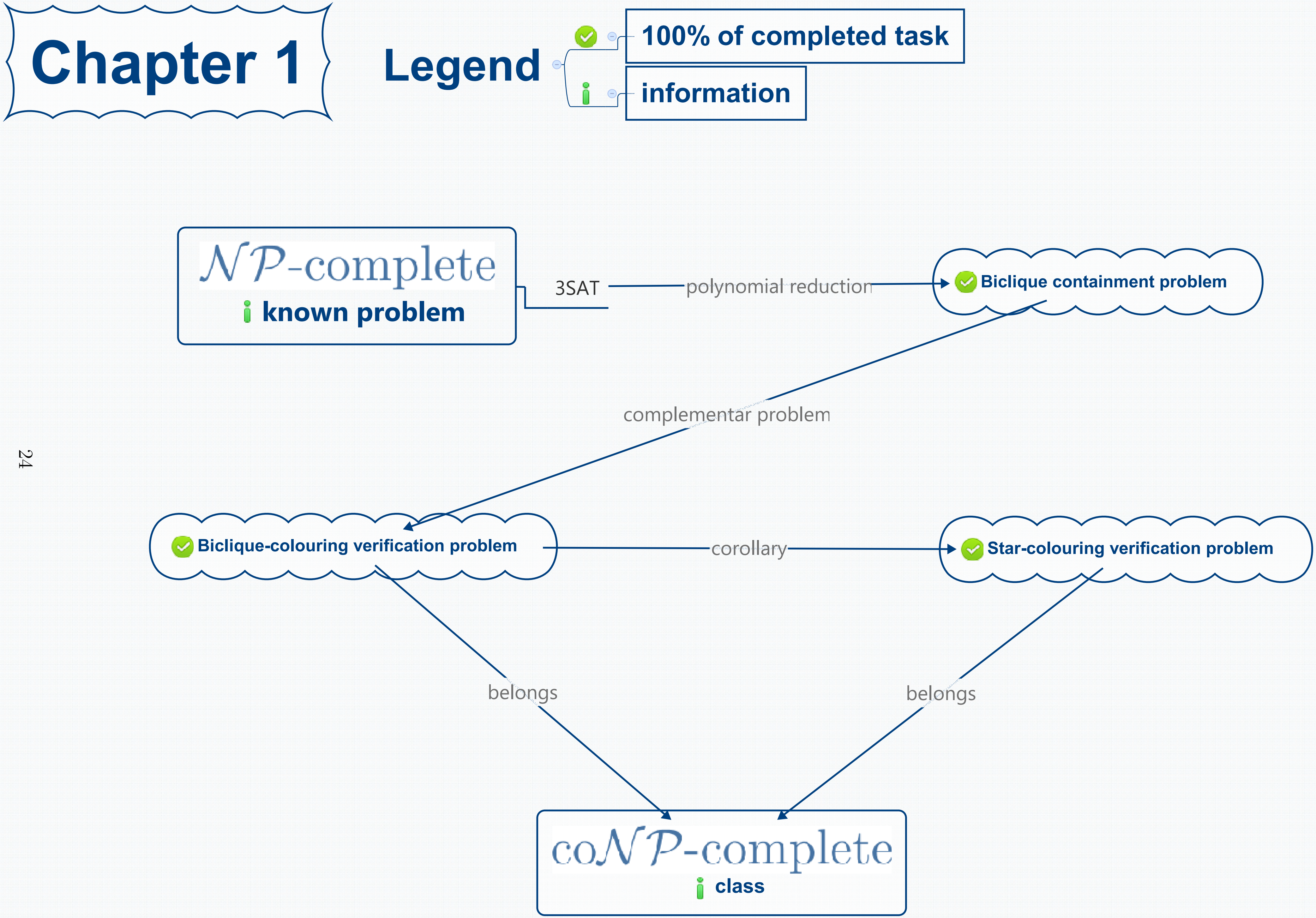
- Split graphs [46].[18]
- $\{diamond, K_{i,i}\}$-free graphs, for $i \in O(1)$ [46].
- $\overline{K_3}$-free graphs, for every $k \geq 3$ [46].

On the other hand, the k-biclique-colouring problem (resp. k-star-colouring problem) is Σ_2^P-complete, when restricted to $\{C_4, K_{k+2}\}$-free graphs [46].

We also remark that the computation of clique-chromatic number is $\mathcal{NP}$-complete for triangle-free graphs, since it is $\mathcal{NP}$-complete to compute chromatic number for triangle-free graphs [72] and triangle-free graphs have chromatic number equals to clique-chromatic number.

Future work is going to be discussed throughout the text, mainly in the end of each chapter.

[18] A split graph is a chordal graph [38]. In contrast to biclique- and star-colourings of split graphs, it is rather easy to give a 2-clique-colouring of a chordal graph. See Chapter 4.



Chapter 1
Legend
100% of completed task
information
NP-complete
known problem
3SAT
polynomial reduction
Biclique containment problem
complementar problem
Biclique-colouring verification problem
corollary
Star-colouring verification problem
belongs
belongs
coNP-complete
class

Chapter 2

Unichord-free Graphs

This chapter is devoted to the results below.

- *Clique-chromatic number of a unichord-free graph is at most 3.*
- *2-clique-colourable unichord-free graphs are precisely perfect unichord-free graphs.*
- *We describe a polynomial-time optimal clique-colouring algorithm for unichord-free graphs.*
- *Biclique-, star-, and star-biclique-chromatic numbers coincide for unichord-free graphs.*
- *Star-biclique-chromatic number of unichord-free graphs is either the increment of or exactly the size of the largest twin set.*
- *We describe polynomial-time optimal biclique-colouring algorithm for unichord-free graphs.*
- *Linear-time algorithm to fully decompose every input biconnected graph on proper 2-cutset. As an application of our algorithm, we settle an open problem posed by Trotignon and Vušković to recognize a unichord-free graph in linear-time [96, Section 5] and infer other linear-time algorithms, which are asymptotically faster than their predecessors.*

We start with the detached definition of unichord-free since it is the centerpiece of this chapter. We remark that most of our results established in this chapter are handled in a high-level approach and reader is invited to check omitted proofs and details in Appendix A. On the other hand, there are results only handled here. For instance, a proposed linear-time algorithm to fully decompose biconnected graphs on proper 2-cutset is not included in any appendix. Last, but not least, reader can get a big picture of this chapter results at Page 65 and Page 66.

Definition 2.1 (*Unichord-free* [96])**.** *Unichord-free* are graphs that do not contain, as an induced subgraph, a cycle with a unique chord.

In Figure 2.1, there are some examples of graphs to better understand the definition of unichord-free graphs.

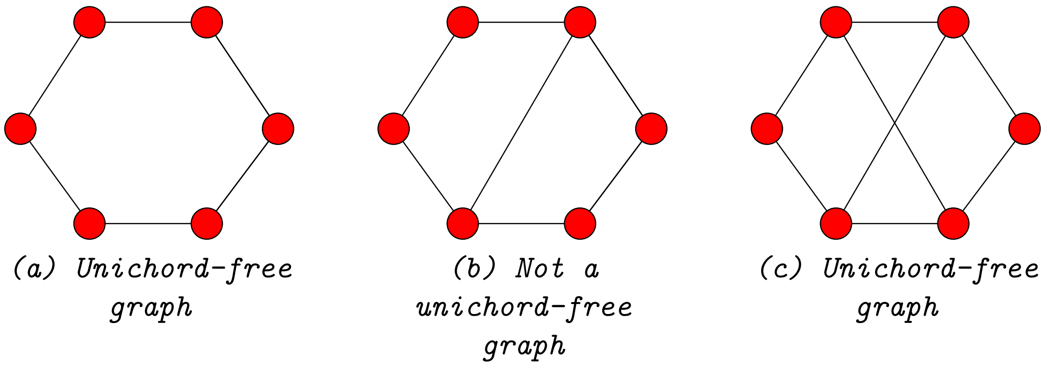


Figure 2.1: Unichord-free graphs and a not unichord-free graph

Unichord-free graphs were recently studied by Trotignon and Vušković [96]. They have showed a structural result, which is very useful to provide algorithms in the class. Every unichord-free graph is in a very restricted set of graphs or it has a decomposition. Decomposition operations are in a set of constant size. They are used to build up a graph from the basic graphs. A famous example of a structured result has been obtained for the class of perfect graphs, proving a long standing conjecture, that was open for more than four decades, called Strong Perfect Graph Theorem [22]. This thoerem contributed to the recognition of perfect graphs in $O(n^9)$-time [21].

The structural result of unichord-free graphs could be used to obtain important results concerning the time-complexity of recognition and colouring problems. For instance, one consequence of the structural result was the recognition of unichord-free graphs in $O(nm)$-time by Trotignon and Vušković [96]. At the end of this chapter, we intend to adapt theirs algorithm to infer an asymptotically faster algorithm in $O(n + m)$-time.

Machado *et al.* [70] researched whether the structural study of unichord-free could be applied for edge-colouring problem. They have showed that edge-colouring problem is $\mathcal{NP}$-complete for unichord-free graphs. Moreover, an interesting dichotomy has been obtained for edge-colouring unichord-free graphs with squares forbidden as induced subgraphs. If the maximum degree is not three, then edge-colouring problem is solvable in polynomial-time. Otherwise, edge-colouring problem is $\mathcal{NP}$-complete. Regarding total-colouring, polynomial-time algorithm is given for {unichord, square}-free graphs [68]. It is worth to point out that {unichord, square}-free graphs are the first example of class in which edge-colouring is $\mathcal{NP}$-complete and total-colouring is polynomial-time solvable [71].

We consider clique-, biclique-, and star-colourings of unichord-free graphs in this chapter. Regarding clique-colouring, the construction of an efficient algorithm for clique-colouring unichord-free graphs is straightforward due to results obtained by Trotignon and Vušković for vertex-colouring [96]. Regarding biclique- and star-colourings, we first notice that the tough decomposition for biclique-colouring unichord-free graphs is surprisingly on the articulation vertex. As we shall see, the roughness of this particular decomposition is the main reason to consider biclique- and star-colourings at the same time in this chapter. Reader may think that a biclique-colouring that is also a star-colouring would need more colours than the one that is not a star-colouring. For instance, we shall see in Chapter 3 that star-colouring a power of a cycle C_{11}^4 needs more colours than biclique-colouring it. Nevertheless, we will show that biclique-, star-, and biclique-star-chromatic numbers coincide for unichord-free graphs. It is quite interesting to notice that a further restriction makes our lives easier, since we are free to put biclique-colourings together along articulation vertices and no further colour is needed. Then, from now on, we do not consider biclique- and star-colourings separately; we consider star-biclique-colourings, which are biclique-colourings that are also star-colourings.

First, we define a *star-biclique-colouring* of a graph as a function that associates a colour to each vertex such that no (maximal) biclique and no (maximal) star, both with at least one edge, are monochromatic. Then, a star-biclique-colouring is a biclique-colouring that is also a star-colouring. A *k-star-biclique-colouring* of a graph G is a star-biclique-colouring of G with at most k colours.

Problem 2.1. k-star-biclique-colouring

Input: Graph G, Integer k

Output: A star-biclique-colouring of G with at most k colours?

An optimization problem related to star-biclique-colouring is the star-biclique-chromatic number of a graph G, which corresponds to the fewest number of colours among all star-biclique-colourings of G. More formally, *star-biclique-chromatic number* of G, denoted by $\kappa_{SB}(G)$, is the least k for which G has a k-star-biclique-colouring.

Problem 2.2. Star-biclique-chromatic number

Input: Graph G

Output: $\kappa_{SB}(G)$, which is the least k for which G has a k-star-biclique-colouring.

Regarding star-biclique-colouring and star-biclique-chromatic number, an *optimal star-biclique-colouring* of a graph G corresponds to a $\kappa_{SB}(G)$-star-biclique-colouring of G, i.e. an optimal star-biclique-colouring of G is a star-biclique-colouring of G with the least number of colours.

Problem 2.3. Optimal star-biclique-colouring

Input: Graph G

Output: A $\kappa_{SB}(G)$-star-biclique-colouring of G.

This chapter is organized as follows. Section 2.1 reviews the structure of unichord-free graphs according to the decomposition defined by Trotignon and Vušković [96]. This is very useful towards clique-, biclique- and star-colourings unichord-free graphs throughout this paper. Section 2.2 shows an overview of our strategy to obtain clique-, biclique-, and star-colourings algorithms for unichord-free graphs. Section 2.3 contains clique-colouring results for unichord-free graphs, while Section 2.4 contains biclique- and star-colourings results for unichord-free graphs. We remark that we obtain a polynomial-time algorithm that recognizes perfect unichord-free graphs as a consequence of our clique-colouring results for unichord-free graphs. Finally, Section 2.5 contains the concluding remarks and the state-of-the-art about a procedure for speeding up recognition and colouring problems algorithms related to unichord-free graphs and to other wheel-free graph subclasses.[1]

We refer reader to Table 2.1, which highlights the computational complexity of colouring problems restricted to classes related to unichord-free graphs. Notice that shadowed cells indicate our results regarding time-complexity of clique-, biclique-, and star-colourings for unichord-free graphs.

Table 2.1: Time-complexity of colourings for unichord-free graphs and subclasses

Problem \ Class	general	unichord-free	{□,unichord}-free	{△,unichord}-free
vertex-colouring	$\mathcal{NPC}$ [54]	$\mathcal{P}$ [96]	$\mathcal{P}$ [96]	$\mathcal{P}$ [96]
edge-colouring	$\mathcal{NPC}$ [52]	$\mathcal{NPC}$ [70]	$\mathcal{NPC}$ [70]	$\mathcal{NPC}$ [70]
total-colouring	$\mathcal{NPC}$ [74]	$\mathcal{NPC}$ [69]	$\mathcal{P}$ [68, 69]	$\mathcal{NPC}$ [69]
clique-colouring	$\Sigma_2^p\mathcal{C}$ [73]	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$ [96]
biclique-colouring	$\Sigma_2^p\mathcal{C}$ [95]	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$ ($\kappa_B = 2$)
star-colouring	$\Sigma_2^p\mathcal{C}$ [95]	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$ ($\kappa_S = 2$)

2.1 Preliminary Results

We review the decomposition results and its consequences for unichord-free graphs and {square,unichord}-free graphs. We start by defining basic graphs.

Definition 2.2 (Petersen graph)**.** *Petersen graph* is the cubic graph on vertices $\{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ so that both $a_1a_2a_3a_4a_5a_1$ and $b_1b_2b_3b_4b_5b_1$ are chordless cy-

[1] A *wheel graph* is a graph with $n \geq 5$ vertices, formed by connecting a single vertex to at least three vertices of a cycle of size $n-1$.

cles. Moreover, the only edges between a_i and b_j are a_1b_1, a_2b_4, a_3b_2, a_4b_5, a_5b_3. See Figure 2.2a.

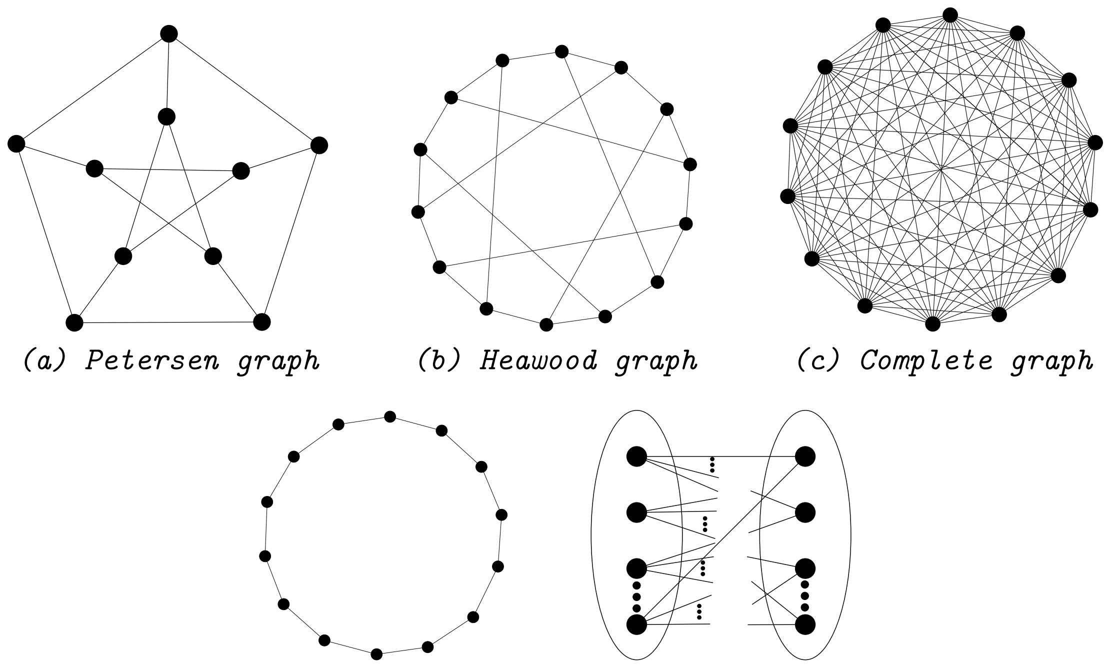
Definition 2.3 (Heawood graph)**.** *Heawood graph* is the cubic bipartite graph on vertices $\{a_1, \ldots, a_{14}\}$, so that $a_1a_2 \ldots a_{14}a_1$ is a cycle. Besides, the only other edges are a_1a_{10}, a_2a_7, a_3a_{12}, a_4a_9, a_5a_{14}, a_6a_{11}, a_8a_{13}. See Figure 2.2b.

We invite reader to check that both Petersen and Heawood graphs are unichord-free.

Definition 2.4 (*Strongly 2-bipartite graph*)**.** A graph is *strongly 2-bipartite* if it is square-free and also bipartite with bipartition (X, Y), in which every vertex in X has degree 2 and every vertex in Y has degree at least 3. See Figure 2.2e.

A strongly 2-bipartite graph is unichord-free because every chord of a cycle is an edge between two vertices of degree at least three. Then, every cycle in a strongly 2-bipartite graph is chordless.

Definition 2.5 (*Basic graph*)**.** A graph G is called *basic* if it is a complete graph, a cycle with at least five vertices, a strongly 2-bipartite graph, or an induced subgraph (not necessarily proper) of Petersen or Heawood graphs. See Figure 2.2.



(a) Petersen graph (b) Heawood graph (c) Complete graph

(d) Cycle (e) Strongly 2-bipartite graph

Figure 2.2: Unichord-free basic graphs

Now, we introduce the definitions towards the decomposition.

Definition 2.6 (*Cutset*)**.** A *cutset* S of a connected graph G is a set of vertices or edges in which its removal disconnects G.

Definition 2.7 (*Decomposition*)**.** A decomposition of a graph is the systematical 1) removal of a cutset S, and 2) a possible addition of some vertices and edges to the obtained connected components. We use this to get even smaller graphs until no graphs have cutset S.

The goal of decomposing a graph is to try a solution on the original graph by combining the solutions on decomposed graphs. The cutsets below are used in the decomposition theorems for unichord-free graphs [96]. We depict all of them in Figure 2.3. Notice that dashed lines represent non-edges. If there are dashed lines forming a $K_{2,2}$, it means that there are not any edges between those sets of vertices that the dashed lines connect.

Definition 2.8 (*1-cutset*)**.** A *1-cutset* of a connected graph $G = (V, E)$ is a vertex v such that V can be partitioned into sets X, Y and $\{v\}$, in a way that there is no edge between X and Y. We say that (X, Y, v) is a *split* of this 1-cutset. See Figure 2.3a.

Notice that 1-cutset is very known in the literature as *articulation vertex*.

Definition 2.9 (*Proper 2-cutset*)**.** A *proper 2-cutset* of a connected graph $G = (V, E)$ is a pair of non-adjacent vertices a and b, both of degree at least three, such that V can be partitioned into sets X, Y, and $\{a, b\}$ with the below properties.

- $|X|, |Y| \geq 2$;
- There is no edge between X and Y, and
- Both $G[X \cup \{a, b\}]$ and $G[Y \cup \{a, b\}]$ contain an ab-path.

We say that (X, Y, a, b) is a *split* of this proper 2-cutset. See Figure 2.3b.

Definition 2.10 (*Proper 1-join*)**.** An *1-join* of a graph $G = (V, E)$ is a partition of V into sets X and Y such that

- $\emptyset \neq A \subseteq X$, $\emptyset \neq B \subseteq Y$;
- $|X|, |Y| \geq 2$;
- There are all possible edges between A and B; and
- There is no other edge between X and Y.

A *proper 1-join* is an 1-*join* in which A and B are stable sets of G of size at least 2. We say that (X, Y, A, B) is a *split* of this (proper) 1-join. See Figure 2.3c.

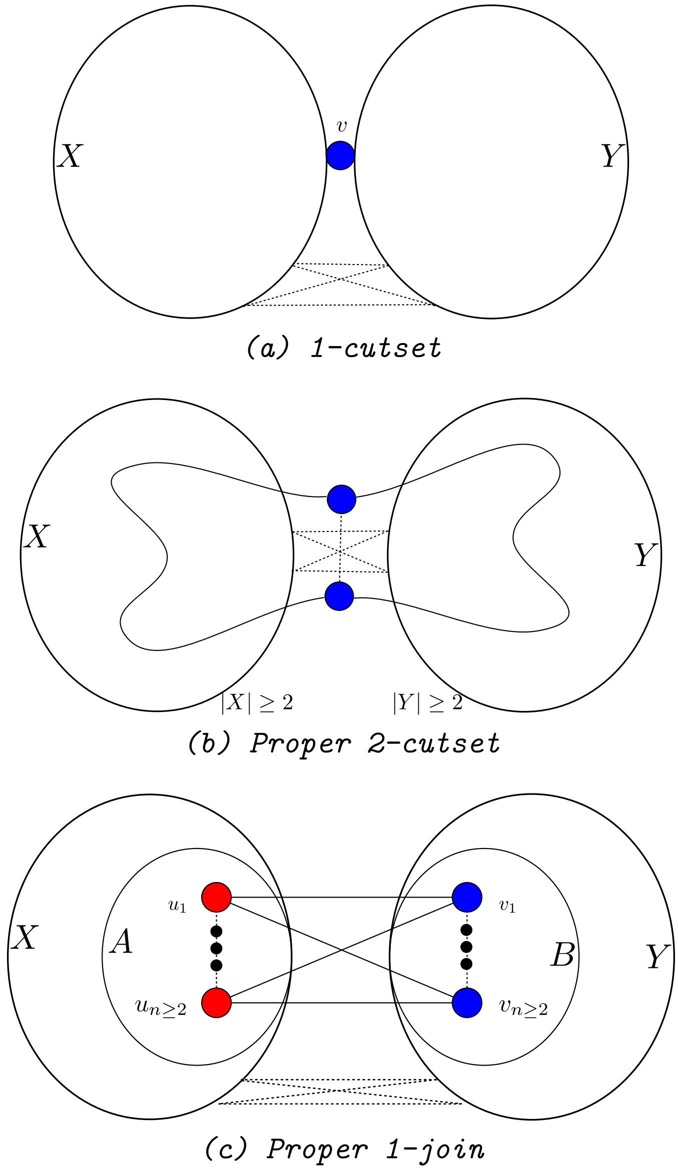


Figure 2.3: Unichord-free decompositions

The smaller graphs obtained with decompositions are called *blocks of decomposition*. They are constructed in such a way that they remain unichord-free [96] in order to be very useful to use induction in the proof of many statements.

Definition 2.11 (Blocks of decomposition G_X and G_Y).

- The *block of decomposition* G_X (resp. G_Y) of a graph G w.r.t. 1-cutset with split (X, Y, v) is $G[X \cup \{v\}]$ (resp. $G[Y \cup \{v\}]$).

- The *block of decomposition* G_X (resp. G_Y) of a graph G w.r.t. proper 1-join with split (X, Y, A, B) is the graph obtained by taking $G[X]$ (resp. $G[Y]$) and adding a vertex y adjacent to every vertex of A (resp. x adjacent to every vertex of B). Vertices x and y are called *markers*.

- The *blocks of decomposition* G_X and G_Y of a graph G w.r.t. proper 2-cutset with split (X, Y, a, b) are defined as follows. If there is a vertex c of G such that $N_G(c) = \{a, b\}$, then $G_X = G[X \cup \{a, b, c\}]$ and $G_Y = G[Y \cup \{a, b, c\}]$. Otherwise, the block of decomposition G_X (resp. G_Y) is a graph obtained by taking $G[X \cup \{a, b\}]$ (resp. $G[Y \cup \{a, b\}]$) and adding a new vertex c adjacent to a and b. Vertices c is called a *marker*.

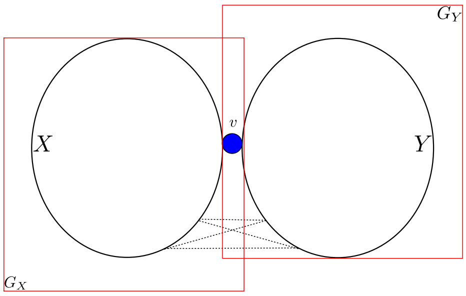
See Figure 2.4 for blocks of decomposition w.r.t. 1-cutset, proper 1-join, and proper 2-cutset. Notice that dashed lines represent non-edges. If there are dashed lines forming a $K_{2,2}$, it means that there are not any edges between those sets of vertices that the dashed lines connect. Now, we consider blocks of decomposition in a higher level of abstraction, as follows.

Definition 2.12 (Decomposition tree [96])**.** A *decomposition tree* of a graph is a rooted tree in which each node corresponds either to G or to a block of decomposition of its parent.

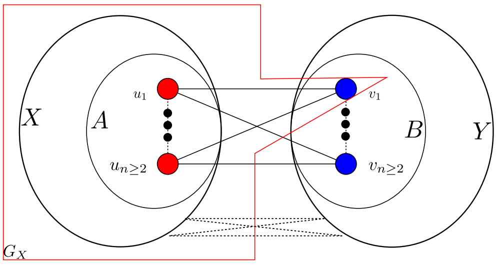
We strongly use the decomposition tree defined by Trotignon and Vušković as *proper decomposition tree* [96]. The proper decomposition tree of a connected unichord-free graph G is a rooted tree T_G such that the following holds.

1. G is the root of T_G.
2. Every node of T_G is a connected graph.
3. Every leaf of T_G is basic.
4. Every non-leaf node H of T_G is of one of the following types:
    - Type 1. The children of H in T_G are the blocks of decomposition w.r.t. 1-cutset or proper 1-join.
    - Type 2. H and all its descendants are {Petersen, triangle, square}-free and have no 1-cutset and no proper 1-join. Moreover, the children of H in T_G are the blocks of decomposition w.r.t. proper 2-cutset, and every non-leaf descendant of H is of type 2.
5. If a node of T_G is a triangle-free graph, then all its descendants are triangle-free graphs.

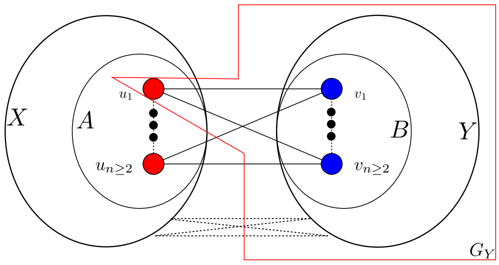
Such proper decomposition tree of a unichord-free graph is proved to always exist, being suitable for vertex-colouring of unichord-free graphs [96]. We require that, on type 2 non-leaf node H of T_G, (at least) one block of decomposition is basic. This is always possible since Machado *et al.* proved that every non-basic



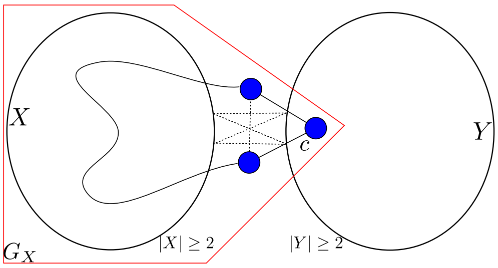
(a) Blocks of decomposition w.r.t. an 1-cutset



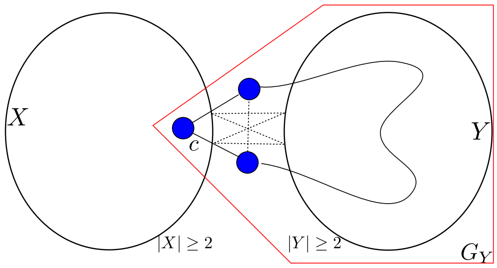
(b) Block of decomposition G_X w.r.t. a proper 1-join



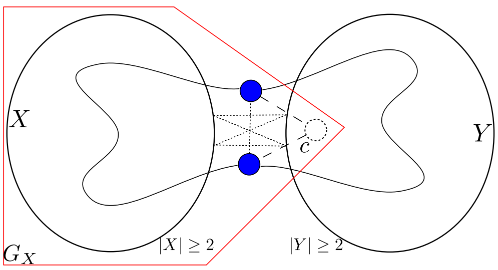
(c) Block of decomposition G_Y w.r.t. a proper 1-join



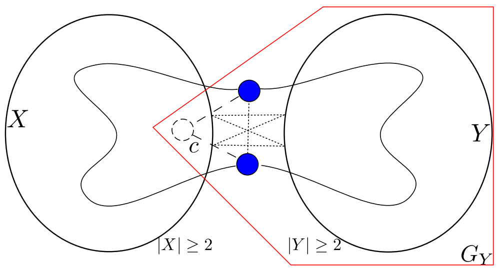
(d) Block of decomposition G_X w.r.t. a proper 2-cutset, if there is a vertex c of G such that $N_G(c) = \{a, b\}$



(e) Block of decomposition G_Y w.r.t. a proper 2-cutset, if there is a vertex c of G such that $N_G(c) = \{a, b\}$



(f) Block of decomposition G_X w.r.t. a proper 2-cutset, if is not a vertex c of G such that $N_G(c) = \{a, b\}$



(g) Block of decomposition G_Y w.r.t. a proper 2-cutset, if there is not a vertex c of G such that $N_G(c) = \{a, b\}$

Figure 2.4: Blocks of decomposition w.r.t. unichord-free graphs

biconnected {square,unichord}-free graph has the so-called *extremal* decomposition, which is a decomposition whereby every non-leaf has (at least) one basic block of decomposition [70]. Extremal decomposition is suitable to extend the colouring of a non-leaf type 2 block of decomposition to the basic block of decomposition. This approach is useful to optimal star-biclique-colourings of {triangle, square, unichord}-free graphs.

Another decomposition result concerns complete graphs. If a unichord-free graph G contains a triangle, then either G is a complete graph, or one vertex of the clique that contains this triangle is a 1-cutset of G [96]. Equivalently, we have the following theorem, which is a useful tool towards both clique- and star-biclique-colourings throughout this chapter.

Theorem 2.1 (Trotignon and Vušković [96], rephrased)**.** *Let G be a unichord-free graph. Every biconnected component of G is either a complete graph, or a triangle-free graph.*

Finally, Theorem 2.2 states an algorithm that computes an optimal vertex-colouring of a unichord-free graph. This algorithm is a black box to solve optimally clique-colouring {triangle, unichord}-free graphs.

Theorem 2.2 (Trotignon and Vušković [96])**.** *Let G be a unichord-free graph. The chromatic number of G is at most $\max\{3, \omega(G)\}$. Moreover, there is an $O(nm)$-time algorithm that computes an optimal vertex-colouring for unichord-free graph.*

2.2 Colourings Strategy Overview

Clique-, biclique-, and star-colourings algorithms developed here, as you shall see in Pages 35 – 54, follow the same general strategy that is frequently used to obtain vertex-colouring algorithms in classes defined by forbidden subgraphs. A specific structure F is chosen in such a way that one of the following cases holds.

1. A graph in the class does not contain F and so belongs to a more restricted subclass for which the solution is already known; or

2. A graph contains F and the existence of such structure entails a decomposition into smaller subgraphs in the same class.

The chosen structure for the clique-colouring algorithm is the triangle. If there is a triangle in the unichord-free graph, we have decompositions along 1-cutsets. Otherwise, the graph is triangle-free and clique-colouring reduces to vertex-colouring. Based on an efficient algorithm for vertex-colouring unichord-free graphs (Theorem 2.2), the construction of an efficient algorithm for clique-colouring unichord-free graphs is straightforward.

The composition of colourings along 1-cutsets is surprisingly tough in the context of biclique-colouring, while it is straightforward in the context of vertex-, clique-, and, of course, star-colourings. In order to alleviate the 1-cutset composition, we assign a biclique-colouring to a graph that is also a star-colouring, i.e. a star-biclique-colouring. Recall that we shall prove that biclique-, star-, and biclique-star-chromatic numbers coincide for unichord-free graphs. Then, we infer optimal biclique- and star-colourings from an optimal star-biclique-colouring.

The star-biclique-colouring algorithm makes a deeper use of the decomposition results of Trotignon and Vušković [96]. The first chosen structure for star-biclique-colouring unichord-free graphs is the triangle. The second chosen structure for star-biclique-colouring algorithm {triangle, unichord}-free graphs is the square. Finally, the proper 2-cutset extremal decomposition is used to star-biclique-colour {triangle, square, unichord}-free graphs.

2.3 Clique-colouring

If a graph is triangle-free, then clique-colouring reduces to vertex-colouring and Theorem 2.2 handles this case. If the unichord-free graph contains a triangle, we entail a decomposition by 1-cutset given by Theorem 2.1. The following lemma states that we just need to put optimal clique-colourings of blocks of decomposition together along an 1-cutset to get an optimal clique-colouring of a graph. The key observation of its proof is that a subset of vertices of every graph G is a clique in G if, and only if, it is a clique (properly contained) in one block of decomposition.

Lemma 2.3 (Lemma 1, Appendix A)**.** *Let G be a graph. An optimal clique-colouring of G can be obtained from optimal clique-colourings of its blocks of decomposition w.r.t. 1-cutset.*

If we join Theorem 2.1 and Theorem 2.2, we have that biconnected components of unichord-free graphs are either 2-clique-colourable complete graphs or 3-clique-colourable {triangle, unichord}-free graphs. Hence, as a consequence of Lemma 2.3, the clique-chromatic number of every unichord-free graph is at most 3.

Theorem 2.4 (Theorem 3, Appendix A)**.** *Every unichord-free graph is 3-clique-colourable.*

Now, we have that 2-clique-colourable unichord-free graphs are exactly perfect unichord-free graphs.

Theorem 2.5 (Theorem 4, Appendix A)**.** *A unichord-free graph is 2-clique-colourable if, and only if, it is perfect.*

It is really interesting to notice that by restricting the 2-clique-colouring problem from perfect graphs to perfect unichord-free graphs, the time-complexity falls from Σ_2^P-complete [27] to $\mathcal{P}$.

2.3.1 Algorithmic Aspects

Our proof of Lemma 2.3 gives a constant-time algorithm to combine optimal clique-colourings of blocks of decomposition in order to obtain an optimal clique-colouring for their parent. Moreover, recall that we inherit an $O(nm)$-time optimal clique-colouring algorithm for {triangle, unichord}-free graphs from an $O(nm)$-time algorithm optimal vertex-colouring algorithm for unichord-free graphs. Finally, blocks of decomposition have only one vertex in common. These 3 ingredients altogether imply that the overall time-complexity to give an optimal clique-colouring for unichord-free graphs is $O(nm)$.

Theorem 2.6 (Theorem 5, Appendix A)**.** *There is an $O(nm)$-time optimal clique-colouring algorithm for unichord-free graphs.*

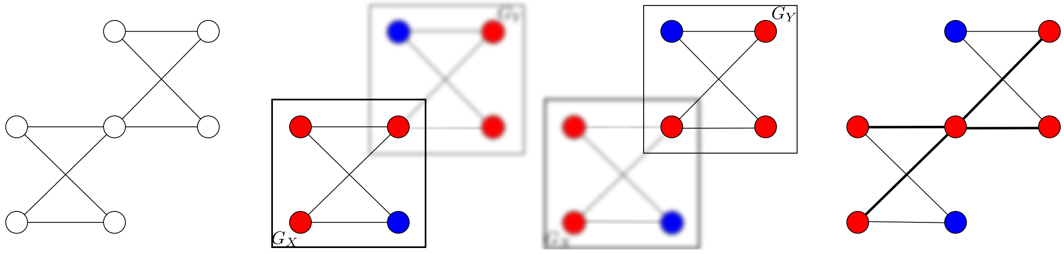
As a corollary of Theorem 2.6, we have that the overall time-complexity to recognize perfect unichord-free graphs is $O(nm)$.

Corollary 2.7 (Implicit, Appendix A)**.** *There is an $O(nm)$-time algorithm to recognize perfect unichord-free graphs.*

Proof. There is an algorithm to recognize unichord-free graphs in $O(nm)$-time [96], say X, and an $O(nm)$-time algorithm to compute an optimal clique-colouring of a unichord-free graph, say Y. Moreover, we know that a unichord-free graph is 2-clique-colourable if, and only if, it is perfect. Hence, we have an $O(nm)$-time algorithm to recognize perfect unichord-free graphs, as follows. Given a graph G, run X on G. If the answer is no, then return *no*. If the answer is yes, run Y on G. If the number of colours assigned is not 2, then return *no*. Otherwise, output *yes*. □

2.4 Biclique- and Star-colourings

We now turn our attention to the biclique-colouring problem restricted to unichord-free graphs. In contrast to clique-colouring, there is not an analogous of Lemma 2.3. Indeed, optimal biclique-colourings of blocks of decomposition does not necessarily determine an optimal biclique-colouring of their parent. An example is illustrated in Figure 2.5. Notice that the monochromatic star biclique of graph G shown in Figure 2.5c is not a biclique in all blocks of decomposition of G. Hence, the key idea of this section follows. We overcome monochromatic star bicliques when biclique-colourings are put together along 1-cutsets restricting biclique-colourings to be also star-colourings. Fortunately, we show in Theorem 2.12 that biclique-, star-, and star-biclique-chromatic numbers coincide for unichord-free graphs. It is quite interesting to notice that this further restriction makes our lives easier, since we are free to put (star-)biclique-colourings together along 1-cutsets, and no further colour is needed.



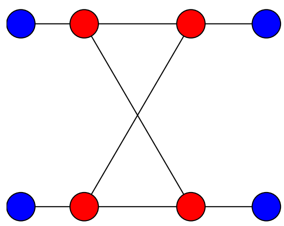
(a) Unichord-free graph

(b) Optimal biclique-colourings of blocks of decomposition

(c) Invalid colouring

Figure 2.5: Gluing biclique-colourings along 1-cutset may not be a good idea

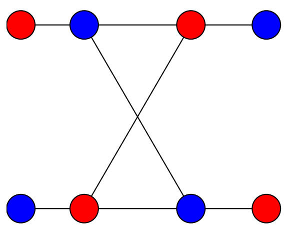
As a remark, notice that there are biclique-colourings that are not star-colourings and *vice-versa*. For instance, see Figure 2.6. We invite reader to check Figure 2.7 for corresponding star-biclique-colourings of graphs shown in Figure 2.6. Notice that all colourings in Figure 2.6 and in Figure 2.7 have the same amount of colours.



(a) Star-colouring which is not biclique-colouring

(b) Biclique-colouring which is not star-colouring

Figure 2.6: Biclique-colouring which is not star-colouring and *vice-versa*



(a) Star-biclique-colouring of Figure 2.6a

(b) Star-biclique-colouring of Figure 2.6b

Figure 2.7: Star-biclique-colourings for graphs of Figure 2.6

We divide this section into two parts. Subsection 2.4.1 starts with a 2-star-biclique-colouring algorithm for biconnected unichord-free graphs. This result is very important to begin Subsection 2.4.2 with a constructive proof that biclique-, star-, and star-biclique-chromatic numbers coincide for unichord-free graphs. Then, Subsection 2.4.2 develops an optimal star-biclique-colouring algorithm for non-biconnected unichord-free graphs, which, in turn, is divided into two more parts. First, we define our proposed extremal decomposition tree, and establish that the star-biclique-chromatic number of a unichord-free graph G is either $\beta(G)$ or $\beta(G) + 1$, in which $\beta(G)$ is the maximum cardinality of a true twin set of graph G. Then, we finish by describing an algorithm that decides between those two possible values.

2.4.1 Biconnected Unichord-free Graphs

From now on, we consider K_1 and K_2 biconnected components as a complete graph, and not as a triangle-free graph. This assumption helps us with case analysis when we consider biconnected {triangle, unichord}-free graphs and cliques as two distinct cases.

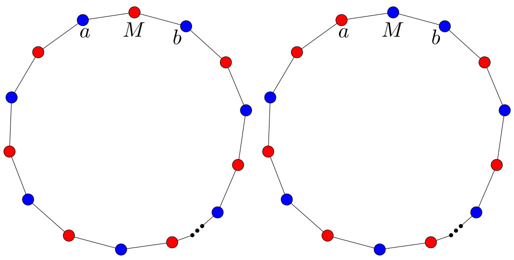
In order to construct an algorithm to combine colourings along 1-cutsets, we start dealing with a biconnected unichord-free graph G, as follows. If G is a complete graph, then the optimal star-biclique-colouring uses $|V(G)|$ colours. Next, we consider biconnected {triangle, unichord}-free graphs with at least four vertices.

Optimal star-biclique-colouring algorithm for biconnected {triangle, unichord}-free graphs strongly relies on proper decomposition tree defined in Section 2.2.

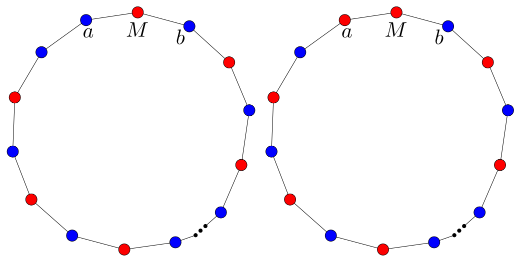
We consider a bottom-up approach in order to construct such optimal star-biclique-colouring algorithm. We start by showing how to assign a 2-star-biclique-colouring for biconnected {square, triangle, unichord}-free graphs. By definition, biclique- and star-colourings coincide for square-free graphs. We show in Lemma 2.8 that every basic graph has a 2-star-biclique-colouring. In fact, we give a slightly stronger result. A basic graph has a 2-star-biclique-colouring even if colours of two arbitrary vertices at distance 2 are fixed. As we shall see, this stronger result makes very easy to assign 2-star-biclique-colourings along proper 2-cutsets. This is the first part in order to construct a 2-star-biclique-colouring algorithm for biconnected {triangle, unichord}-free graphs.

Lemma 2.8 (Lemma 2, Appendix A)**.** *Let G be a basic graph, M be a vertex of degree at least 2, a and b be two neighbors of M, and $\alpha, \beta \in \{1, 2\}$ be any two colours. There is a 2-star-biclique-colouring of G in which a and b are coloured α and β, respectively.*

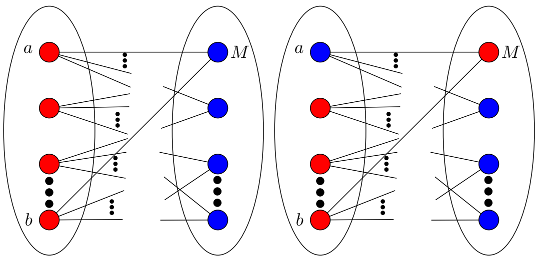
See Figure 2.8 for 2-star-biclique-colourings of basic graphs assuming that colours of two arbitrary vertices at distance 2 are fixed. Notice that Petersen graph is vertex-transitive and so the colouring given in Figure 2.8e works for every induced subgraph properly mapped. On the other hand, the colouring given in Figure 2.8d for Heawood graph can be properly adapted for induced subgraphs.



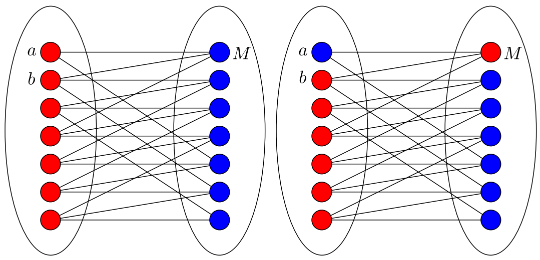
(a) 2-star-biclique-colouring of even cycles with any 2 colours at a and b



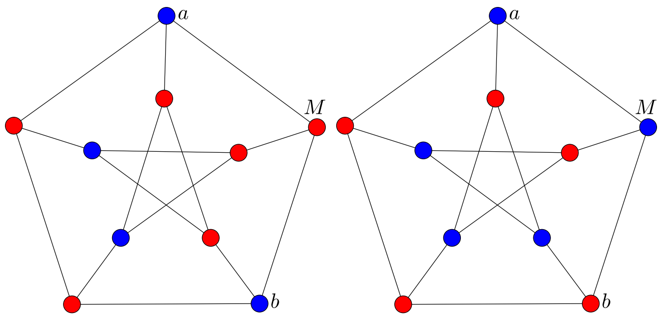
(b) 2-star-biclique-colouring of odd cycles with any 2 colours at a and b



(c) 2-star-biclique-colouring of strongly 2-bipartites with any 2 colours at a and b



(d) 2-star-biclique-colouring of an induced subgraph of Heawood graph with any 2 colours at a and b



(e) 2-star-biclique-colouring of an induced subgraph of Petersen graph with any 2 colours at a and b

Figure 2.8: 2-star-biclique-colourings of basic graphs with any 2 colours at a and b

Lemma 2.8 is suitable to extend star-biclique-colourings from biconnected {square, triangle, unichord}-free graphs to every basic graph, and *vice-versa*. As a consequence of this strategy, we are able to show, in Lemma 2.9, that every biconnected {triangle, square, unichord}-free graph has a 2-star-biclique-colouring obtained from 2-star-biclique-colourings of its blocks of decomposition w.r.t. proper 2-cutset, in which (at least) one of the blocks of decomposition is a basic graph. The proof is done by induction on the number of vertices. We also use the extremal decomposition for non-basic biconnected {square, unichord}-free graphs proposed by Machado *et al.* [70]. This is the second part in order to construct a 2-star-biclique-colouring algorithm for biconnected {triangle, unichord}-free graphs.

Lemma 2.9 (Lemma 3, Appendix A)**.** *Let G be a biconnected {triangle, square, unichord}-free graph. A 2-star-biclique-colouring of G can be obtained from 2-star-biclique-colourings of its blocks of decomposition w.r.t. proper 2-cutset, such that one of the blocks of decomposition is basic.*

Now, we show, at Lemma 2.10, that every biconnected {triangle, unichord}-free graph has a 2-star-biclique-colouring obtained from 2-star-biclique-colourings of its blocks of decomposition w.r.t. proper 1-join. Again, the proof is done by induction on the number of vertices. This is the third and final part in order to construct a 2-star-biclique-colouring algorithm for biconnected {triangle, unichord}-free graphs.

Lemma 2.10 (Lemma 4, Appendix A)**.** *Let G be a biconnected {triangle, unichord}-free graph. A 2-star-biclique-colouring of G can be obtained from 2-star-biclique-colourings of its blocks of decomposition w.r.t. a proper 1-join.*

If we join Lemma 2.8, Lemma 2.9, and Lemma 2.10 according to the proper decomposition tree, we have a 2-star-biclique-colouring algorithm for {triangle, unichord}-free graphs.

Theorem 2.11 (Theorem 6, Appendix A)**.** *There is a 2-star-biclique-colouring of a biconnected {triangle, unichord}-free graph.*

Clearly, a 2-star-biclique-colouring algorithm implies an optimal star-colouring, an optimal biclique-colouring, and an optimal star-biclique-colouring. In particular, biclique-, star-, and star-biclique-chromatic number coincide for {triangle, unichord}-free graphs. In what follows, we shall see that those parameters coincide for general unichord-free graphs.

2.4.2 Non-biconnected Unichord-free Graphs

We start this subsection by showing that biclique-, star-, and star-biclique-chromatic numbers coincide for unichord-free graphs. This fact implies that an optimal star-biclique-colouring algorithm for unichord-free graphs is also optimal biclique- and star-colourings algorithms for them. In order to show what was abovementioned, we transform a given optimal (not necessarily star-colouring) biclique-colouring into an optimal star-biclique-colouring, both with the same amount of colours. Analogously, we transform a given optimal (not necessarily biclique-colouring) star-colouring into an optimal star-biclique-colouring, both with the same amount of colours. Our proof is constructive and strongly relies on 2-star-biclique-colouring {triangle, unichord}-free graphs obtained in Subsection 2.4.1. We use it as a black box to recolour {triangle, unichord}-free biconnected components, but monochromatic stars (resp. monochromatic bicliques) may appear. Then, we apply some colour swaps intercalated with those recolourings to make all stars (resp. bicliques) polychromatic.

Theorem 2.12 (Theorem 7, fixed, Appendix A)**.** *Biclique-, star-, and star-biclique-chromatic numbers coincide for unichord-free graphs.*

From now on, we are interested in determining an optimal star-biclique-colouring for unichord-free graphs, since it is also optimal star- and biclique-colourings.

Extremal Decomposition Tree for Non-biconnected Graphs

We apologize to the readers for the following (very heavy) technical terminology that we need towards optimal star-biclique-colouring. Unfortunately, this is a very dense topic. For the sake of clarity, reader can refer to Figure 2.9, which shows examples of the terminology that will be introduced. Notice that each non-highlighted biconnected component does not belong to any of the defined biconnected components types, namely $\mathcal{F}$, $\mathcal{S}$, and $\mathcal{S}^*$.

Definition 2.13 ($\mathcal{C}(B)$ set)**.** $\mathcal{C}(B)$ is the set of 1-cutsets of a graph G that is in a biconnected component B of G.

Definition 2.14 (Type $\mathcal{F}$ biconnected component)**.** A *type* $\mathcal{F}$ biconnected component is a biconnected component B with $|\mathcal{C}(B)| = 1$.

Definition 2.15 ($\Gamma_G(B, v)$ set)**.** $\Gamma_G(B, v)$ is the set of biconnected components of a graph G that shares vertex v with a biconnected component B of G.

Definition 2.16 (Type $\mathcal{F}$ set)**.** $\Gamma_G(B, v)$ set is called *type* $\mathcal{F}$ *set* when every biconnected component of $\Gamma_G(B, v)$ is type $\mathcal{F}$.

Definition 2.17 ($\mathcal{C}_{\mathcal{F}}(B)$ set)**.** $\mathcal{C}_{\mathcal{F}}(B)$ is the subset of $\mathcal{C}(B)$, in which for each vertex $v \in \mathcal{C}_{\mathcal{F}}(B)$, $\Gamma_G(B, v)$ is a type $\mathcal{F}$ set.

Definition 2.18 (Type $\mathcal{S}$ biconnected component)**.** A *type* $\mathcal{S}$ biconnected component is a biconnected component B with $|\mathcal{C}_{\mathcal{F}}(B)| \geq 1$.

Definition 2.19 ($\mathcal{C}_{\overline{\mathcal{F}}}(B)$ set)**.** $\mathcal{C}_{\overline{\mathcal{F}}}(B)$ is the subset of $\mathcal{C}(B)$ without vertices in $\mathcal{C}_{\mathcal{F}}(B)$, i.e. $\mathcal{C}_{\overline{\mathcal{F}}}(B) = \mathcal{C}(B) \setminus \mathcal{C}_{\mathcal{F}}(B)$.

Definition 2.20 (Type $\mathcal{S}^*$ biconnected component)**.** A *type* $\mathcal{S}^*$ biconnected component is a type $\mathcal{S}$ biconnected component B with $|\mathcal{C}_{\overline{\mathcal{F}}}(B)| \leq 1$.

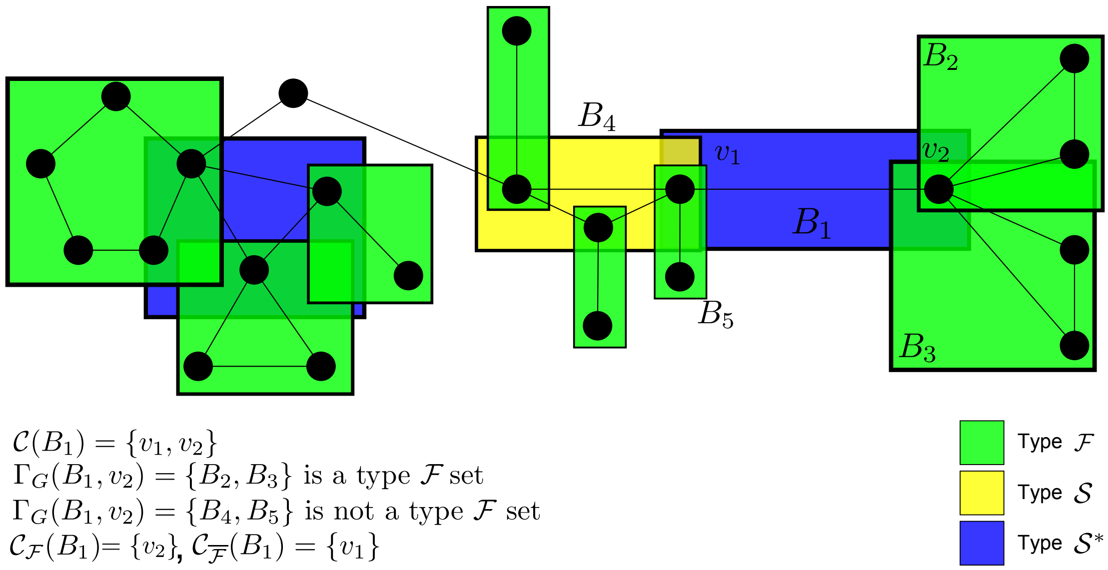


Figure 2.9: Unichord-free graph enhancing biconnected component types

We remark that a type $\mathcal{S}^*$ biconnected component is a type $\mathcal{S}$, but the converse is not necessarily true. Clearly, for every non-biconnected graph, if we have one type $\mathcal{S}^*$, then we have one type $\mathcal{S}$ and one type $\mathcal{F}$. Nevertheless, we prove that there is (at least) one type $\mathcal{F}$, one type $\mathcal{S}$, and one type $\mathcal{S}^*$ biconnected components, respectively, for every non-biconnected graph.

Lemma 2.13 (Lemma 5, Appendix A)**.** *Every non-biconnected graph has at least one type $\mathcal{F}$, one type $\mathcal{S}$, and one type $\mathcal{S}^*$ biconnected components.*

Now, we introduce an extremal decomposition for a non-biconnected graph G via type $\mathcal{S}^*$ biconnected component B^*. Graph G can be decomposed into subgraphs G_1 and G_2, as follows.

$$G_1 = B^* \cup \left(\bigcup_{v \in \mathcal{C}_{\mathcal{F}}(B^*)} \Gamma_G(B^*, v) \right) \text{ and } G_2 = G[V(G \setminus G_1) \cup V(B^*)]$$

The decomposition algorithms of this section have the following general strategy. First, we examine G_1 and based on it, we determine possible values for the number of colours needed in the vertices of biconnected component B^* for a star-biclique-colouring of G. Then, we modify G_2 in order to record that information and, after that, we apply recursion on new G_2. The base case is the prime graph, defined as follows.

Definition 2.21 (Prime graph)**.** Let G be a graph and B^* be a type $\mathcal{S}^*$ biconnected component. We say that G is as *prime graph* if

$$G = B^* \cup \left(\bigcup_{v \in \mathcal{C}_{\mathcal{F}}(B^*)} \Gamma_G(B^*, v) \right)$$

In contrast to the structural result of unichord-free graphs, a prime graph is a basic graph for the proposed extremal decomposition. Notice that $|\mathcal{C}_{\overline{\mathcal{F}}}(B^*)| = 0$ if G is a prime graph. Otherwise, G is non-prime, then $|\mathcal{C}_{\overline{\mathcal{F}}}(B^*)| = 1$.

Bounds for Star-biclique-chromatic Number of Unichord-free Graphs

Let u be a vertex of a graph G. The *open neighbourhood* of u is $N(u) = \{v \in V(G) \mid uv \in E(G)\}$, and the *closed neighbourhood* of u is $N[u] = N(u) \cup \{u\}$. Two distinct vertices u and v are *true twins*, if $N[u] = N[v]$. This equivalence relation on the vertex set $V(G)$ of a graph defines a partition of $V(G)$ into *twin sets*. Notice that a twin set induces a complete subgraph that is not necessarily a clique.

Let $\beta(G)$ be the size of the largest twin set. Clearly, every twin set requires distinct colours for each of its vertices in order to give a biclique-, a star-, or a star-biclique-colouring. Now, we are now ready to establish a $\beta(G)$-lower bound for the biclique-, star-, and star-biclique-chromatic numbers of a graph G.

Lemma 2.14 (Lemma 6, fixed, Appendix A)**.** *Biclique-, star-, and star-biclique-chromatic numbers of every graph G is at least $\beta(G)$.*

In the terms of the following definition, we characterize all twin sets of a unichord-free graph at Lemma 2.15.

Definition 2.22 ($\overline{\mathcal{C}}(B)$ set)**.** $\overline{\mathcal{C}}(B)$ is the subset of $V(B)$ without vertices in $\mathcal{C}(B)$, i.e. $\overline{\mathcal{C}}(B) = V(B) \setminus \mathcal{C}(B)$.

Lemma 2.15 (Lemma 7, Appendix A)**.** *Let G be a unichord-free graph. Hence, all twin sets are precisely*

- *$\overline{\mathcal{C}}(B)$, for every complete biconnected component B;*
- *$\{v\}$, if v is an 1-cutset of G or a vertex of a triangle-free biconnected component.*

An example that shows all twin sets of a unichord-free graph is given in Figure 2.10. Notice that the size of the largest twin set is 2.

Now, we turn our attention to establish an upper bound of $(\beta(G)+1)$ for the star-biclique-chromatic number of a unichord-free graph G. We start by showing that every non-biconnected unichord-free prime graph G has a $(\beta(G) + 1)$-star-biclique-colouring. Then, regarding a unichord-free graph G with a decomposition via type $\mathcal{S}^*$ biconnected component B^* and $\mathcal{C}_{\overline{\mathcal{F}}}(B^*) = \{v^*\}$, we proceed as follows.

1. Assign a special $(\beta(G) + 1)$-star-biclique-colouring to the prime graph G_1;

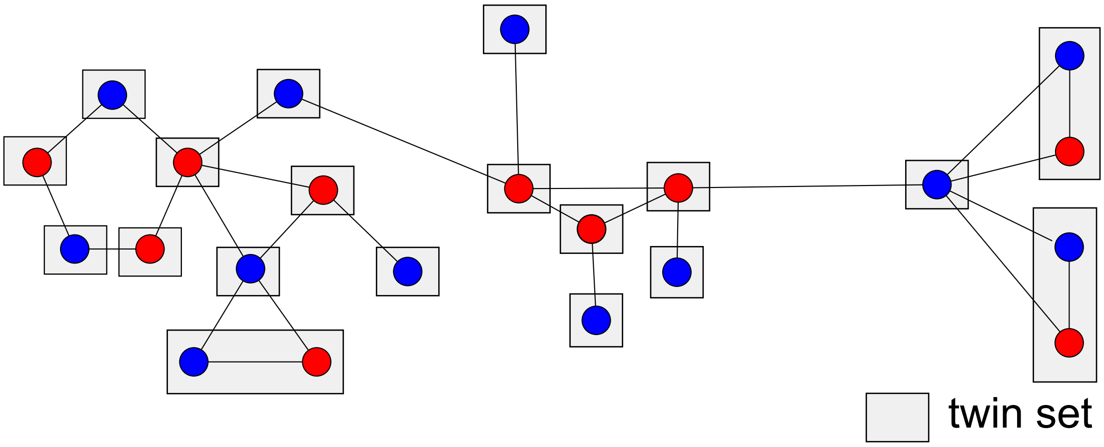


Figure 2.10: Twin sets of a unichord-free graph G, with $\beta(G) = 2$ and $\kappa_B(G) = 2$

2. Modify G_2 to $\widetilde{G_2}$ by shrinking B^* to

$$\widetilde{B} \simeq \begin{cases} K_2, & \text{if } B^* \text{ is triangle-free} \\ K_{|V(B^*)\setminus\mathcal{C}(B^*)|+1}, & \text{if } B^* \text{ is complete} \end{cases}$$

   with $\mathcal{C}_{\overline{\mathcal{F}}}(\widetilde{B}) = \{v^*\}$.

3. Assign recursively a $(\beta(\widetilde{G_2}) + 1)$-star-biclique-colouring to $\widetilde{G_2}$;

4. Combine star-biclique-colourings of G_1 and $\widetilde{G_2}$ to obtain a $(\beta(G) + 1)$-star-biclique-colouring for G.

This concludes the general idea of the proof of Theorem 2.16.

Theorem 2.16 (Theorem 8, Appendix A)**.** *Star-biclique-chromatic number of a unichord-free graph G is at most $\beta(G) + 1$.*

In fact, our proof of Theorem 2.16 assigns a $(\beta(G) + 1)$-star-biclique-colouring to every unichord-free graph G. See Figure 2.11 for an example of an application of our algorithm given by the constructive proof of Theorem 2.16. As a remark, Theorem 2.16 yields an optimal star-biclique-colouring algorithm for every unichord-free graph G that is not $\beta(G)$-star-biclique-colourable, in particular, for every unichord-free graph G with $\beta(G) = 1$. Then, from now on, we consider only unichord-free graphs with $\beta(G) \geq 2$.

Optimal Star-biclique-colouring Algorithm

Let G be a unichord-free graph with $\kappa_{SB}(G) = \beta(G) \geq 2$. We turn our attention to try our best to use, when possible, only $\beta(G)$ colours in a star-biclique-colouring of G, which is the lower bound of biclique-, star-, and star-biclique-chromatic numbers.

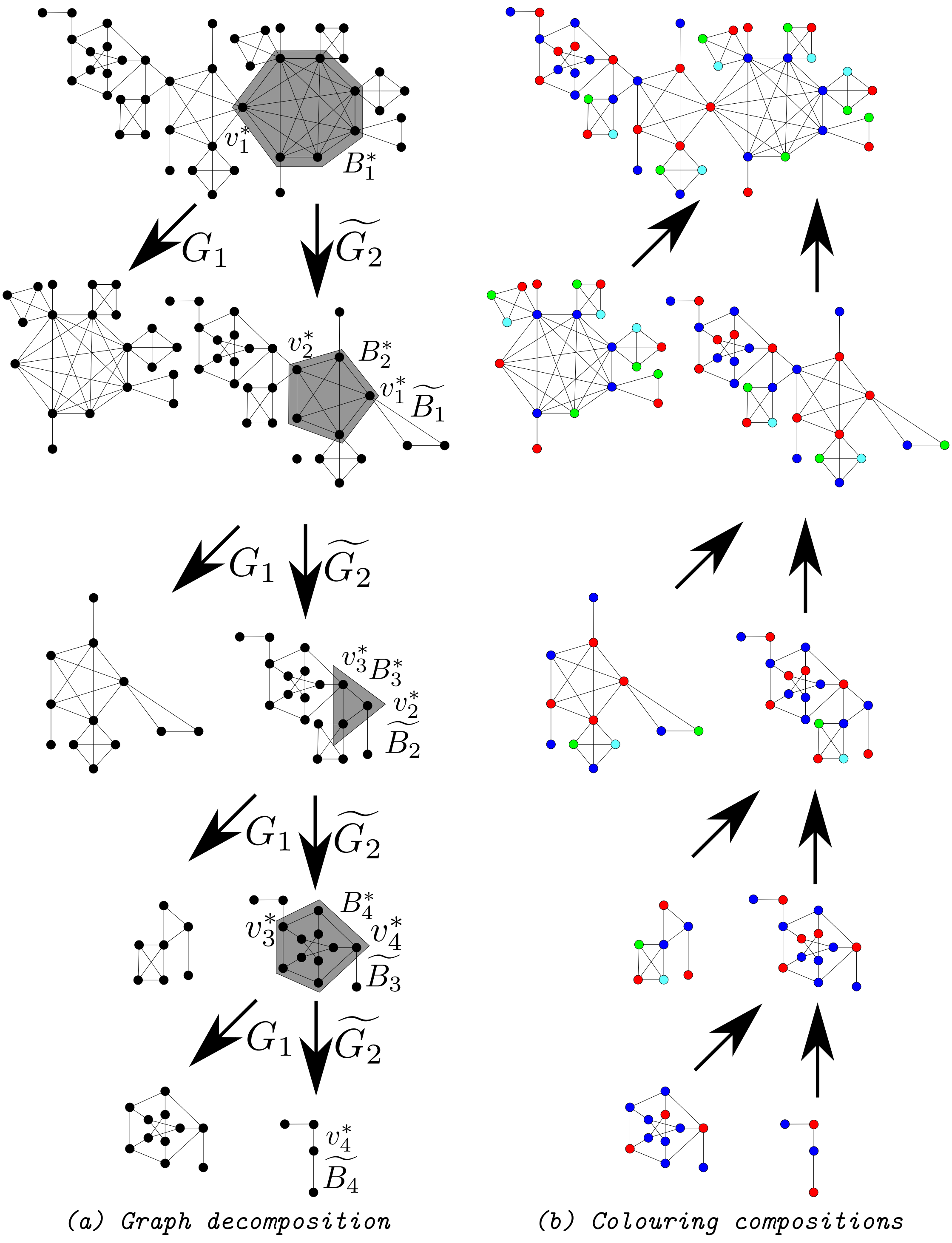


Figure 2.11: Application of our $(\beta + 1)$-star-biclique-colouring algorithm

In our $\beta(G)$-star-biclique-colouring, we proceed recursively on G, by always looking at type $\mathcal{S}^*$ biconnected components, seeking for a certificate that G is **not** $\beta(G)$-star-biclique-colourable, denoted by *bad certificate*. Graph G is $\beta(G)$-star-biclique-colourable if we do not find any bad certificate in all decompositions via type $\mathcal{S}^*$ biconnected components.

We remark that triangle-free biconnected components are never a bad certificate. Indeed, {triangle, unichord}-free graphs are 2-star-biclique-colourable and $\beta(G) \geq 2$. Nevertheless, complete biconnected components can be a bad certificate. As we shall see, a complete type $\mathcal{S}^*$ biconnected component B can demand more than $\beta(G)$ colours in a star-biclique-colouring of G if we assign at most $\beta(G)$ colours to those biconnected components that share a vertex with B.

Let B^* be a complete type $\mathcal{S}^*$ biconnected component of G. Consider the partition of $\mathcal{C}_{\mathcal{F}}(B^*)$ into the following two disjoint sets.

- $\mathcal{T}(B^*) = \{v \in \mathcal{C}_{\mathcal{F}}(B^*) \mid \forall B \in \Gamma_G(B^*, v), B \simeq K_{\beta(G)+1}\}$;
- $\overline{\mathcal{T}}(B^*) = \mathcal{C}_{\mathcal{F}}(B^*) \setminus \mathcal{T}(B^*)$.

We have the following property about $\mathcal{T}(B^*)$.

Lemma 2.17 (Lemma 8, Appendix A)**.** *Let G be a non-biconnected unichord-free graph, B^* be a complete type $\mathcal{S}^*$ biconnected component, and π be a $\beta(G)$-star-biclique-colouring of G. If $v \in \mathcal{T}(B^*)$, then $\pi(v) \neq \pi(u)$ for every $u \in B^*$, $u \neq v$.*

Now, consider that G is a $\beta(G)$-star-biclique-colourable non-biconnected prime unichord-free graph with $\beta(G) \geq 2$. Call $f_P(B)$ the least number of colours in a biconnected component B of G, for all $\beta(G)$-star-biclique-colourings of G. The partition of the vertices of a type $\mathcal{S}^*$ biconnected component B^* into subsets allows us to precisely determine $f_P(B^*)$. See Lemma 2.18.

Lemma 2.18 (Implicit, Appendix A)**.** *Let G be a $\beta(G)$-star-biclique-colourable non-biconnected prime unichord-free graph with $\beta(G) \geq 2$, and B^* be a type $\mathcal{S}^*$ biconnected component. Then,*

$$f_P(B^*) = \begin{cases} 2, & \textit{if } B^* \textit{ is triangle-free} \\ |\mathcal{T}(B^*)|, & \textit{if } B^* \textit{ is complete, } \overline{\mathcal{T}}(B^*) = \emptyset, \textit{ and } \overline{\mathcal{C}}(B^*) = \emptyset \\ |\mathcal{T}(B^*)| + 1, & \textit{if } B^* \textit{ is complete, } \overline{\mathcal{T}}(B^*) \neq \emptyset, \textit{ and } \overline{\mathcal{C}}(B^*) = \emptyset \\ |\mathcal{T}(B^*)| + |\overline{\mathcal{C}}(B^*)|, & \textit{if } B^* \textit{ is complete, and } \overline{\mathcal{C}}(B^*) \neq \emptyset \end{cases}$$

Proof. Let G be a non-biconnected prime unichord-free graph with $\beta(G) \geq 2$ and B^* be a type $\mathcal{S}^*$ biconnected component. Recall that a triangle-free biconnected component is 2-star-biclique-colourable. Then, $f_P(B^*) \leq 2$, if B^* is triangle-free. Without loss of generality, we can suppose that $f_P(B^*) = 2$, since $\beta(G) \geq 2$.

Now, consider that B^* is complete and that we have a $\beta(G)$-star-biclique-colouring of G. By Lemma 2.17, $f_P(B^*) \geq |\mathcal{T}(B^*)|$.

Clearly, $f_P(B^*) \leq |\mathcal{T}(B^*)|$, if B^* is complete, $\overline{\mathcal{T}}(B^*) = \emptyset$, and $\overline{\mathcal{C}}(B^*) = \emptyset$.

Let v be a vertex of B^*. If $v \in \overline{\mathcal{T}}(B^*)$, one can check that we can recolour all biconnected components in $\Gamma_G(B^*, v)$ in order to 1) keep G star-biclique-colourable with the same number of colours and 2) make all stars and bicliques in $\Gamma_G(B^*, v)$ polychromatic.

Then, every $u \neq v$ in $\overline{\mathcal{T}}(B^*)$ can have the same colour as that assigned to v. In particular, we can assume that all vertices of $\overline{\mathcal{T}}(B^*)$ have the same colour. Nevertheless, all vertices of $\overline{\mathcal{T}}(B^*)$ cannot have the same colour of any vertex of $\mathcal{T}(B^*)$. Then, $f_P(B^*) = |\mathcal{T}(B^*)| + 1$, if B^* is complete, $\overline{\mathcal{T}}(B^*) \neq \emptyset$, and $\overline{\mathcal{C}}(B^*) = \emptyset$.

Now, notice that every pair of vertices in $\overline{\mathcal{C}}(B^*)$ induces a biclique of G. Then, all vertices in $\overline{\mathcal{C}}(B^*)$ must have distinct colours. On the other hand, all vertices of $\overline{\mathcal{T}}(B^*)$ can have the same colour of every vertex of $\overline{\mathcal{C}}(B^*)$. Then, $f_P(B^*) = |\mathcal{T}(B^*)| + |\overline{\mathcal{C}}(B^*)|$, if B^* is complete and $\overline{\mathcal{C}}(B^*) \neq \emptyset$. □

By definition, $f_P(B^*) \leq \beta(G)$ is a necessary condition for graph G to be $\beta(G)$-star-biclique-colourable. Notice that B^* is a bad certificate, if $f_P(B^*) > \beta(G)$. Now, we can see that $f_P(B^*) \leq \beta(G)$ is also a sufficient condition for G to be $\beta(G)$-star-biclique-colourable.

Lemma 2.19 (Lemma 10, rephrased, Appendix A)**.** *Let G be a prime non-biconnected unichord-free graph with $\beta(G) \geq 2$ and let B^* be a type $\mathcal{S}^*$ biconnected component of G. Graph G is $\beta(G)$-star-biclique-colourable if, and only if, $f_P(B^*) \leq \beta(G)$.*

Now, consider that G is a $\beta(G)$-star-biclique-colourable non-prime unichord-free graph with $\beta(G) \geq 2$. Call $f_N(B)$ the least number of colours in a biconnected component B of G, for all $\beta(G)$-star-biclique-colourings of G. The partition of the vertices of a type $\mathcal{S}^*$ biconnected component B^* into subsets allows us to determine bounds for the possible values of $f_N(B^*)$. See Lemma 2.20.

Lemma 2.20 (Lemma 9, rephrased, Appendix A)**.** *Let G be a $\beta(G)$-star-biclique-colourable non-prime unichord-free graph, B^* be a type $\mathcal{S}^*$ biconnected component. Then,*

$$f_N(B^*) = \begin{cases} 2, & \textit{if } B^* \textit{ is triangle-free} \\ |\mathcal{T}(B^*)| + 1, & \textit{if } B^* \textit{ is complete, } \overline{\mathcal{T}}(B^*) = \emptyset, \textit{ and } \overline{\mathcal{C}}(B^*) = \emptyset \\ |\mathcal{T}(B^*)| + 1 + c, & \textit{if } B^* \textit{ is complete, } \overline{\mathcal{T}}(B^*) \neq \emptyset, \textit{ and } \overline{\mathcal{C}}(B^*) = \emptyset \\ |\mathcal{T}(B^*)| + |\overline{\mathcal{C}}(B^*)| + c, & \textit{if } B^* \textit{ is complete, and } \overline{\mathcal{C}}(B^*) \neq \emptyset \end{cases}$$

for some $c \in \{0, 1\}$.

By definition, $f_N(B^*) \le \beta(G)$ is a necessary condition for graph G to be $\beta(G)$-star-biclique-colourable. Since we do not know $f_N(B^*)$ precisely, we denote by $g(B^*)$ the largest lower bound of $f_N(B^*)$. By Lemma 2.20,

$$g(B^*) = \begin{cases} 2, & \text{if } B^* \text{ is triangle-free} \\ |\mathcal{T}(B^*)| + 1, & \text{if } B^* \text{ is complete and } \overline{\mathcal{C}}(B^*) = \emptyset \\ |\mathcal{T}(B^*)| + |\overline{\mathcal{C}}(B^*)|, & \text{if } B^* \text{ is complete and } \overline{\mathcal{C}}(B^*) \neq \emptyset \end{cases}$$

Consider that G has a decomposition via type $\mathcal{S}^*$ biconnected component B^* into subgraphs G_1 and G_2, and that we have a $\beta(G)$-star-biclique-colouring of G with $f_N(B^*)$ colours in B^*. Reader is invited to check our proof of Lemma 2.20 to see that $g(B^*)$ describes precisely the number of colours in $B^* \setminus v^*$, in which $\{v^*\} = \mathcal{C}_{\overline{\mathcal{F}}}(B^*)$. The range of two possible values given by Lemma 2.20 for $f_N(B^*)$ is due to the fact that, by just looking to G_1, 1) one can determine the number of colours in $B^* \setminus v^*$, but 2) one cannot determine whether v^* must have a distinct colour in B^*.

Strictly speaking, vertex v^* could have the same colour as some other vertex in B^* as long as we do not have any monochromatic star centered in v^*. Notice that vertex v^* is absent in prime graphs, so we could precisely determine $f_P(B^*)$ at Lemma 2.18. Moreover, notice that B^* is a bad certificate, if $g(B^*) > \beta(G)$. In the light of Lemma 2.20, we try our best to use only $\beta(G)$ colours, by proceeding as follows.

1. Assign a special $\max(\beta(G), g(B^*))$-colouring to the prime graph G_1;
2. Modify G_2 to $\widehat{G_2}$ by possibly shrinking B^* to

$$\widehat{B} \simeq \begin{cases} K_2, & \text{if } B^* \text{ is triangle-free} \\ K_{|\mathcal{T}(B^*)|+1}, & \text{if } B^* \text{ is complete, } \overline{\mathcal{T}}(B^*) = \emptyset, \text{ and } \overline{\mathcal{C}}(B^*) = \emptyset \\ K_{|\mathcal{T}(B^*)|+2}, & \text{if } B^* \text{ is complete, } \overline{\mathcal{T}}(B^*) \neq \emptyset, \text{ and } \overline{\mathcal{C}}(B^*) = \emptyset \\ K_{|\mathcal{T}(B^*)|+|\overline{\mathcal{C}}(B^*)|+1}, & \text{if } B^* \text{ is complete, and } \overline{\mathcal{C}}(B^*) \neq \emptyset \end{cases}$$

with $\mathcal{C}_{\overline{\mathcal{F}}}(\widehat{B}) = \{v^*\}$.

3. Assign recursively a $\kappa_{SB}(\widehat{G_2})$-star-biclique-colouring to $\widehat{G_2}$;

4. Combine colourings of G_1 and $\widehat{G_2}$ to obtain a $\max(\beta(G),\ g(B^*),\ \kappa_{SB}(\widehat{G_2}))$-star-biclique-colouring for G.

Clearly, graph G has a $\beta(G)$-star-biclique-colouring if $g(B^*), \kappa_{SB}(\widehat{G_2}) \leq \beta(G)$. In fact, the converse is also true. Then, we are ready to characterize $\beta(G)$-star-biclique-colourable non-prime non-biconnected unichord-free graphs with $\beta(G) \geq 2$. This concludes the general idea of Theorem 2.21.

Theorem 2.21 (Theorem 9, rephrased, Appendix A)**.** *Let G be a non-prime unichord-free graph with $\beta(G) \geq 2$ and B^* be a type $\mathcal{S}^*$ biconnected component. Graph G is $\beta(G)$-star-biclique-colourable if, and only if, $g(B^*) \leq \beta(G)$ and $\kappa_{SB}(\widehat{G_2}) \leq \beta(G)$.*

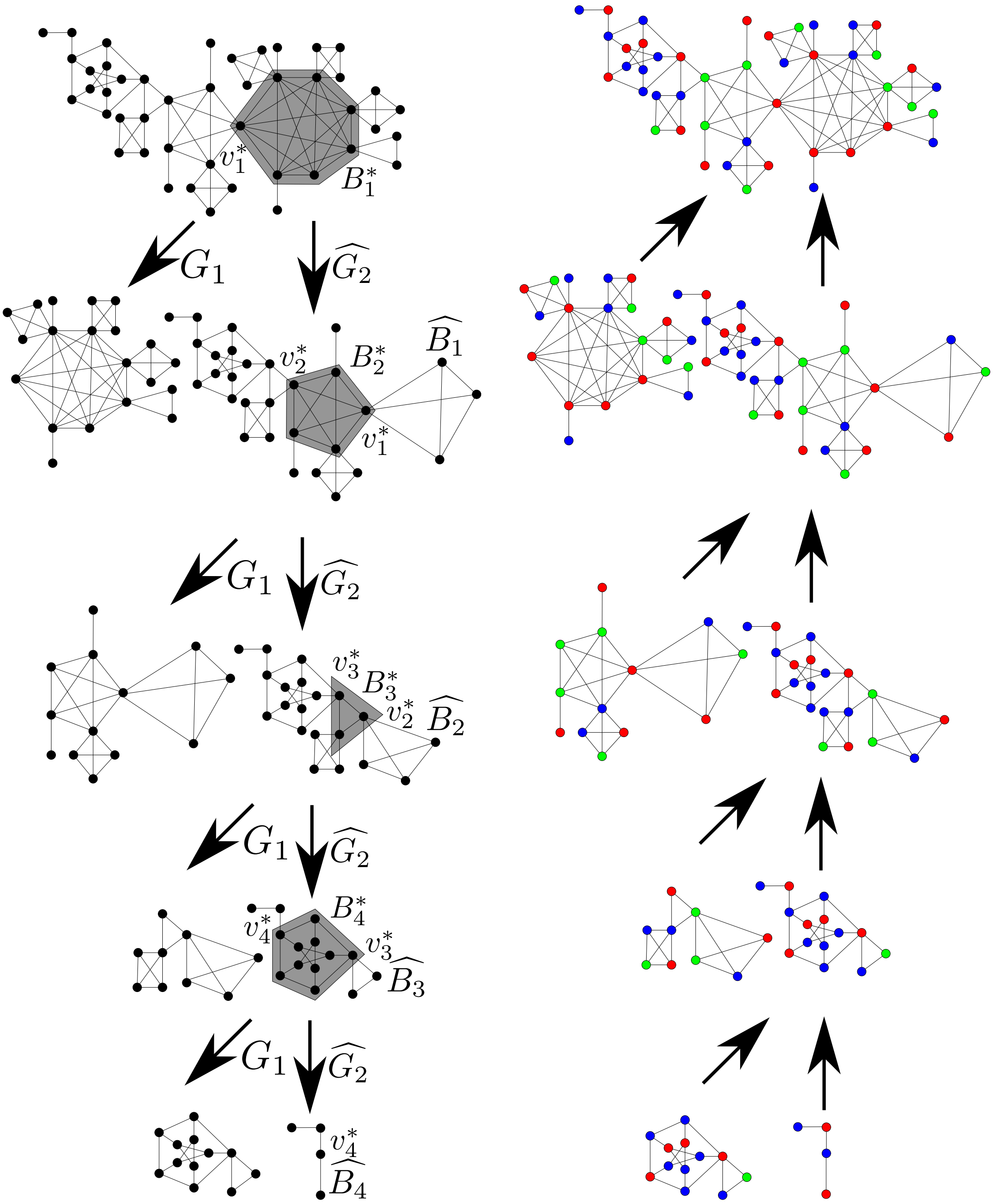
In fact, our proof of Theorem 2.21 assigns a $\beta(G)$-star-biclique-colouring to every $\beta(G)$-star-biclique-colourable unichord-free graph G. See Figure 2.12 for an example of an application of our algorithm given by the constructive proof of Theorem 2.16.

As a remark, recall our discussion about the range of two possible values for $f(B^*)$, which is due to the unique vertex of $\mathcal{C}_{\overline{\mathcal{F}}}(B^*)$. In our $\beta(G)$-star-biclique-colouring algorithm, as soon as the recursive call on $\widehat{G_2}$ returns, we know whether v^* must have a distinct colour in B^*. Then, we can also precisely determine $f_N(B^*)$.

2.4.3 Algorithmic Aspects

Let G be a biconnected unichord-free graph. If G is a complete graph, then an optimal star-biclique-colouring uses $|V(G)|$ colours. Now, we consider G as a biconnected {triangle, unichord}-free graph. An optimal star-biclique-colouring algorithm of G strongly relies on the proper decomposition tree defined in Section 2.1, in which (at least) one block of decomposition of the type 2 non-leaf node is basic.

First, we construct a proper decomposition tree T_G of the input graph G in time $O(n^2m)$. It is shown by Trotignon and Vušković an $O(nm)$-time algorithm to output a proper decomposition tree of a unichord-free graph [96]. In their algorithm, we replace the $O(n+m)$-time algorithm to find a proper 2-cutset (if some) in a unichord-free graph with no 1-cutset and no proper 1-join with the following $O(nm)$-time algorithm. Consider all possible 2-cutset decompositions of G and pick a proper 2-cutset S that has a block of decomposition B which size is the smallest possible. Machado *et al.* [70] showed that B must be basic. All proper 2-cutsets (and the size of its blocks of decomposition) can be found in $O(nm)$-time. Indeed, for every vertex v, find all 1-cutsets and all blocks of decomposition of $G\backslash\{v\}$ with depth-first search. For every such block of decomposition, check whether the corresponding 1-cutset u



(a) Graph decomposition (b) Colouring compositions

Figure 2.12: Application of our β-star-biclique-colouring algorithm

is such that $\{u, v\}$ is a proper 2-cutset. Keep in memory the size of the blocks of decomposition regarding $\{u, v\}$. Then, choose among all proper 2-cutsets, one with a block of decomposition of minimum size. We now have an algorithm to output a proper decomposition tree such that every proper 2-cutset subtree decomposition is extremal. On the other hand, the time-complexity of the algorithm to output such proper decomposition tree is raised to $O(n^2m)$, because we replaced an $O(n+m)$-time algorithm with an $O(nm)$-time one to find a proper 2-cutset.

Now, we discuss time-complexity to combine solutions of blocks of decomposition of a given node H of T_G in order to assign a 2-star-biclique-colouring to G. We have three cases, as follows.

- H is a non-leaf node of type 1. Proof of Lemma 2.10 shows how to proceed (in constant-time) in order to find a 2-star-biclique-colouring of H by asking recursively for the appropriate chosen 2-star-biclique-colouring of its children.
- H is a non-leaf node of type 2. Proof of Lemma 2.9 shows how to proceed (in constant-time) in order to find a 2-star-biclique-colouring of H by asking recursively for the appropriate chosen 2-star-biclique-colouring of its children.
- H is a leaf node. Lemma 2.8 shows how to proceed (in linear-time) in order to find a 2-star-biclique-colouring of H.

We combine solutions of blocks of decomposition at each non-leaf node of T_G in $O(1)$-time. It is proved that T_G is $O(n)$ [96], and that the time-complexity to assign 2-star-biclique-colourings to all leaves of T_G is $O(n+m)$. It means that the time-complexity to process the tree is $O(n+m)$. Finally, notice that the bottleneck of this algorithm is the construction of the proper decomposition tree, which has the time-complexity of $O(n^2m)$. Therefore, according to the abovementioned, we have just proved Lemma 2.22.

Lemma 2.22 (Lemma 11, rephrased, Appendix A)**.** *There is an $O(n^2m)$-time 2-star-biclique-colouring algorithm for $\{$triangle, unichord$\}$-free graphs.*

Now, we determine the largest size of a twin set and how to find a $\mathcal{S}^*$ biconnected component. This shall be useful to compute the optimal star-biclique-colouring for unichord-free graphs. The proposed algorithms rely on Tarjan's linear-time algorithm used to determine all biconnected components of a graph [93]. We add two integers i_1^v and i_2^v for each vertex v of the input graph. Integer i_1^v stores the number of biconnected components containing v. We can easily modify Tarjan's linear-time algorithm to compute i_1^v. Integer i_2^v stores the number of type $\mathcal{F}$ biconnected components containing v. We identify every type $\mathcal{F}$ biconnected component in the graph by searching for those biconnected components that contain exactly only one vertex with $i_1^v \geq 2$. We identify every type $\mathcal{S}^*$ biconnected component in a unichord-free graph, as follows.

- Search for type $\mathcal{F}$ biconnected components that contain exactly one vertex with $i_1^v = i_2^v \geq 2$, or
- Search for non-type $\mathcal{F}$ biconnected components that contain at least one vertex with $i_1^v = i_2^v + 1 \geq 2$ and at most one vertex with $i_1^v \geq i_2^v + 2$.

We invite reader to check that the abovementioned steps find all type $\mathcal{S}^*$ biconnected components in every unichord-free graph. It is easy to see that the given algorithm has linear-time complexity.

Lemma 2.23 (Lemma 12, rephrased, Appendix A)**.** *Let G be a unichord-free graph. There is a linear-time algorithm to find all type $\mathcal{S}^*$ biconnected components of G.*

Consider a unichord-free graph decomposition via a type $\mathcal{S}^*$ biconnected component B^* into subgraphs G_1 and G_2. We invite reader to check how to obtain all type $\mathcal{S}^*$ biconnected components of G_2 by just looking to B^* and biconnected components that share a vertex with B^*. Hence, it is linear-time to find all type $\mathcal{S}^*$ biconnected components in a sequence of decompositions via type $\mathcal{S}^*$ biconnected components until we have only prime graphs.

In order to compute $\beta(G)$ for a given unichord-free graph G, we proceed as follows. Use Tarjan's linear-time algorithm to determine all biconnected components of G [93]. According to Lemma 2.15, we have the twin sets showed bellow. We also show how to collect them in linear-time.

- $\{v\}$, if v is an 1-cutset of G or a vertex of a triangle-free biconnected component. Notice that every 1-cutset v of G has $i_1^v \geq 2$. Moreover, it is rather easy to check whether a vertex is in a triangle-free biconnected component or not.
- $\overline{\mathcal{C}}(B)$, for every complete biconnected component B. It is rather easy to check whether a vertex is in a complete biconnected component B or not. Moreover, every vertex $v \in \overline{\mathcal{C}}(B)$ has $i_1^v = 1$. It is rather easy to compute the union of every vertex u with $i_1^u = 1$ in the same complete biconnected component.

Keep in memory the size of each twin set and choose the largest one. Clearly, it is a linear-time algorithm.[2]

Lemma 2.24 (Lemma 13, rephrased, Appendix A)**.** *Let G be a unichord-free graph. There is a linear-time algorithm to compute $\beta(G)$.*

[2]The interested reader can construct a linear-time algorithm to output all twin sets for general graphs with modular decomposition. Modular decomposition is done in linear-time [94].

Now, we briefly discuss our constructive proof of Theorem 2.16, which yields an $O(n^2m)$-time $(\beta(G)+1)$-star-biclique-colouring algorithm for every unichord-free graph G with n vertices and m edges. Lemma 2.22 states that there is an $O(n^2m)$-time $(\beta(G)+1)$-star-biclique-colouring algorithm for every biconnected unichord-free graph G. Now, consider that G is a non-biconnected prime unichord-free graph. We assign a $(\beta(G)+1)$-star-biclique-colouring to G, such that the bottleneck is the algorithm cited in Lemma 2.22 that works as a black box for {triangle, unichord}-free biconnected components. Now, consider that G is a non-prime unichord-free graph. We entail a decomposition via type $\mathcal{S}^*$ biconnected component B^* into subgraphs G_1 and G_2. We assign a special $(\beta(G)+1)$-star-biclique-colouring to G_1, such that the bottleneck is again the algorithm cited in Lemma 2.22. Subgraph G_2 is modified to $\widetilde{G_2}$. Then, we apply recursion on $\widetilde{G_2}$ in order to assign a $(\beta(\widetilde{G})+1)$-star-biclique-colouring for $\widetilde{G_2}$. Finally, we combine star-biclique-colourings of G_1 and $\widetilde{G_2}$ to obtain a $(\beta(G)+1)$-star-biclique-colouring for G. Graphs G_1 and $\widetilde{G_2}$ have at least K_2 and at most B^* in common. Hence, we have two equations.

$$|V(G_1)| + |V(\widetilde{G_2})| - |V(B^*)| \leq n \leq |V(G_1)| + |V(\widetilde{G_2})| - 2$$
$$|E(G_1)| + |E(\widetilde{G_2})| - |E(B^*)| \leq m \leq |E(G_1)| + |E(\widetilde{G_2})| - 1$$

It follows from above equations that the overall time-complexity to assign a $(\beta(G)+1)$-star-biclique-colouring of G_1 and G_2, and their combination is $O(n^2m)$.

Lemma 2.25 (Lemma 14, rephrased, Appendix A)**.** *Let G be a unichord-free graph. There is an $O(n^2m)$-time $(\beta(G)+1)$-star-biclique-colouring algorithm for G.*

The constructive proofs of Lemma 2.26 and Theorem 2.21 yield an $O(n^2m)$-time algorithm to compute a $\beta(G)$-star-biclique-colouring for every $\beta(G)$-star-biclique-colourable unichord-free graph G. The time-complexity case analysis is very similar to that of Lemma 2.25.

Lemma 2.26 (Lemma 15, rephrased, Appendix A)**.** *Let G be a $\beta(G)$-star-biclique-colourable unichord-free graph. There is an $O(n^2m)$-time $\beta(G)$-star-biclique-colouring algorithm for G.*

We finish this section by showing an optimal star-biclique-colouring algorithm for unichord-free graphs. First, check whether the given unichord-free graph G is $\beta(G)$-star-biclique-colourable using Lemma 2.19 and Theorem 2.21. Recall that the star-biclique-chromatic number of G is at most $\beta(G)+1$. If G is $\beta(G)$-star-biclique-colourable, then Lemma 2.26 yields a $\beta(G)$-star-biclique-colouring of G. Otherwise, Lemma 2.25 yields a $(\beta(G)+1)$-star-biclique-colouring of G. All algorithms here have $O(n^2m)$-time complexity. Therefore, we have the following theorem.

Theorem 2.27 (Theorem 10, rephrased, Appendix A)**.** *There is an $O(n^2m)$-time optimal star-biclique-colouring algorithm for unichord-free graphs.*

2.5 Final Considerations

In our extended abstract presented at LATIN 2012, we showed that the biclique-chromatic number of every unichord-free graph is at most its clique-number. Unfortunately, this upper bound may be very large compared to actual biclique-chromatic number. Let H_n be a graph on vertices $a_1, \ldots, a_{\frac{n}{2}}, b_1, \ldots, b_{\frac{n}{2}}$, such that $\{a_1, \ldots, a_{\frac{n}{2}}\}$ is a complete set, $\{b_1, \ldots, b_{\frac{n}{2}}\}$ is a stable set, and the only edges between a_i and b_j are $a_k b_k$, for each $1 \leq k \leq \frac{n}{2}$. Since H_n is square-free, every biclique is a star, and H is clearly 2-biclique-colourable, but the clique-number of H_n is $\frac{n}{2}$. In this chapter, we strengthen the bounds by showing that the biclique-chromatic number of a unichord-free graph is the increment of or exactly the size of the largest twin set. Note that $\beta(H_n) = 1$.

A *block graph* is a graph in which every biconnected component is a clique. The graph H_6 is also known as *net*, which is a block graph. Groshaus *et al.* gave an optimal star-biclique-colouring algorithm for net-free block graphs [46]. A *cactus graph* is a graph in which every nontrivial biconnected component is a cycle. In our extended abstract presented at LATIN 2012, we gave an optimal biclique-colouring algorithm for cacti graphs. The class of net-free block graphs is incomparable to the class of cacti graphs. Indeed, a complete graph with four vertices K_4 and a chordless cycle with four vertices C_4 are witnesses. Nevertheless, both classes are unichord-free subclasses. In this chapter, we have given an optimal biclique-colouring algorithm for unichord-free graphs.

We have showed that 2-star-biclique-colouring $\{K_3$, unichord$\}$-free graphs is polynomial-time solvable. On one hand, our 2-star-biclique-colouring $\{K_3$, unichord$\}$-free graphs leads to an optimal biclique-colouring polynomial-time algorithm for unichord-free graphs. On the other hand, it is an open problem to determine biclique-colouring complexity for K_3-free graphs [46]. We remark that Groshaus *et al.* [46] gave a polynomial-time 2-star-colouring algorithm for K_3-free graphs.

Last, but not least, we have an ongoing work about speeding up an algorithm to fully decompose a graph by proper 2-cutsets. This would lead us to optimize our clique-, biclique-, and star-colourings algorithms for unichord-free graphs. Moreover, we would be able to settle an open problem posed by Trotignon and Vušković to recognize unichord-free graphs in linear-time [96, Section 5] and to infer other linear-time algorithms that are asymptotically faster than their predecessors, such as vertex-colouring unichord-free graphs [96] and edge-colouring chordless graphs [71].

2.5.1 A Procedure for Speeding Up Algorithms

The polynomial-time algorithms obtained so far in this chapter for clique- and biclique-colouring problems do not have a linear-time complexity. The bottleneck of the time-complexity of those algorithms is a superlinear-time complexity that fully decomposes a graph by proper 2-cutsets. Indeed, several subclasses of wheel-free graphs, that admit structural characterization through proper 2-cutsets, have this issue [1, 61, 71, 96]. We remark the following open problems.

- Lévêque *et al.* [61, Section 12] proposed as an open problem to make their (decomposition) algorithms of {ISK4, wheel}-free graphs rely on classical decomposition along 2-cutsets such as Hopcroft and Tarjan's algorithm.

- Trotignon and Vušković [96, Section 5] proposed as an open problem whether it is possible to use full strenght of sophisticated algorithm by Hopcroft and Tarjan to recognize a unichord-free graph in linear-time.

To address both open problems, we find suitable to consider SPQR-tree [31], which is a tree data structure developped by Di Battista and Tamassia to describe 2-cutsets among triconnected components of a graph. Moreover, this construction is done in linear-time [49]. Gutwenger [48] presents a thorough introduction into theory of triconnected components and SPQR-tree data structure. We highly recommend the interested reader in basic definitions and properties of SPQR-trees to read Gutwenger's PhD thesis [48], mainly Chapter 3.

In this section, we outline our approach to construct a proper 2-cutset decomposition tree of a biconnected graph G in linear-time. We use a depth-first search based algorithm that relies on SPQR-tree [31] of G to ouput a proper decomposition tree. Each of its node has (at least) one subtree which corresponds to an indecomposable graph by proper 2-cutsets. For instance, consider graph G of Figure 2.13. Figure 2.14 shows an SPQR-tree of G built as an intermediary step.[3] Figure 2.15 shows an extremal proper 2-cutset decomposition tree of G obtained by our proposed depth-first search based algorithm that relies on SPQR-tree of G.

If the combinatorial problem that we intend to solve has a linear-time algorithm for an indecomposable graph by proper 2-cutsets, then we consider the extremal proper 2-cutset decomposition tree to extend *with the same asymptotic time-complexity* the abovementioned algorithm to decomposable graphs by proper 2-cutsets. Indeed, many linear-time algorithms that work for triconnected graphs could only be extended to work for biconnected graphs using SPQR-trees (see Bertolazzi *et al.* [12] and Kant's [55] papers).

[3] Figure 2.13 and Figure 2.14 were constructed using OGDF library (see Listing 2.1, Page 63).

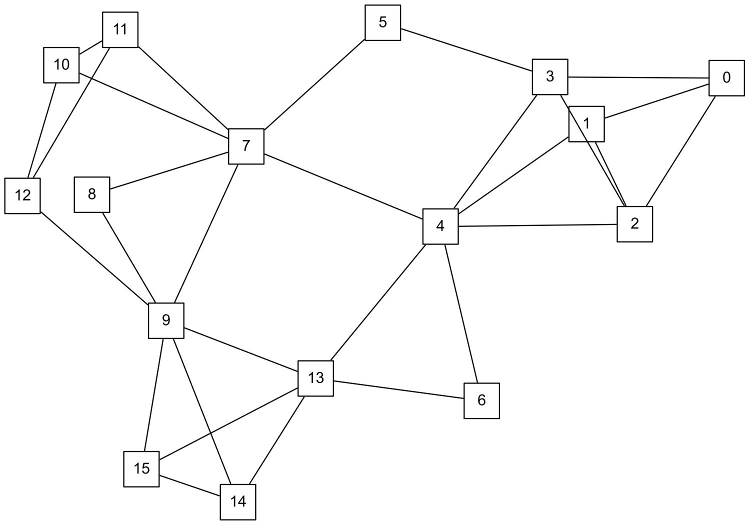


Figure 2.13: Arbitrary biconnected graph given by OGDF [20]

Now, we define SPQR-tree of a biconnected graph G, as follows. It is an unrooted tree that has four different types of nodes, namely S-, P-, Q- and R-nodes, in which initials S, P, and R stand for *series*, *parallel*, and *rigid*. Main definitions and properties of SPQR-trees follows.

Definition 2.23 (Skeleton)**.** Each internal node of an SPQR-tree has an associated biconnected multigraph $\mathcal{S}$, its *skeleton*.

We have the following properties about skeletons of S-, P-, and R-nodes.

Lemma 2.28 (Di Battista and Tamassia [30])**.** *Skeleton of*

- *an S-node is a cycle of length at least 3,*
- *a P-node consists of at least 3 parallel edges, and*
- *an R-node is a triconnected graph.*

Due to Theorem 2.29, we always omit Q-nodes. Moreover, the last item of this theorem states that skeletons of an SPQR-tree of a biconnected graph G are in one-to-one correspondence to triconnected components of G.

Theorem 2.29 (Gutwenger [48])**.** *Let G be a biconnected graph and $\mathcal{T}$ be an SPQR-tree of G. Then, following statements are true.*

- *Q-nodes of $\mathcal{T}$ are in one-to one correspondence to edges of G.*
- *Graph obtained from $\mathcal{T}$ by removing the Q-nodes is a tree formed by all triconnected components.*

An important fact about the neighbourhood of S-nodes and P-nodes follows.

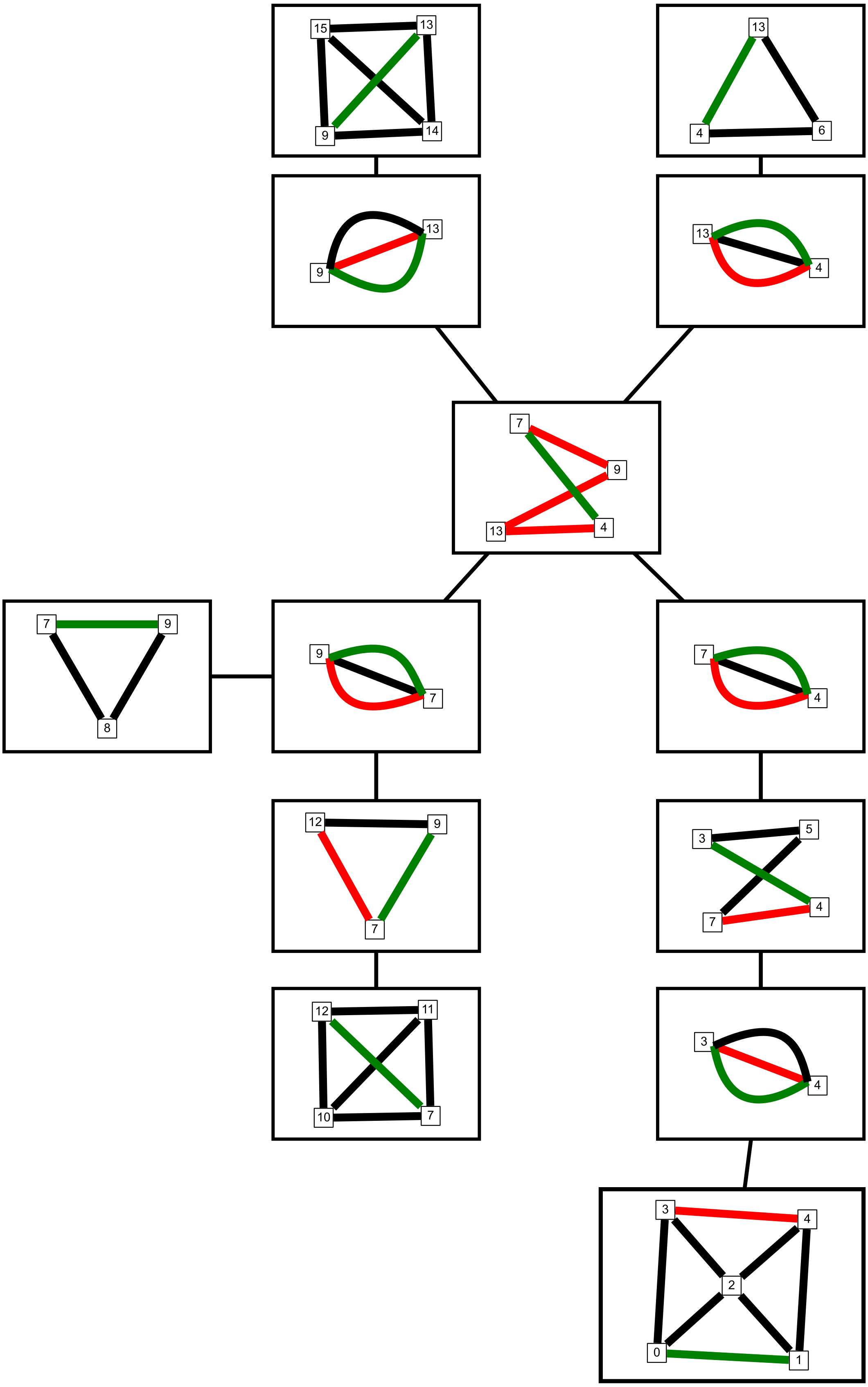


Figure 2.14: SPQR-tree based on Figure 2.13 and given by OGDF [20]

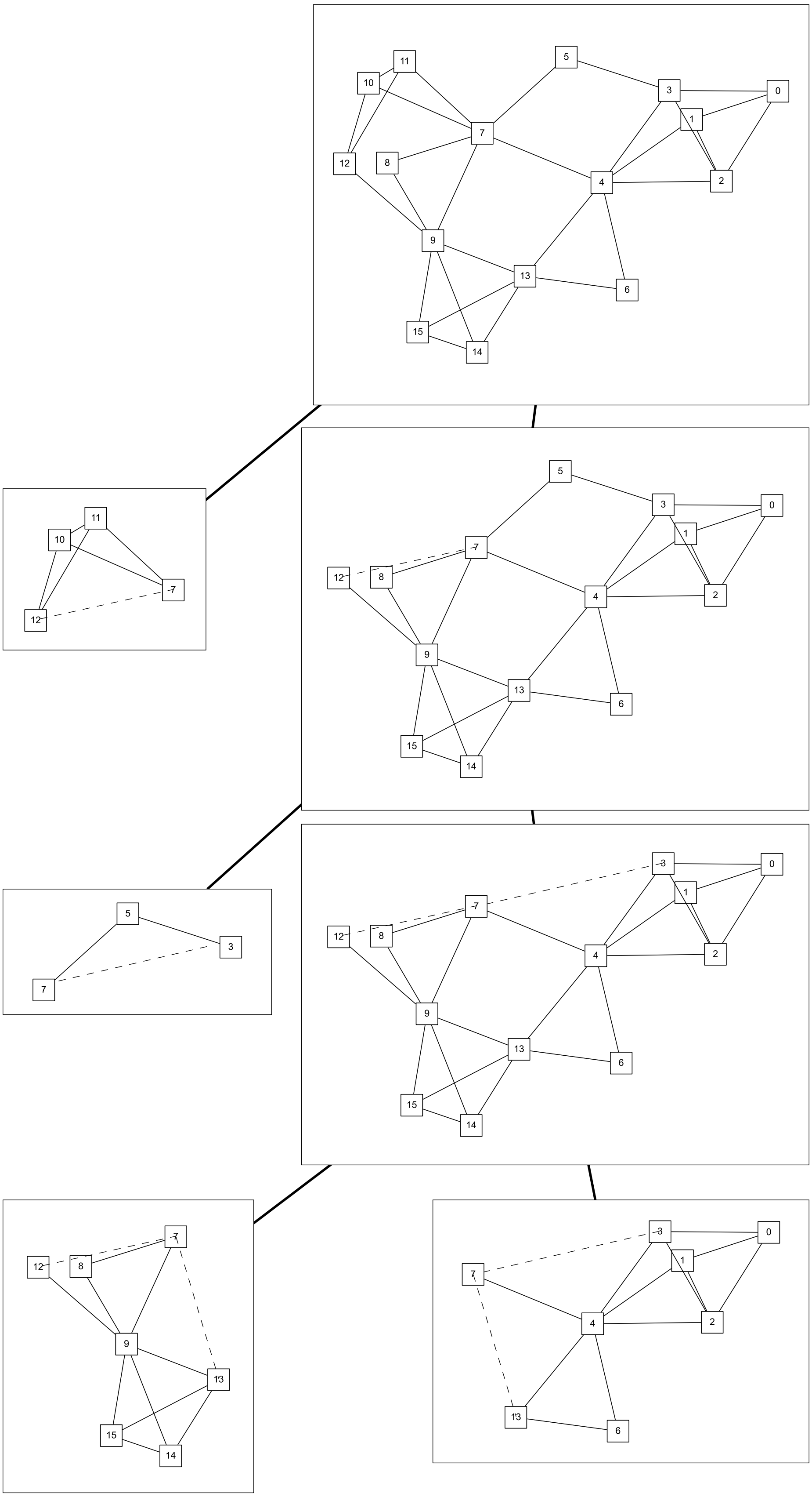


Figure 2.15: Extremal proper 2-cutset decomposition tree of Figure 2.13

Lemma 2.30 (Di Battista and Tamassia [30]). *Two S-nodes (resp. P-nodes) cannot be adjacent in an SPQR-tree.*

Now, we turn our attention to definitions and properties of SPQR-trees that are closely related to finding all 2-cutsets of a biconnected graph.

Definition 2.24 (Virtual edge). A *virtual edge* establishes a relationship between two adjacent nodes of an SPQR-tree of a graph G. Virtual edge is a placeholder for skeletons of both adjacent nodes. Moreover, two vertices (of a skeleton) join by a virtual edge forms a 2-cutset of G.

Subtrees $\mathcal{T}_1, \ldots, \mathcal{T}_k$ of an SPQR-tree $\mathcal{T}$ determine a decomposition of G into edge-disjoint subgraphs $G_1, \ldots, G_k$, such that each subgraph is connected and G_i shares exactly two vertices u_i and v_i, $1 \leq i \leq k$, with the graph corresponding to $\mathcal{T} \setminus \mathcal{T}_i$ [53]. Moreover, there is a virtual edge joinning u_i and v_i.

The following theorem says that every two non-adjacent vertices of an S-node and every pair of vertices which contains a virtual edge in an S-, P-, or R-node of an SPQR-tree of a biconnected graph G are exactly 2-cutsets of G.

Theorem 2.31 (Di Battista and Tamassia [30]). *The 2-cutsets of a graph G are exactly 2-cutsets of nodes' skeletons of an SPQR-tree of G.*

Now, we consider SPQR-trees rooted at one of its nodes in order to perform post-order depth-first algorithm on them. For instance, consider graph G of Figure 2.13. Figure 2.14 shows a rooted SPQR-tree of G. Red and green edges are virtual edges. Green edges connect one skeleton to its parent, as well as red edges connect one skeleton to its child. The root is a Q-node on vertices 0 and 1. Since we omit Q-nodes, root of Figure 2.14 is a wheel graph on vertices 0, 1, 2, 3, and 4.

Our approach starts with a preprocessing step in which SPQR-tree is manipulated to represent only proper 2-cutsets for suitable class of graphs. Then, our approach finishes as follows. Given the proper 2-cutset decomposition tree, we use a post-order depth-first algorithm to build an extremal proper 2-cutset decomposition tree. In preprocessing step, we perform *merge* and *split* operations, defined as follows.

Definition 2.25 (Merge operation). Let $e = xy$ be an edge of an SPQR-tree $\mathcal{T}$ and $e_1 = ab$ (resp. $e_2 = a'b'$) be the virtual edge associated with e in skeleton contained in one endpoint of e (resp. the other endpoint of e). Merge operation consists of the following steps.

1. Collapse vertices a and a' (resp. b and b') to a single vertex.

2. Delete virtual edges e_1 and e_2. Hence, we obtain a multigraph μ.

3. Collapse nodes x and y of $\mathcal{T}$ to a single node w.

4. Let multigraph μ be the skeleton contained in node w.

We refer reader to Figure 2.16 for an example of merge operation's application on SPQR-tree depicted in Figure 2.14.

Definition 2.26 (Split operation)**.** Let $e = xy$ be a non-edge of an S-node S and P_1 (resp. P_2) be one path (resp. the other path) starting at x and finishing at y in the skeleton of S. Split operation consists of the following steps.

1. Split node S into nodes S_1 and S_2.

2. Let P_1 be the skeleton of S_1 and P_2 be the skeleton of S_2.

3. Add virtual edge $e = xy$ to P_1 and P_2, making nodes S_1 and S_2 adjacent.

We refer reader to Figure 2.16 for an example of an application of split operation on SPQR-tree depicted in Figure 2.14.

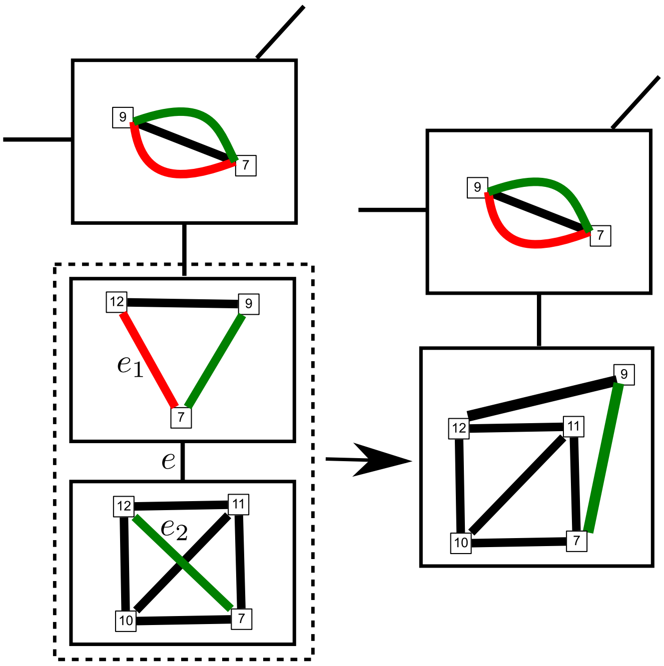


Figure 2.16: Merge operation

Our preprocessing step consists of two applications of the procedure below (described in general terms) to each SPQR-tree node type at a time, namely S-nodes and remaining nodes, respectively. In other words, we first apply the following procedure to S-nodes. As soon as we finish it, we apply to all remaining nodes.

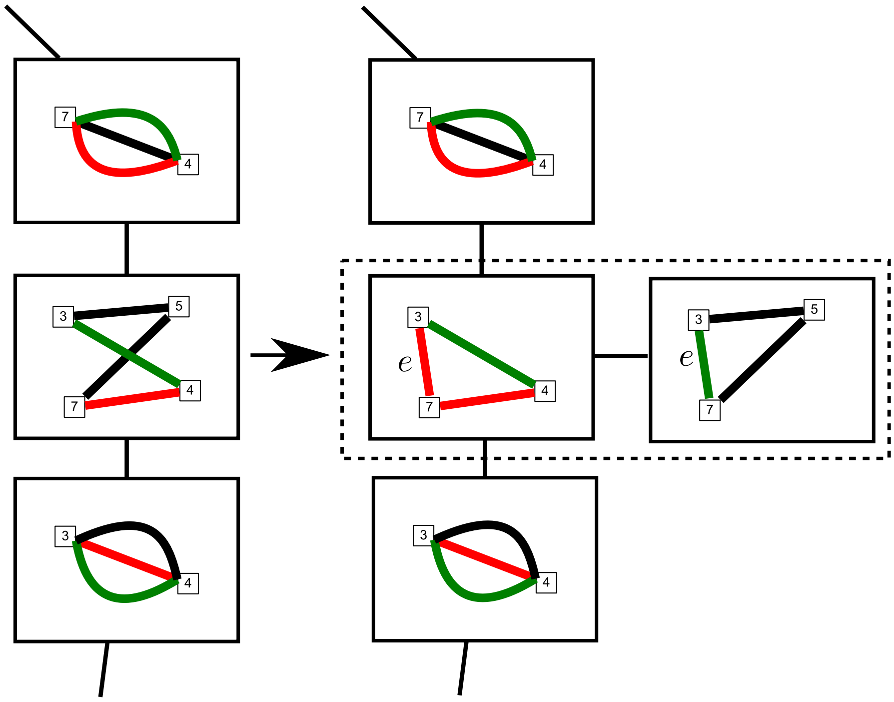


Figure 2.17: Split operation

let $\mathcal{L} \longleftarrow \{\text{root of } \mathcal{T}\}$
while $\mathcal{L} \neq \emptyset$ **do**
  remove a node μ from $\mathcal{L}$
  check type node of μ
    perform either merge or split operations
    for each obtained node N
      for each non-empty subtree $\mathcal{T}_i$ of N **add** root of $\mathcal{T}_i$ to $\mathcal{L}$

Consider an application of the above procedure to check whether μ is an S-node. We perform **split** operations iteratively on non-edges of skeleton of μ, **until** no skeleton obtained so far has a 2-cutset that is not a proper 2-cutset.[4] In other words, we seek for a 2-cutset, which is not a proper 2-cutset, and apply split operation on it. For each obtained node, we apply recursively the same idea until no 2-cutset, which is not a proper 2-cutset, is found.

Now, consider an application of the above procedure to check whether μ is an S-node or not. According to Theorem 2.31, we only have to scan for 2-cutsets joinned by virtual edges. Then, for each virtual edge uv of μ's skeleton, we perform **merge** operation iteratively on the skeleton of both nodes sharing vertices u and v, **if** $\{u, v\}$ is not a proper 2-cutset. In other words, we seek for a 2-cutset, which is

[4]It is easy to see that a cycle with n vertices has $O(n)$ non-intersecting non-edges, i.e. we can apply iteratively at most $O(n)$ split operations on an S-node with a skeleton of size n.

not a proper 2-cutset, and apply merge operation on it and on its neighbors with the same 2-cutset. For the obtained node, we apply recursively the same idea until no 2-cutset, which is not a proper 2-cutset, is found.

We output the obtained tree as a proper 2-cutset decomposition tree. Now, we proceed as follows to build an extremal proper 2-cutset decomposition tree. Consider decomposition tree $\mathcal{T}^*$ (output by preprocessing step) with root μ^*. In order to build an extremal decomposition tree $\mathcal{T}^e$, we perform a post-order depth-first traversal on $\mathcal{T}^*$, as follows.

- If μ has no subtrees, then we output $\mathcal{T}^e$ as a tree with μ as a left subtree and with an empty node as a right subtree.
- If μ has $k \geq 1$ subtrees, for each non-empty subtree $\mathcal{T}_i^*$, $1 \leq i \leq k$, we build recursively extremal decomposition tree $\mathcal{T}_i^e$. Then, we replace unique empty node of $\mathcal{T}_{i+1}^e$ by $\mathcal{T}_i^e$, $1 \leq i \leq k-1$. Finally, we output $\mathcal{T}^e$ as a tree with μ (without its subtrees) as a left subtree and $\mathcal{T}_k^e$ as a right subtree.

We shall prove that our approach fully decomposes graphs by proper 2-cutsets in linear-time. Indeed, we shall prove a slightly stronger version, as follows.

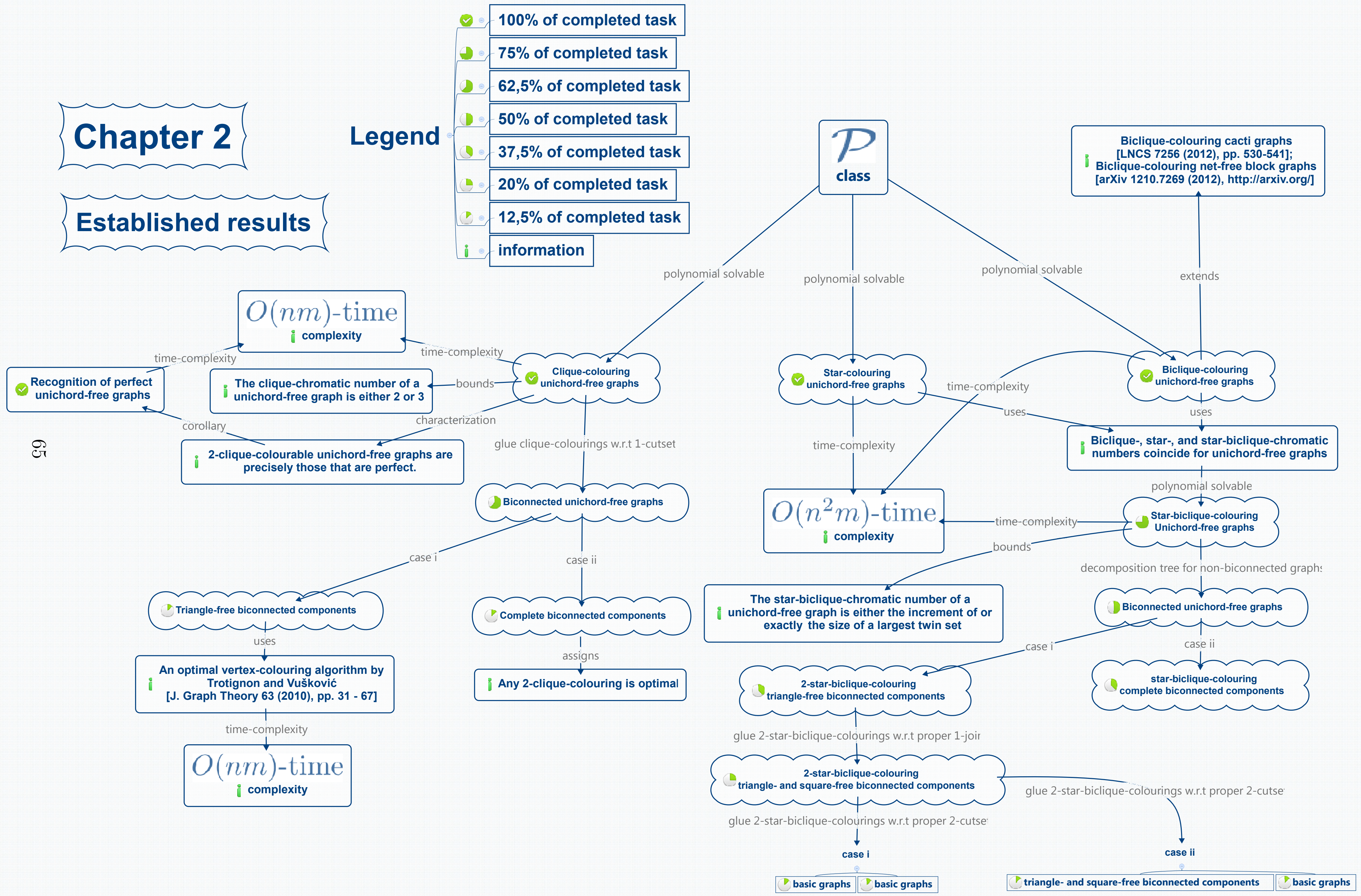
Theorem 2.32 (To be proved)**.** *There is a linear-time algorithm to build an extremal proper 2-cutset decomposition tree for graphs.*

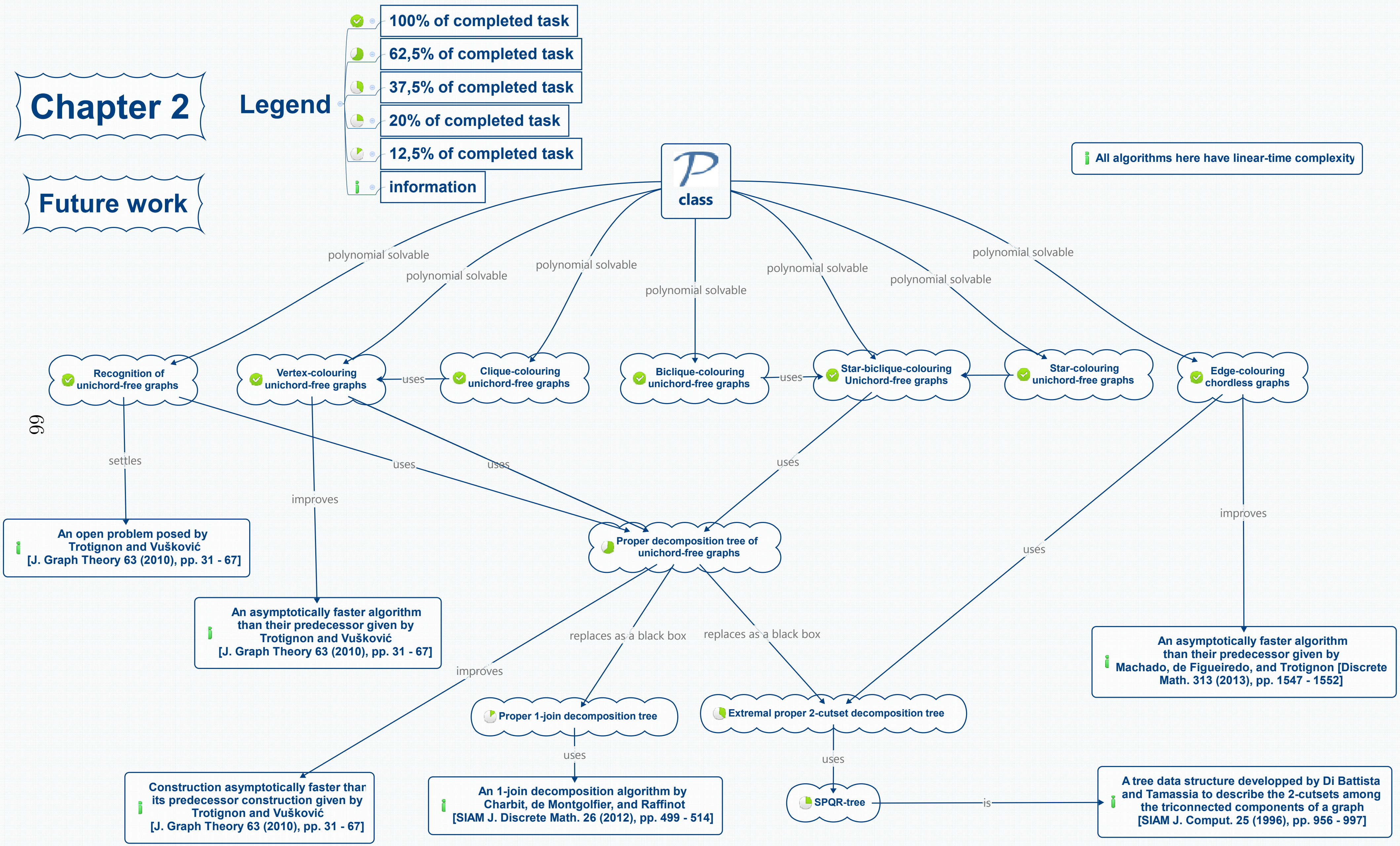
This theorem would lead us to optimize our clique-, biclique-, and star-colourings algorithms for unichord-free graphs. Moreover, we would be able to settle an open problem posed by Trotignon and Vušković to recognize unichord-free graphs in linear-time [96, Section 5]. In order to achieve such improvement, it is enough to replace their algorithm to build a proper 2-cutset decomposition tree by ours. This is also enough to improve their $O(nm)$-time optimal vertex-colouring algorithm to linear-time complexity [96], which implies also improving our $O(nm)$-time optimal clique-colouring algorithm to linear-time complexity. We could also infer other linear-time algorithms that are asymptotically faster than their predecessors, such as edge-colouring chordless graphs [71] and some algorithms given by Lévêque *et al.* [61].

Listing 2.1: How to build an SPQR-tree with OGDF [20]

```
#include <ogdf/basic/Graph.h>
#include <ogdf/energybased/FMMMLayout.h>
#include <ogdf/decomposition/StaticSPQRTree.h>
#include <stdlib.h>
#include <stdio.h>

using namespace ogdf;
using namespace std;

Array<edge> aSimpleBiconnectedGraph(Graph &G)
{
        // Insert here the procedure to create the biconnected graph G.
        // Assign the edges of G to proper variable edges.

        return edges;
}

int main()
{
        char buffer[20];

        // Create layout and set graph drawing algorithm
        FMMMLayout fmmm;
        fmmm.useHighLevelOptions(true);
        fmmm.unitEdgeLength(100.0);
        fmmm.newInitialPlacement(true);
        fmmm.pageFormat(FMMMLayout::pfLandscape);
        fmmm.qualityVersusSpeed(FMMMLayout::qvsGorgeousAndEfficient);

        //Create graph G;
        Graph G;
        Array<edge> edges = aSimpleBiconnectedGraph(G);
        GraphAttributes GA1(G, GraphAttributes::nodeGraphics |
                        /* insert other graph attributes here */ );
        GA1.directed(false);
        fmmm.call(GA1);
        GA1.writeGML("Graph-G.gml");

        // Create SPQR Tree T of G (this is where magic happens)
        StaticSPQRTree *SPQR = new StaticSPQRTree(G);
        Graph T = SPQR->tree();
        GraphAttributes GA2(T, GraphAttributes::nodeGraphics |
                        /* insert other graph attributes here */ );
        GA2.directed(false);
        fmmm.call(GA2);
```

```
        sprintf (buffer, "SPQR-of-G.gml");
        GA2.writeGML(buffer);

        return 0;
}
```


Chapter 2
Established results
Legend
100% of completed task
75% of completed task
62,5% of completed task
50% of completed task
37,5% of completed task
20% of completed task
12,5% of completed task
information
$\mathcal{P}$ class
polynomial solvable
polynomial solvable
polynomial solvable
Biclique-colouring cacti graphs [LNCS 7256 (2012), pp. 530-541]; Biclique-colouring net-free block graphs [arXiv 1210.7269 (2012), http://arxiv.org/]
extends
$O(nm)$-time complexity
time-complexity
time-complexity
Recognition of perfect unichord-free graphs
The clique-chromatic number of a unichord-free graph is either 2 or 3
bounds
Clique-colouring unichord-free graphs
corollary
characterization
2-clique-colourable unichord-free graphs are precisely those that are perfect.
glue clique-colourings w.r.t 1-cutset
Biconnected unichord-free graphs
case i
case ii
Triangle-free biconnected components
Complete biconnected components
uses
assigns
An optimal vertex-colouring algorithm by Trotignon and Vušković [J. Graph Theory 63 (2010), pp. 31 - 67]
Any 2-clique-colouring is optimal
time-complexity
$O(nm)$-time complexity
Star-colouring unichord-free graphs
time-complexity
time-complexity
$O(n^2m)$-time complexity
Biclique-colouring unichord-free graphs
uses
uses
Biclique-, star-, and star-biclique-chromatic numbers coincide for unichord-free graphs
polynomial solvable
Star-biclique-colouring Unichord-free graphs
time-complexity
bounds
The star-biclique-chromatic number of a unichord-free graph is either the increment of or exactly the size of a largest twin set
decomposition tree for non-biconnected graphs
Biconnected unichord-free graphs
case i
case ii
2-star-biclique-colouring triangle-free biconnected components
star-biclique-colouring complete biconnected components
glue 2-star-biclique-colourings w.r.t proper 1-join
2-star-biclique-colouring triangle- and square-free biconnected components
glue 2-star-biclique-colourings w.r.t proper 2-cutset
glue 2-star-biclique-colourings w.r.t proper 2-cutset
case i
case ii
basic graphs
basic graphs
triangle- and square-free biconnected components
basic graphs



Chapter 2
Future work
Legend
100% of completed task
62,5% of completed task
37,5% of completed task
20% of completed task
12,5% of completed task
information
P class
All algorithms here have linear-time complexity
polynomial solvable
Recognition of unichord-free graphs
Vertex-colouring unichord-free graphs
Clique-colouring unichord-free graphs
Biclique-colouring unichord-free graphs
Star-biclique-colouring Unichord-free graphs
Star-colouring unichord-free graphs
Edge-colouring chordless graphs
uses
settles
improves
An open problem posed by Trotignon and Vušković [J. Graph Theory 63 (2010), pp. 31 - 67]
An asymptotically faster algorithm than their predecessor given by Trotignon and Vušković [J. Graph Theory 63 (2010), pp. 31 - 67]
Proper decomposition tree of unichord-free graphs
replaces as a black box
Proper 1-join decomposition tree
Extremal proper 2-cutset decomposition tree
An asymptotically faster algorithm than their predecessor given by Machado, de Figueiredo, and Trotignon [Discrete Math. 313 (2013), pp. 1547 - 1552]
Construction asymptotically faster than its predecessor construction given by Trotignon and Vušković [J. Graph Theory 63 (2010), pp. 31 - 67]
An 1-join decomposition algorithm by Charbit, de Montgolfier, and Raffinot [SIAM J. Discrete Math. 26 (2012), pp. 499 - 514]
SPQR-tree
is
A tree data structure developped by Di Battista and Tamassia to describe the 2-cutsets among the triconnected components of a graph [SIAM J. Comput. 25 (1996), pp. 956 - 997]

Chapter 3

Powers of Cycles and Paths

This chapter is devoted to the results below.

- *We determine 2-biclique- and 2-star-colourable powers of cycles by using a very similar idea of Euclid's algorithm.*
- *Constant-time algorithms to compute star- and biclique-chromatic numbers of powers of cycles and powers of paths.*
- *Linear-time optimal biclique- and star-colourings algorithms for powers of cycles and for powers of paths that address connections with number theory and colourings of graphs in which no induced P_3 is monochromatic.*

We start with the detached definition of powers of cycles and powers of paths since they are the centerpiece of this chapter. We remark that the results established in this chapter are handled in a high-level approach and reader is invited to check omitted proofs and details in Appendix B, Page 156. Nevertheless, we invite reader to check that we rephrased most of the results of Appendix B to focus on algorithmic aspects that are not handled there. All algorithm analysis, that are not given in the proofs of Appendix B, are given before their respective statement at this chapter. Last, but not least, the reader can get a big picture of this chapter results at Page 86.

Definition 3.1 (Power of a cycle)**.** A *power of a cycle* C_n^k, for $n, k \geq 1$, is a simple graph on n vertices with $V(G) = \{v_0, \ldots, v_{n-1}\}$ and $\{v_i, v_j\} \in E(G)$ if, and only if, $\min\{(j-i) \bmod n, (i-j) \bmod n\} \leq k$.

In a power of a cycle C_n^k, we take $(v_0, \ldots, v_{n-1})$ to be a *cyclic order* on the vertex set and we always perform arithmetic modulo n on vertex indices. Notice that C_n^1 is the induced cycle C_n on n vertices and C_n^k, $n \leq 2k+1$, is the complete graph K_n on n vertices.

Definition 3.2 (Power of a path)**.** A *power of a path* P_n^k, for $n, k \geq 1$, is a simple graph on n vertices with $V(G) = \{v_0, \ldots, v_{n-1}\}$ and $\{v_i, v_j\} \in E(G)$ if, and only if, $|i - j| \leq k$.

In a power of a path P_n^k, we take $(v_0, \ldots, v_{n-1})$ to be a *linear order* on the vertex set and we always perform arithmetic modulo n on vertex indices. Notice that P_n^1 is the induced path P_n on n vertices and P_n^k, $n \leq k+1$, is the complete graph K_n on n vertices.

A very interesting (and useful) remark is that a power of a path P_n^k can be seen as the subgraph of C_{n+3k}^k with n consecutive vertices in common, *e.g.* $v_0, \ldots, v_{n-1}$.

Definition 3.3 (Reach)**.** In a power of a cycle C_n^k, the *reach* of an edge $\{v_i, v_j\}$ is $\min\{(i-j) \bmod n, (j-i) \bmod n\}$. In a power of a path P_n^k, the *reach* of an edge $\{v_i, v_j\}$ is $|i - j|$.

Notice that the reach of every edge of a power of a cycle C_n^k and of a power of a path P_n^k is at most k. The definition of reach is extended to an induced path to be the sum of the reach of its edges.

Definition 3.4 (Block)**.** A *block* is a maximal set of consecutive vertices with the same colour w.r.t cyclic ordering (resp. linear ordering). The *size* of a block is the number of vertices in the block.

Powers of cycles and powers of paths have been recently investigated in the context of well studied variations of colouring problems. For instance, we have the following results related to powers of cycles of order n raised to the k-th power.

- Their b-chromatic number [36] is

$$\begin{cases} n, & \text{if } n \leq 2k+1 \\ k+1, & \text{if } n = 2k+2 \\ \text{at least } \min(n-k-1, k+1+\lfloor\frac{n-k-1}{3}\rfloor), & \text{if } 2k+3 \leq n \leq 3k \\ k+1+\lfloor\frac{n-k-1}{3}\rfloor, & \text{if } 3k+1 \leq n \leq 4k \\ 2k+1, & \text{if } n \geq 4k+2 \end{cases}$$

- Their chromatic number and their choice number are both

$$k + 1 + \lceil r/q \rceil,$$

  in which $n = q(k+1) + t$ with $q \geq 1$, $0 \leq t \leq k$ and $n \geq 2k+1$ [85].

- Their total-chromatic number is at most their maximum degree plus 2, when n is even and $n \geq 2k+1$. Still, total-colouring is an open problem [16].

- Their clique-chromatic number is [17]

$$\begin{cases} 3, & \text{for odd cycles of size at least 5} \\ 2, & \text{otherwise} \end{cases}$$

Related to powers of paths of order n and raised to the k-th power, their b-chromatic number is precisely [36]

$$\begin{cases} n, & \text{if } n \leq k+1 \\ k+1+\lfloor \frac{n-k-1}{3} \rfloor, & \text{if } k+2 \leq n \leq 4k+1 \\ 2k+1, & \text{if } n \geq 4k+2 \end{cases}$$

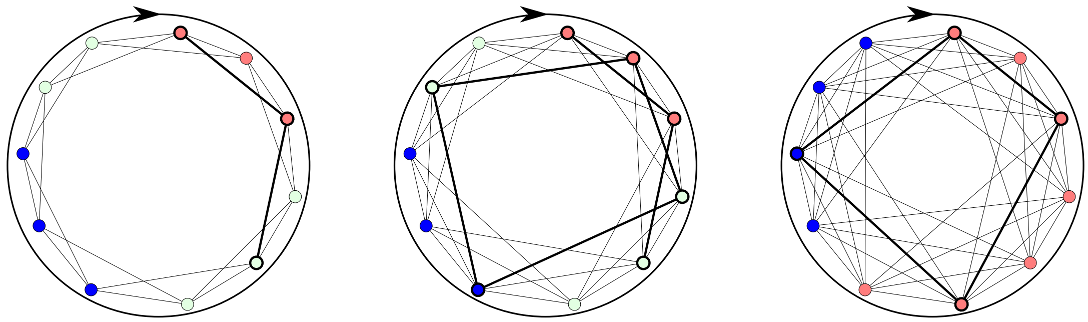
Notice that other significant works have been done in power graphs [15, 18]. In particular, works in powers of cycles and powers of paths [11, 13, 58, 62, 63, 99].

Now, we turn to an overview of our strategy to solve biclique-colouring of powers of cycles and powers of paths. We say that a biclique of size 2 is a P_2 biclique, that a biclique of size 3 is a P_3 biclique, that a star of size 3 is a P_3 star, and that a biclique of size 4, other than star, is a C_4 biclique.

Regarding complete powers of cycles and powers of paths, both with n vertices, it is easy to see that they have biclique-chromatic number n.

Regarding non-complete powers of cycles and powers of paths, we address connections with number theory in order to solve biclique-colouring problem. In particular, we use the division algorithm to provide 3-colourings for powers of cycles, in which no induced P_3 is monochromatic. At first glance, this special colouring may sound unrelated to our purposes, but we invite the reader to check that every biclique is either a P_2 or contains (not necessarily properly) an induced P_3 as a subgraph. Then, for graphs without P_2 biclique, we get an upper bound of k for the biclique-chromatic number as long as we have a k-colouring, in which no induced P_3 is monochromatic. Indeed, this is our strategy to provide a linear-time algorithm to assign a 3-biclique-colouring for powers of cycles without P_2 biclique, which also yields a tight upper bound of 3 to their biclique-chromatic number. As we shall see, powers of cycles without P_2 biclique are precisely non-complete powers of cycles. We refer the reader to see Figure 3.1 for distinct non-complete powers of cycles.

Regarding powers of paths, we recall that a power of a path P_n^k is an induced subgraph of a power of a cycle C_{n+3k}^k both with n consecutive vertices in common. Then, we can also infer a biclique-colouring algorithm and an upper bound for the biclique-chromatic number for powers of paths without P_2 biclique. An interesting fact about our 3-colouring, in which no induced P_3 is monochromatic, follows. One colour class is properly contained in $V(C_{n+3k}^k) \setminus V(P_n^k)$, so we can actually infer a 2-



(a) Power of a cycle C_{11}^2 has only P_3 bicliques

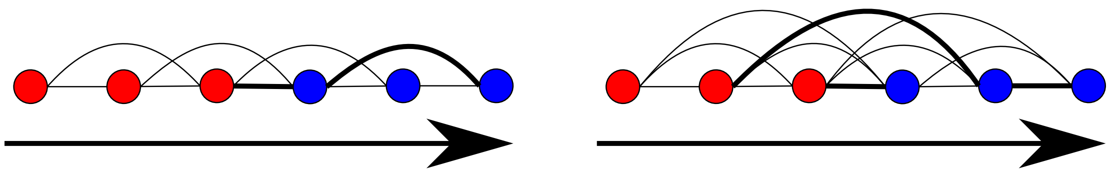
(b) Power of a cycle C_{11}^3 has only P_3 bicliques and C_4 bicliques

(c) Power of a cycle C_{11}^4 has only C_4 bicliques

Figure 3.1: Non-complete powers of cycles

colouring for powers of paths, in which no induced P_3 is monochromatic. It provides us not only an upper bound for the biclique-chromatic number of powers of paths without P_2 biclique, but also an optimal biclique-colouring algorithm for them. We refer the reader to see Figure 3.2a for a non-complete power of a path without P_2 biclique.

In contrast to non-complete powers of cycles, powers of paths without P_2 biclique is a proper subset of non-complete powers of paths. We refer the reader to see Figure 3.2b for a non-complete power of a path with P_2 biclique.



(a) Power of a path P_6^2 has only P_3 bicliques

(b) Power of a path P_6^3 has only P_2 bicliques and P_3 bicliques

Figure 3.2: Non-complete powers of paths

Regarding non-complete powers of paths with P_2 bicliques, we have an optimal biclique-colouring algorithm by first using our 2-colouring in which no induced P_3 is monochromatic and, then, assigning the least number of new and distinct colours on vertices to make each P_2 biclique polychromatic.

Now, let me recap our results so far. We gave an upper bound of 3 to biclique-chomatic number of non-complete powers of cycles and an optimal biclique-colouring algorithm for powers of paths. If you go back to the very beggining of this overview and replace all biclique words by star words, you can see that we also have an upper bound of 3 to the star-chomatic number of non-complete powers of cycles and an optimal star-colouring algorithm for powers of paths.

A fast-paced reader may think that optimal biclique- and star-colourings coincide for powers of cycles, but this is not true. We refer the reader to see Figure 3.1c and to check that such power of a cycle C_{11}^4 is 2-biclique-colourable, but it is not 2-star-colourable. Indeed, we shall see that powers of cycles with only C_4 bicliques are 2-biclique-colourable, but not necessarily 2-star-colourable. For now, keep powers of cycles with only C_4 bicliques in the pocket. We will be coming back to this case after looking very carefully at non-complete powers of cycles with P_3 bicliques.

Regarding non-complete powers of cycles with P_3 biclique, we use a restricted version of that colouring in which no induced P_3 is monochromatic. The restricted version assumes that we use only 2 colours and every monochromatic block has size either k or $k+1$. We shall prove that, for a given non-complete power of a cycle C_n^k with P_3 biclique, this restricted version is possible if, and only if, C_n^k is 2-biclique-colourable. Then, we address another connection with number theory. We infer a system of equations to decide if the restricted version is possible and we use a very similar idea of Euclid's algorithm to infer an algorithm to decide the biclique-chromatic number for non-complete powers of cycles with P_3 biclique. This algorithm has the surprisingly $O(1)$-time complexity, since we use only operations that are executed in a constant amount of time by the random-access machine model of computation [24, Page 23]. Moreover, we infer an optimal algorithm for biclique-colouring non-complete powers of cycles with P_3 biclique, as follows. We use the restricted version in those non-complete powers of cycles with P_3 biclique which are 2-biclique-colourable, while we use the non-restricted version in those that are not 2-biclique-colourable. We refer the reader to see Figure 3.1a and Figure 3.1b for non-complete powers of cycles with P_3 bicliques.

Now, get non-complete powers of cycles with only C_4 bicliques off the pocket. A very interesting fact is that star- and biclique-colourings now fall apart. We shall prove that, for a given non-complete power of a cycle C_n^k with P_3 star, the restricted version is possible if, and only if, C_n^k is 2-star-colourable. Notice that non-complete powers of cycles with only C_4 bicliques have P_3 stars. Then, the restricted version availability is a characterization that works for star-colouring *all* non-complete powers of cycles and for biclique-colouring *only* non-complete powers of cycles without C_4 biclique. Notice that the optimal algorithm for biclique-colouring non-complete powers of cycles with P_3 biclique is the same for star-colouring non-complete powers of cycles. We refer the reader to see Figure 3.1c for a non-complete power of a cycle with only C_4 bicliques.

Finally, we refer the reader to Figure 3.3 for our results about biclique- and star-chromatic numbers of powers of cycles and powers of paths.

This chapter is organized as follows. Section 3.1 identifies all bicliques of powers of cycles and powers of paths. Section 3.2 determines linear-time optimal biclique-

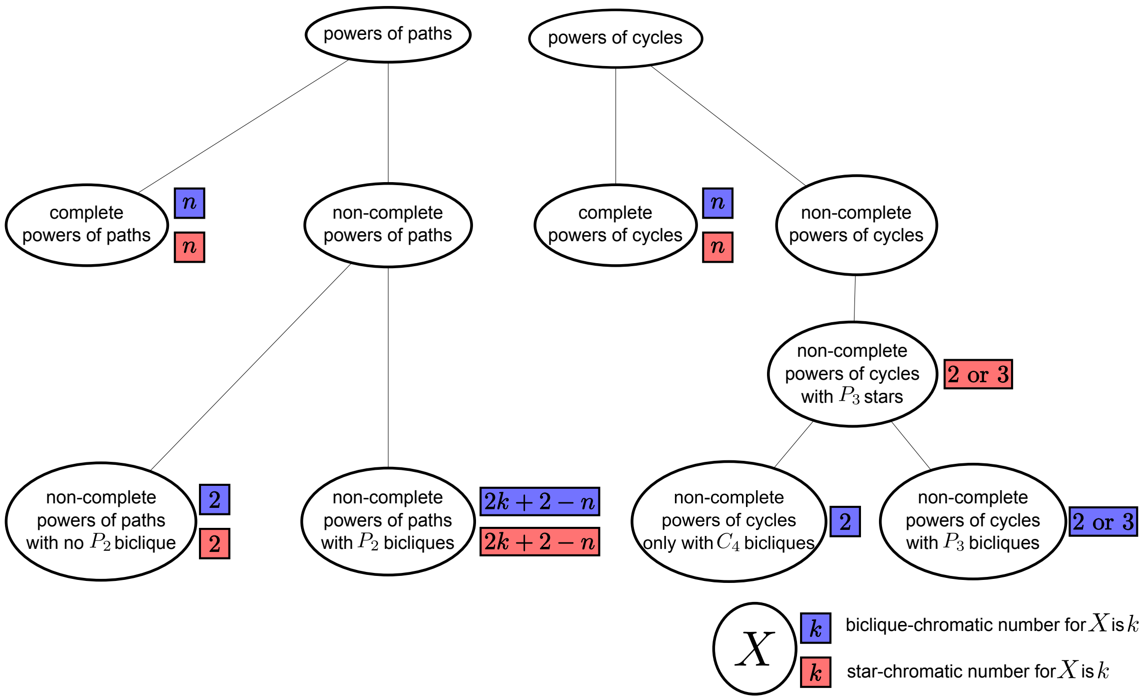


Figure 3.3: Biclique- and star-chromatic numbers of powers of cycles and of paths

algorithms for powers of cycles and powers of paths by addressing connections with number theory and colourings of graphs, in which no induced P_3 is monochromatic. Moreover, we determine constant-time algorithm to compute the biclique-chromatic number of powers of cycles and powers of paths. Section 3.3 states results about star-colouring powers of cycles and powers of paths by addressing connections with all results related to biclique-colouring. We finish this chapter with Section 3.4, which contains our concluding remarks and future work.

3.1 Bicliques

We start by identifying a key property about bicliques of powers of cycles and of paths. In a general graph, it is easy to see that a pair of vertices is a P_2 biclique if, and only if, these two vertices are true twins. In particular, every pair of universal vertices defines a P_2 biclique, and every biclique, which is not a P_2 biclique, contains a P_3 as an induced subgraph.

Claim 3.1 (Claim 3, Appendix B). *Every P_2 biclique of a power of a cycle is a pair of universal vertices, as well as every P_2 biclique of a power of a path.*

As a consequence, a non-complete power of a cycle has no P_2 bicliques, as well as a power of a path P_n^k with $n \geq 2k+1$. Considering that powers of cycles and powers of paths are $K_{1,3}$-free, and that powers of paths are C_4-free, we remark that bicliques of non-complete powers of cycles and non-complete powers of paths are very restricted. Now, we explicitly identify bicliques of powers of cycles and powers of paths. Notice that, for each value of n in the considered range, every biclique in Claim 3.2 and in Claim 3.3 always exist. We refer to Figure 3.1 to illustrate distinct biclique structures for each considered case of non-complete powers of cycles, as well as we refer to Figure 3.2 to illustrate distinct biclique structures for each considered case of non-complete powers of paths.

Claim 3.2 (Inline, without proof, Appendix B). *Bicliques of a non-complete power of a cycle C_n^k are precisely*

$$\begin{cases} C_4, & \text{if } 2k+2 \leq n \leq 3k+1 \\ P_3 \text{ and } C_4, & \text{if } 3k+2 \leq n \leq 4k \\ P_3, & \text{if } n \geq 4k+1 \end{cases}$$

Proof. A power of a cycle is $K_{1,3}$-free. Thus, bicliques of a power of a cycle are possibly P_2, P_3 or C_4 bicliques. Let C_n^k be a power of a cycle with $n \geq 2k+2$. We shall prove that there is no P_2 biclique, i.e. every P_2 is properly contained in an induced P_3. Let v_i and v_j be two adjacent vertices in C_n^k such that $i < j$, and let v_ℓ be the last consecutive vertex after v_j and adjacent to v_i along the cyclic order. It follows that $v_{\ell+1}$ is not adjacent to v_i, but $v_{\ell+1}$ is adjacent to v_j, and that vertices v_i, v_j, and $v_{\ell+1}$ define a P_3. Thus, in what follows, each biclique is possibly P_3 or C_4 biclique.

Let G be a power of a cycle C_n^k with $2k+2 \leq n \leq 4k$. Since $2k+2 \leq n \leq 4k$, the subset of vertices $H = \{v_0, v_{\lceil \frac{n}{4} \rceil}, v_{\lceil \frac{n}{2} \rceil}, v_{\lceil \frac{3n}{4} \rceil}\}$ is a C_4 biclique. Hence, graph G has a C_4 biclique.

Let G be a power of a cycle C_n^k with $2k+2 \leq n \leq 3k+1$. Suppose $P' = \{v_{h'}, v_{s'}, v_{r'}\}$ is a P_3. By symmetry, assume that the missing edge is $\{v_{h'}, v_{r'}\}$ and that $h' < s' < r'$. Since $2k+2 \leq n \leq 3k+1$, vertices $v_{h'}$ and $v_{r'}$ have a common neighbour – with index at most $h'-1$ and at least $r'+1$ – which is not a neighbour of $v_{s'}$. We conclude that every P_3 is contained in a C_4, and so, graph G contains only C_4 biclique.

Now, let G be a power of a cycle C_n^k with $n \geq 3k+2$. Consider P_3 induced by vertices v_0, v_k, and v_{k+1}. Since $n \geq 3k+2$, vertices v_0 and v_{k+1} have no common neighbour with index at least $k+2$. Hence, graph G has a P_3 biclique.

Finally, let G be a power of a cycle C_n^k with $n \geq 4k+1$. Suppose $P = \{v_h, v_s, v_r\}$ is an induced P_3. By symmetry, assume that the missing edge is $\{v_h, v_r\}$ and that $h < s < r$. Since $n \geq 4k+1$, vertices v_h and v_r have no common neighbour with index at most $h-1$, and at least $r+1$. Hence, graph G has no C_4 biclique. □

Claim 3.3 (Inline, without proof, Appendix B). *Bicliques of a non-complete power of a path P_n^k are precisely*

$$\begin{cases} P_2 \text{ and } P_3, & \text{if } k+2 \leq n \leq 2k \\ P_3, & \text{if } n \geq 2k+1 \end{cases}$$

Proof. A power of a path is $K_{1,3}$-free and C_4-free. Thus, bicliques of a power of a path are possibly P_2 or P_3 bicliques.

Let G be a power of a path P_n^k with $k+2 \leq n \leq 2k$. First, notice that v_k and v_{n-1-k} are distinct vertices, since $k = n-1-k$ implies $2k+1 = n$ (a contradiction). Moreover, $n \leq 2k$ implies that both v_k and v_{n-1-k} are universal vertices. Recall that a pair of universal vertices induces a P_2 biclique. Clearly, vertices v_0, v_k, and v_{k+1} are distinct and define a P_3 biclique.

Now, let G be a power of a path P_n^k with $n \geq 2k+1$. We shall prove that there is no P_2 biclique, i.e. every P_2 is properly contained in an induced P_3. Let v_i and v_j be two adjacent vertices in G, such that $i < j$. If $j \leq k$, vertices v_i, v_j, v_{j+k} induce a P_3. Otherwise, i.e. $j \geq k+1$, vertices $v_{j-(k+1)}, v_i, v_j$ induce a P_3. Moreover, recall that G is C_4-free. Then, G has only P_3 bicliques. □

3.2 Biclique-colouring

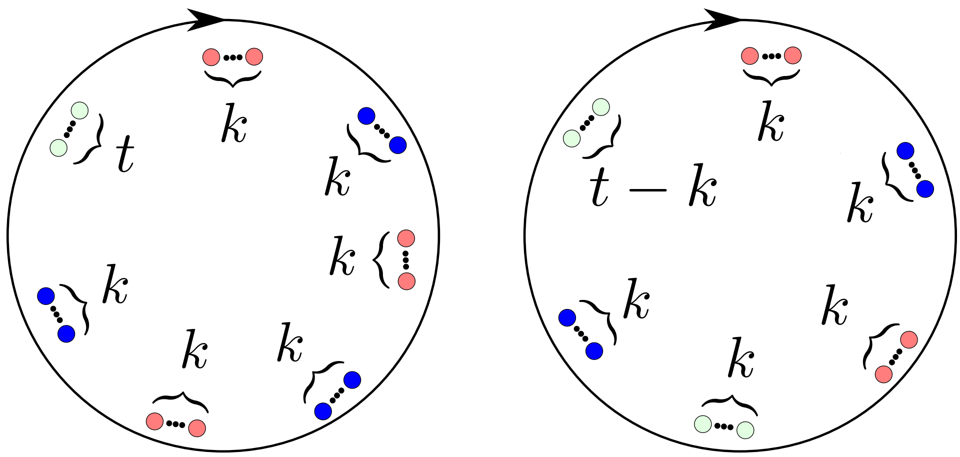
We determine biclique-colouring algorithms for powers of cycles and powers of paths by addressing connections with number theory and the so-called division algorithm. The latter states that every natural number a can be expressed using the equation $a = bq + t$, with a requirement that $0 \leq t < b$. A proof of the division algorithm can be seen in the book of Niven and Zuckerman [78]. We shall use the following equivalent version in which b is even and $0 \leq t < 2b$.

Theorem 3.4 (Division algorithm)**.** *Given two natural numbers n and k, with $n \geq 2k$, there are unique natural numbers a and t such that $n = ak + t$, $a \geq 2$ is even, and $0 \leq t < 2k$.*

We now begin with Lemma 3.5 that will provide us an optimal biclique-colouring algorithm for non-complete powers of paths. It will also establish a 3-biclique-colouring algorithm for non-complete powers of cycles. Lemma 3.5 uses division algorithm and it aims to assign a 3-colouring π for every power of a cycle C_n^k such that no induced P_3 is monochromatic, as follows. Consider a and t given by Theorem 3.4. The random-access machine model of computation assumes that division, floor, modulo, and conditional operations are constant-time [24, Page 23], which implies that we can compute $a = \lfloor \frac{n}{k} \rfloor$ and $t = n \bmod k$ in constant-time. If $0 \leq t \leq k$, we define 3-colouring π with *blue*, *red*, and *green* colours as an even number a of monochromatic-blocks of size k switching *red* and *blue* colours alternately, followed by a monochromatic-block of size t with *green* colour. Otherwise, i.e. $k < t < 2k$, we define 3-colouring π with *blue*, *red*, and *green* colours as an odd number $a-1$ of monochromatic-blocks of size k switching *red* and *blue* colours alternately, followed by a monochromatic-block of size k with *green* colour, a monochromatic-block of size k with *blue* colour, and a monochromatic-block of size $t-k$ with *green* colour. We refer to Figure 3.4a to illustrate the former 3-colouring and to Figure 3.4b to illustrate the latter 3-colouring. Both colourings imply the following lemma.

Lemma 3.5 (Lemma 5, rephrased, Appendix B)**.** *Let G be a non-complete power of a cycle C_n^k. There is a linear-time algorithm to assign a 3-colouring to G such that*

1. *G has **no** monochromatic P_3.*
2. *v_{k-1} and v_k have distinct colours.*
3. *One colour class is contained in $\{v_{-1}, \ldots, v_{-3k}\}$.*



(a) 3-colouring without monochromatic P_3, when $n \geq 2k+2$ and $0 \leq t \leq k$

(b) 3-colouring without monochromatic P_3, when $n \geq 2k+2$ and $k < t < 2k$

Figure 3.4: 3-colourings without monochromatic P_3 given by Lemma 3.5

One of the virtues of Lemma 3.5 is that it can be applied to induced subgraphs of powers of cycles, starting with powers of paths.

3.2.1 Powers of Paths

The extreme cases are easy to compute. The dense case occurs when $n \leq k+1$, which implies that a power of a path P_n^k is the complete graph K_n with biclique-chromatic number n. For the non-complete case, the sparsest case P_n^k occurs when $k = 1$, which implies that a power of a path P_n^k is the chordless path P_n with biclique-chromatic number 2. Now, we consider the sparse case $n \in [k+2, \infty)$, which is precisely the case of non-complete powers of paths. Lemma 3.5 has some interesting consequences on biclique-colouring non-complete powers of paths, showing that there is a linear-time algorithm to assign a 2-colouring to every power of a path such that no P_3 is monochromatic and $\{v_{k-1}, v_k\}$ is polychromatic.

Lemma 3.6 (Lemma 6, rephrased, Appendix B)**.** *Let G be a non-complete power of a path P_n^k. There is a linear-time algorithm to assign a 2-colouring to G in which*

1. *G has **no** monochromatic P_3.*
2. *v_{k-1} and v_k have distinct colours.*

As a corollary of Lemma 3.6, we can settle the case of non-complete powers of paths. Consider case $n \in [k+2, 2k]$. On one hand, vertices $v_{n-1-k}, \ldots, v_k$ are all $2k+2-n$ universal vertices of G, which implies a biclique-chromatic number of at least $2k+2-n$. On the other hand, we use 2-colouring of Lemma 3.6, so all P_3 are polychromatic, and vertices v_{k-1} and v_k have distinct colours. We assign new and distinct colours for vertices $v_{n-1-k}, \ldots, v_{k-2}$ that are all $2k-n$ remaining universal vertices. Then, every P_2 and P_3 bicliques are polychromatic, which implies that G has a $(2k+2-n)$-biclique-colouring. Now, consider the case $n \in [2k+1, \infty]$. We have no pair of twin vertices and no P_2 biclique, which implies that 2-colouring of Lemma 3.6 is enough to assign a 2-biclique-colouring. This paragraph implies the following theorems.[1]

Theorem 3.7 (Corollary 7, Appendix B)**.** *A non-complete power of a path P_n^k has biclique-chromatic number* $\max(2, 2k+2-n)$*.*

Theorem 3.8 (Implicit, Appendix B)**.** *There is a linear-time optimal biclique-colouring algorithm for powers of paths.*

Alternatively, we can define a very similar linear-time $(2k+2-n)$-biclique-colouring algorithm for a non-complete power of a path $G = P_n^k$, when $k+2 \leq n \leq 2k$. Let $\pi : V(G) \to \{1, \ldots, 2k+2-n\}$ be an assignment of (arbitrarily) distinct colours $3, \ldots, 2k+2-n$ to vertices $v_{n-k}, \ldots, v_{k-1}$, respectively. Use colour 1 in the uncoloured vertices with indices in $\{0, \ldots, n-k-1\}$ and colour 2 in the uncoloured vertices with indices in $\{k, \ldots, n-1\}$. We refer to Figure 3.5 to illustrate the given $(2k+2-n)$-biclique-colouring.

The linear-time 2-biclique-colouring algorithm given by Lemma 3.6 for the case $n \in [2k+1, \infty)$ can be seen in Figure 3.6.

[1] Recall that complete powers of paths with n vertices have biclique-chromatic number n, which implies that all vertices should have distinct colours in an optimal biclique-colouring.

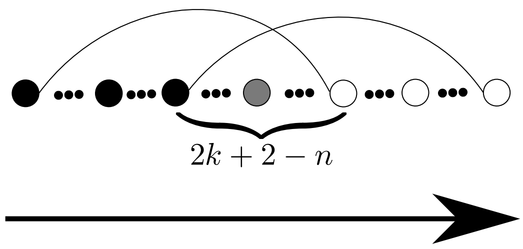


Figure 3.5: Optimal biclique-colouring of a power of a path P_n^k, when $k+2 \leq n \leq 2k$

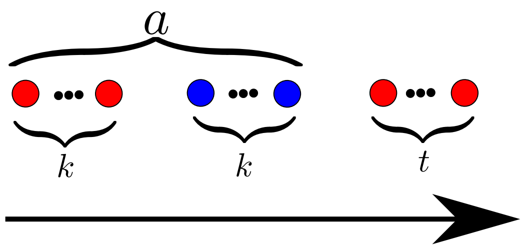


Figure 3.6: Optimal biclique-colouring of a power of a path P_n^k, when $n \geq 2k+1$

We summarize all results about biclique-colouring powers of paths in Table 3.1. In Figure 3.7, we illustrate biclique-chromatic number of a non-complete power of a path for a fixed value of k and an increasing n.

Range of n	Biclique-chromatic number	Optimal biclique-colouring
$[1, k+1]$	n	$O(n)$
$[k+2, 2k]$	$2k+2-n$	
$[2k+1, \infty)$	2	

Table 3.1: Biclique-colouring of powers of paths

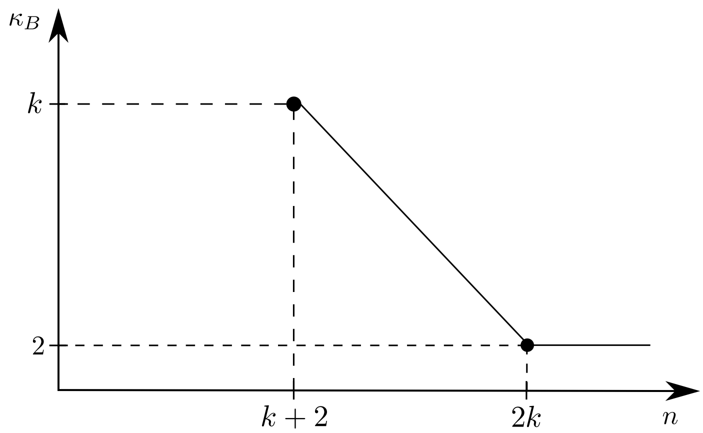


Figure 3.7: Biclique-chromatic numbers for non-complete powers of paths

3.2.2 Powers of Cycles

The extreme cases are easy to compute. The dense case occurs when $n \leq 2k+1$, which implies that a power of a cycle C_n^k is the complete graph K_n with biclique-chromatic number n. For the non-complete case, the sparsest case C_n^k occurs when $k = 1$, which implies that a power of a cycle C_n^k is the chordless cycle C_n with biclique-chromatic number of 2. A non-complete power of a cycle C_n^k has no P_2 biclique (see Claim 3.1) and does have a linear-time 3-colouring algorithm, in which no induced P_3 is monochromatic (see Lemma 3.5). This establishes a linear-time 3-biclique-colouring algorithm, stated next.

Lemma 3.9 (Inline, rephrased, Appendix B)**.** *There is a linear-time 3-biclique-colouring algorithm for non-complete power of cycles.*

As a consequence of Lemma 3.9, every non-complete power of a cycle has biclique-chromatic number 2 or 3, and it is natural to question how to decide between the two values. Indeed, this upper bound of 3 colours is tight. Please, refer to Figure 3.8 for an example of a non-complete power of a cycle with biclique-chromatic number 3.

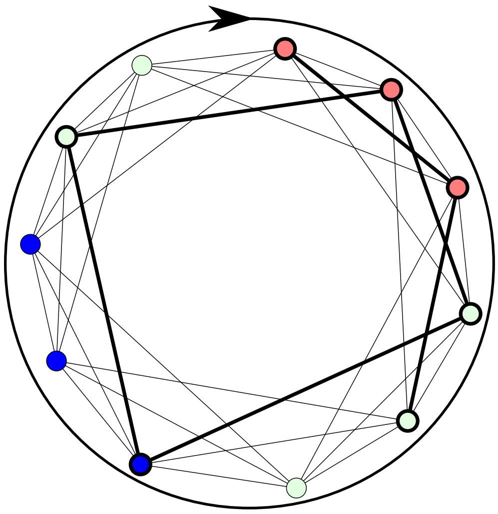


Figure 3.8: Power of a cycle C_{11}^3 with biclique-chromatic number 3

From now on, we do not rely anymore on Lemma 3.5 and we shall use other techniques. According to Claim 3.2, we consider other two cases. The less dense case $n \in [2k+2, 3k+1]$ and the sparse case $n \in [3k+2, \infty)$.

We first settle the less dense case $n \in [2k+2, 3k+1]$ by showing that there is a linear-time 2-biclique-colouring algorithm that assigns a monochromatic-block of size k with *red* colour followed by a monochromatic-block of size $n-k$ with *blue* colour. We refer to Figure 3.9 to illustrate the given 2-biclique-colouring. This colouring implies the following lemma.

Lemma 3.10 (Theorem 8, rephrased, Appendix B)**.** *There is a linear-time 2-biclique-colouring algorithm for every power of a cycle C_n^k, when $2k+2 \leq n \leq 3k+1$.*

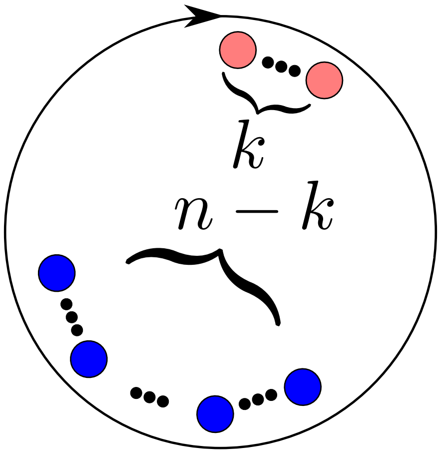


Figure 3.9: 2-biclique-colouring of a power of a cycle C_n^k, when $2k+2 \leq n \leq 3k+1$

The sparse case $n \geq 3k+2$ is more tricky. There could be monochromatic P_3 as long as these induced subgraphs are not bicliques. Nevertheless, we prove that a power of a cycle G in the sparse case has biclique-chromatic number 2 if, and only if, there is a 2-colouring of G such that **no** P_3 is monochromatic, what happens exactly when there is a 2-colouring of G in which every monochromatic-block has size k or $k+1$.

Lemma 3.11 (Lemma 9, rephrased, Appendix B). *Let G be a power of a cycle C_n^k, in which $n \geq 2k+2$, and consider a 2-colouring of its vertices. If every monochromatic-block has size k or $k+1$, then G has **no** monochromatic P_3. Otherwise, if **not** every monochromatic-block has size k or $k+1$, then G has a monochromatic P_3 with reach $k+1$ or $k+2$; in particular, when $n = 3k+2$, G has a monochromatic P_3 with reach $k+1$ or G has a monochromatic C_4.*

Notice the ressemblance of Lemma 3.11 with old Lemma 3.5. Both lemmas consider colourings of powers of cycles such that **no** P_3 is monochromatic in order to infer biclique-colourings.

As a consequence of Lemma 3.5, we can infer a linear-time 3-biclique-colouring algorithm for non-complete powers of cycles and a linear-time optimal biclique-colouring algorithm for non-complete powers of paths. As a consequence of Lemma 3.11, we shall see that we have a characterization for 2-biclique-colourable powers of cycles in the sparse case. Also, we can infer a linear-time 2-biclique-colouring algorithm for those 2-biclique-colourable powers of cycles in the sparse case.

The following theorem decodes the abovementioned characterization as a system of equations. It has a solution if, and only if, the given power of a cycle, in the sparse case, is 2-biclique-colourable.

Theorem 3.12 (Theorem 10, Appendix B). *A power of a cycle C_n^k, when $n \geq 3k+2$, has biclique-chromatic number 2 if, and only if, there are natural numbers a and b, such that $n = ak + b(k+1)$ and $a + b \geq 2$ is even.*

Now, we refer reader to Algorithm 3.1, which describes how to compute biclique-chromatic number of every power of a cycle $G = C_n^k$. Algorithm 3.1 starts by checking if G is in the dense case $n \in [1, 2k+1]$ or in the less dense case $n \in [2k+2, 3k+1]$ (Lines 3 – 8). We already know that G has biclique-chromatic number n in the dense case and biclique-chromatic number 2 in the less dense case. If the system of equations of Theorem 3.12 has a solution, then Algorithm 3.1 returns biclique-chromatic number 2. Otherwise, it returns biclique-chromatic number 3 (Lines 11 – 22).[2] Finally, the random-access machine model of computation assumes that each operation of Algorithm 3.1 is executed in constant-time,[3] which implies that Algorithm 3.1 has $O(1)$-time complexity. Then, we have the following theorem.

Theorem 3.13 (Theorem 11, improved, Appendix B). *There is a constant-time algorithm that computes biclique-chromatic number of powers of cycles.*

input : G, a power of a cycle C_n^k.
output: $\kappa_B(G)$, biclique-chromatic number of G.

```
1  begin
3      if n ≤ 2k + 1 then
4          return n;
5      else
6          if n ≤ 3k + 1 then
8              return 2;
9          else
11             c ⟵ ⌊n/k⌋;
12             b ⟵ n − ck;
13             if c mod 2 = 0 and c ≥ b then
14                 return 2;
15             else
16                 c ⟵ ⌊n/k⌋ − 1;
17                 b ⟵ n − ck;
18                 if c mod 2 = 0 and c ≥ b then
19                     return 2;
20                 else
22                     return 3;
23 end
```

Algorithm 3.1: Biclique-chromatic number of powers of cycles in $O(1)$-time

Algorithm 3.1 computes values $a = c - b$ and b of the system of equations of Theorem 3.12 in constant-time. We use those values in linear-time 2-biclique-

[2]We invite reader to check the proof of Theorem 11, Appendix B, for correctness of Lines 11 – 22, Algorithm 3.1.

[3]The interested reader can see a book of Cormen *et al.* [24, Page 23] for a more comprehensive list of constant-time operations in the random-access machine model of computation.

colouring algorithm for every 2-biclique-colourable power of a cycle in the sparse case, as follows. Consider a number a of monochromatic-blocks of size k plus a number b of monochromatic-blocks of size $k+1$ and assign *red* and *blue* colours alternately to them. We refer to Figure 3.10 to illustrate given 2-biclique-colouring. Lemma 3.11 proves its correctness and we have the following lemma.

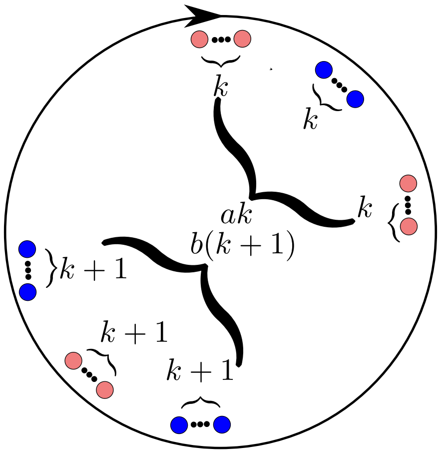


Figure 3.10: 2-biclique-colouring of a 2-biclique-colourable graph, when $n \geq 3k+2$

Lemma 3.14 (Implicit, Appendix B). *There is a linear-time 2-biclique-colouring algorithm for every 2-biclique-colourable power of a cycle C_n^k, when $n \geq 3k+2$.*

Now, a linear-time optimal biclique-colouring algorithm for every power of a cycle $G = C_n^k$ follows. If $n \leq 2k+1$, graph G is a complete graph with biclique-chromatic number n, which implies that all vertices should have distinct colours in an optimal biclique-colouring. If $2k+2 \leq n \leq 3k+1$, apply linear-time algorithm of Lemma 3.10 on G. If $n \geq 3k+2$, apply Algorithm 3.1 on G. If it returns 2, then apply linear-time algorithm of Lemma 3.14 on G. Otherwise, apply linear-time algorithm of Lemma 3.9 on G. It is easy to see that we have Theorem 3.15.

Theorem 3.15 (Implicit, Appendix B). *There is a linear-time optimal biclique-colouring algorithm for powers of cycles.*

We summarize all results about biclique-colouring powers of cycles in Table 3.2. In Figure 3.11, we illustrate the biclique-chromatic number of a non-complete power of a cycle for a fixed value of k and an increasing n. Notice that it is depicted that biclique-chromatic number is 2, when $n \geq 2k^2$. We shall see this fact as Corollary 3.20, Page 84.

3.3 Star-colouring

About star-colouring and the investigated classes of power graphs, we also have a few remarks. On one hand, a power of a path is necessarily C_4-free, what implies

Range of n	Biclique-chromatic number	Optimal biclique-colouring
$[1, 2k+1]$	n	$O(n)$
$[2k+2, 3k+1]$	2	
$[3k+2, \infty)$	2, if there are natural numbers a and b, such that $n = ak + b(k+1)$ and $a+b \geq 2$ is even; 3, otherwise.	

Table 3.2: Biclique-colouring of powers of cycles

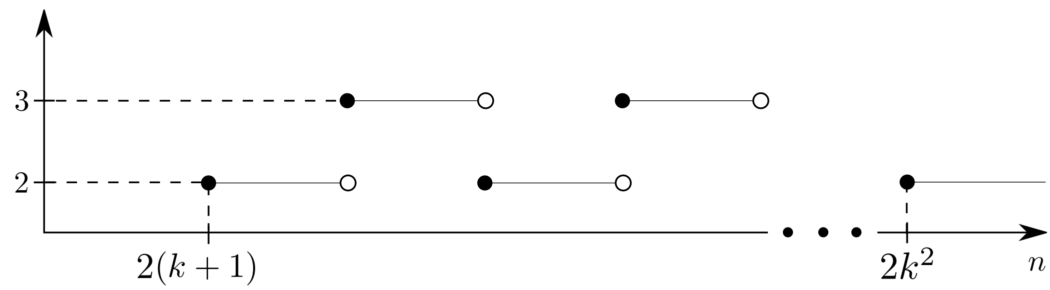


Figure 3.11: Biclique-chromatic numbers for non-complete powers of cycles

that bicliques are precisely stars of the graph. Consequently, all results obtained for biclique-colouring powers of paths hold for star-colouring powers of paths.

Theorem 3.16 (Theorem 14, rephrased, Appendix B). *An optimal biclique-colouring algorithm for powers of paths is also an optimal star-colouring algorithm for them.*

On the other hand, a power of a cycle C_n^k is not necessarily C_4-free, and there are examples of powers of cycles with P_3 stars that are not bicliques due to the fact that such P_3 stars are contained in C_4 bicliques of the graph. This happens, for instance, in the case $n \in [2k+2, 3k+1]$ and one such example is the power of a cycle C_{11}^4 depicted in Figure 3.12. Notice that the highlighted vertices form a monochromatic P_3 star, so the exhibited colouring is not a 2-star-colouring. Nevertheless, the three highlighted vertices together with vertex u form a polychromatic C_4 biclique. Indeed, the exhibited colouring is a 2-biclique-colouring. Following the proof of Theorem 3.12, we can infer the following version for star-colouring, but now we have a wider interval for n, which also includes $[2k+2, 3k+1]$.

Theorem 3.17 (Theorem 15, rephrased, Appendix B). *A non-complete power of a cycle C_n^k has star-chromatic number 2 if, and only if, there are natural numbers a and b, such that $n = ak + b(k+1)$ and $a + b \geq 2$ is even.*

As a consequence, Lines 11 – 22 of Algorithm 3.1 also work for computing star-chromatic number of non-complete powers of cycles, linear-time algorithm of

Lemma 3.9 assigns a 3-star-colouring to non-complete powers of cycles, and the algorithm of Lemma 3.14 assigns a 2-star-colouring to 2-star-colourable non-complete powers of cycles. Then, we have the following theorems.[4]

Theorem 3.18 (Implicit, Appendix B)**.** *There is a constant-time algorithm that computes star-chromatic number for powers of cycles.*

Theorem 3.19 (Implicit, Appendix B)**.** *There is a linear-time optimal star-colouring algorithm for powers of cycles.*

We summarize the results about star-colouring powers of paths and powers of cycles in Table 3.3.

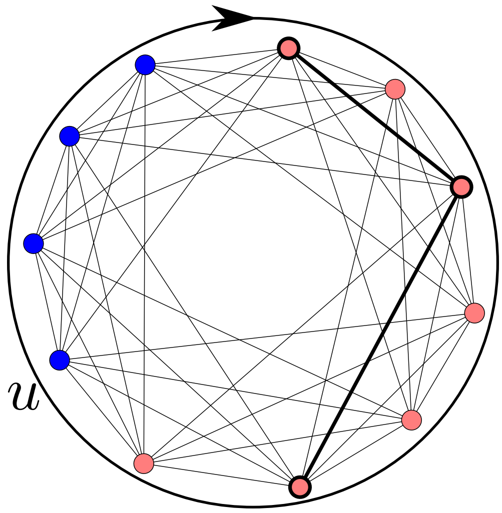


Figure 3.12: 2-biclique-colourable power of a cycle which is not 2-star-colourable

Graph	Range of n	Star-chromatic number	Optimal star-colouring
P_n^k	$[1, k+1]$	n	$O(n)$
	$[k+2, 2k]$	$2k+2-n$	
	$[2k+1, \infty)$	2	
C_n^k	$[1, 2k+1]$	n	$O(n)$
	$[2k+2, \infty)$	2, if there exist natural numbers a and b, such that $n = ak + b(k+1)$ and $a+b \geq 2$ is even; 3, otherwise.	

Table 3.3: Star-colouring of powers of paths and of cycles

[4]Recall that complete powers of cycles with n vertices have biclique-chromatic number n, what implies that all vertices should have distinct colours in an optimal biclique-colouring.

3.4 Final Considerations

As a corollary of Theorem 3.12 and Theorem 3.17, every non-complete power of a cycle C_n^k, with $n \geq 2k^2$, has biclique- and star-chromatic numbers 2. Thus, biclique- and star-chromatic numbers of a power of a cycle C_n^k, for a fixed value of k and an increasing n, do not oscillate forever (see Figure 3.11).

Corollary 3.20 (Corollary 12, Appendix B)**.** *A non-complete power of a cycle C_n^k, with $n \geq 2k^2$, has biclique- and star-chromatic numbers 2.*

We refer reader to Table 3.4, which highlights exact values for biclique- and star-chromatic numbers of the power graphs settled in this chapter. Please, take a look at the line where we consider a power of a cycle with $n \in [2k+2, 3k+1]$ to check the difference between biclique-chromatic number (which is always 2) and star-chromatic number (which depends on n and k). This line gives us examples of graphs in which biclique-chromatic number is not an upper bound of star-chromatic number.

<table>
<tr><th>Graph G</th><th>Range of n</th><th>$\kappa_B(G)$</th><th>$\kappa_S(G)$</th></tr>
<tr><td rowspan="3">P_n^k</td><td>$[1, k+1]$</td><td>n</td><td>n</td></tr>
<tr><td>$[k+2, 2k]$</td><td>$2k+2-n$</td><td>$2k+2-n$</td></tr>
<tr><td>$[2k+1, \infty[$</td><td>2</td><td>2</td></tr>
<tr><td rowspan="4">C_n^k</td><td>$[1, 2k+1]$</td><td>n</td><td>n</td></tr>
<tr><td>$[2k+2, 3k+1]$</td><td>2</td><td></td></tr>
<tr><td>$[3k+2, 2k^2[$</td><td colspan="2">2, if there are natural numbers a and b, such that $n = ak + b(k+1)$ and $a+b \geq 2$ is even;
3, otherwise.</td></tr>
<tr><td>$[2k^2, \infty[$</td><td>2</td><td>2</td></tr>
</table>

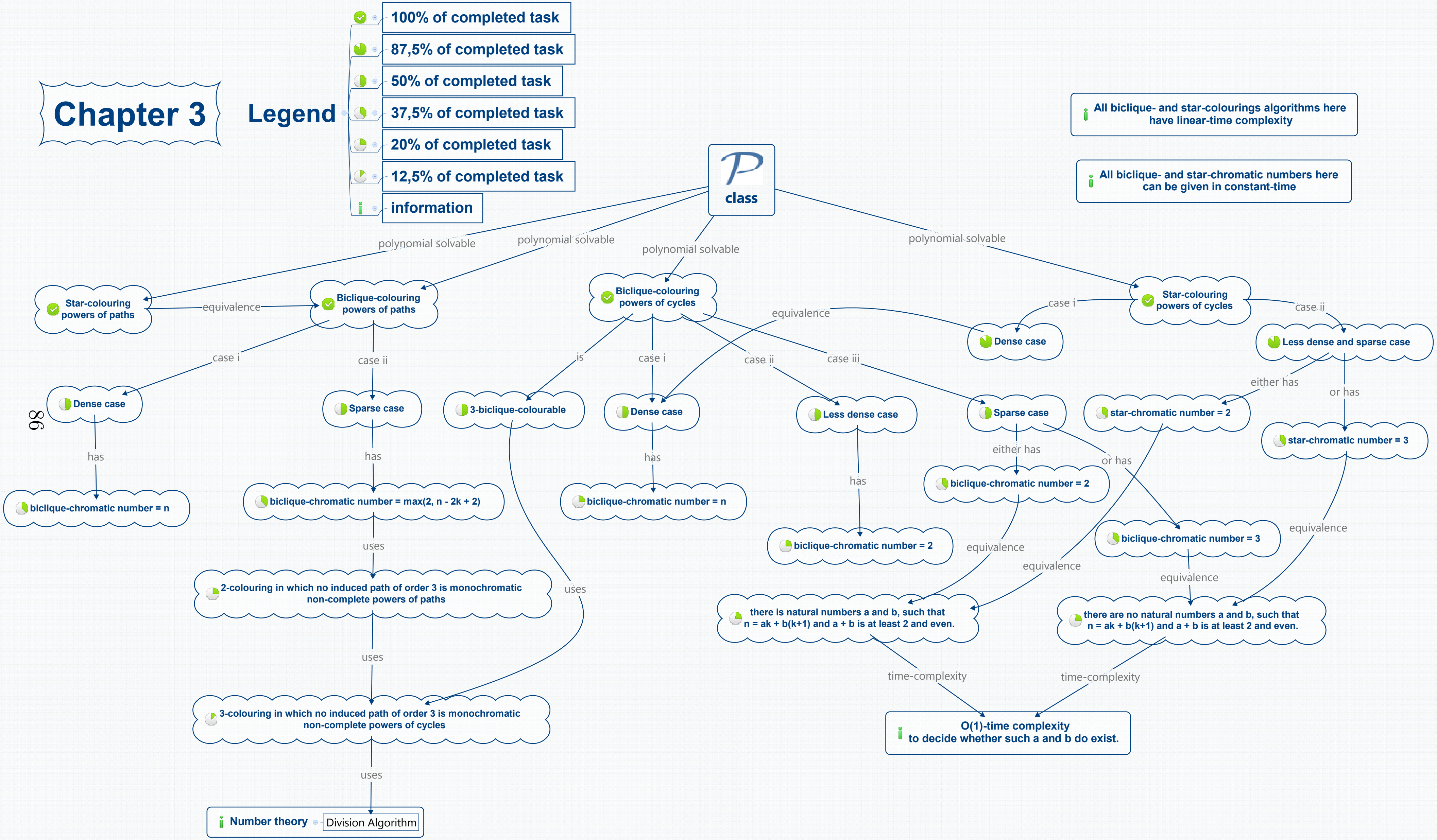
Table 3.4: Biclique- and star-chromatic numbers of powers of paths and of cycles

3.4.1 Future Work

A *circulant graph* $C_n(d_1, \ldots, d_k)$ is a simple graph with $V(G) = \{v_0, \ldots, v_{n-1}\}$ and $E(G) = E^{d_1} \cup \cdots \cup E^{d_k}$, such that $\{v_i, v_j\} \in E^{d_\ell}$ if, and only if, it has reach – in the context of a power of a cycle – d_ℓ. Notice that a circulant graph $C_n(d_1, \ldots, d_k)$ is a power of a cycle if $d_1 = 1$, $d_i = d_{i-1} + 1$, and $d_k < \lfloor \frac{n}{2} \rfloor$. A *distance graph* $P_n(d_1, \ldots, d_k)$ has the same definition as the circulant graph, except by the reach, which, in turn, is in the context of a power of a path. Notice that a distance graph $P_n(d_1, \ldots, d_k)$ is a power of a path if $d_1 = 1$, $d_i = d_{i-1} + 1$, and $d_k < n - 1$. As a future work, we intend to solve biclique-colouring circulant graphs and distance graphs, since colouring problems for circulant graphs and distance graphs graphs have been extensively investigated [7, 89, 101]. Moreover, some results of intractability have been obtained, e.g. determining the chromatic number of circulant graphs in general is an $\mathcal{NP}$-hard problem [23].

Acknowledgement

Last, but not least, we are deeply indebted to an anonymous referee of a submitted paper for publication who gave several insightful suggestions.


Chapter 3
Legend
100% of completed task
87,5% of completed task
50% of completed task
37,5% of completed task
20% of completed task
12,5% of completed task
information
All biclique- and star-colourings algorithms here have linear-time complexity
All biclique- and star-chromatic numbers here can be given in constant-time
P class
polynomial solvable
polynomial solvable
polynomial solvable
polynomial solvable
Star-colouring powers of paths
equivalence
Biclique-colouring powers of paths
case i
case ii
Dense case
has
biclique-chromatic number = n
Sparse case
has
biclique-chromatic number = max(2, n - 2k + 2)
uses
2-colouring in which no induced path of order 3 is monochromatic non-complete powers of paths
uses
3-colouring in which no induced path of order 3 is monochromatic non-complete powers of cycles
uses
Number theory
Division Algorithm
Biclique-colouring powers of cycles
is
3-biclique-colourable
uses
case i
Dense case
has
biclique-chromatic number = n
case ii
Less dense case
has
biclique-chromatic number = 2
case iii
Sparse case
either has
biclique-chromatic number = 2
or has
biclique-chromatic number = 3
equivalence
Star-colouring powers of cycles
case i
Dense case
case ii
Less dense and sparse case
either has
star-chromatic number = 2
or has
star-chromatic number = 3
equivalence
equivalence
equivalence
equivalence
there is natural numbers a and b, such that n = ak + b(k+1) and a + b is at least 2 and even.
there are no natural numbers a and b, such that n = ak + b(k+1) and a + b is at least 2 and even.
time-complexity
time-complexity
O(1)-time complexity to decide whether such a and b do exist.

Chapter 4

Weakly Chordal Graphs and Subclasses

This chapter is devoted to the results below.

- *We strengthen the result of Défossez – 2-clique-colouring of perfect graphs is a Σ_2^P-complete problem [27] – by showing that it is still Σ_2^P-complete for weakly chordal graphs with all cliques having size at least 3.*
- *We determine a hierarchy of nested subclasses of weakly chordal graphs with no known classes lying in between them whereby 2-clique-colouring each graph class is in a distinct complexity class, namely Σ_2^P-complete, $\mathcal{NP}$-complete, and $\mathcal{P}$.*
- *We solve an open problem posed by Kratochvíl and Tuza to determine the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3 [57], showing that it is a Σ_2^P-complete problem.*
- *We determine a hierarchy of nested subclasses of perfect graphs with all cliques having size at least 3 and with no known classes of graphs lying in between them whereby 2-clique-colouring each graph class is in a distinct complexity class, namely Σ_2^P-complete, $\mathcal{NP}$-complete, and $\mathcal{P}$.*

We start with the detached definition of weakly chordal graphs since it is the centerpiece of this chapter. We remark that the results established in this chapter are handled in a high-level approach and reader is invited to check omitted proofs and details in Appendix C. Alternative proofs are given in this chapter. Last, but not least, reader can get a big picture of this chapter results at Page 111 and Page 112.

Throughout this chapter, we assume that polynomial hierarchy does not collapse to its first, second and third level, i.e. $\mathcal{P}$, $\mathcal{NP}$-complete and Σ_2^P-complete are distinct complexity classes.

Definition 4.1 (Weakly chordal graph)**.** A *weakly chordal graph* is a graph which neither it nor its complement contain a chordless cycle with a number of vertices greater than 4.

The 2-clique-colouring problem is a known Σ_2^P-complete problem and Marx was responsible for this major breakthrough in the clique-colouring area. Défossez proved later that 2-clique-colouring perfect graphs, which are a known superclass of weakly chordal graphs, remained a Σ_2^P-complete problem.[1] Both clique-colouring and perfect graphs have attracted much attention due to a conjecture posed by Duffus *et al.*.[2] Related to $\mathcal{NP}$ class, Kratochvíl and Tuza gave a framework to argue that 2-clique-colouring is $\mathcal{NP}$-hard with a reduction from an $\mathcal{NP}$-complete problem called NAE-SAT [57]. As an application of the framework, Kratochvíl and Tuza proved that 2-clique-colouring is $\mathcal{NP}$-complete for K_4-free perfect graphs [57]. Notice that K_3-free perfect graphs are bipartite graphs, which are clearly 2-clique-colourable.

When restricted to chordal graphs – graphs with no chordless cycles with number of vertices greater than 3 – which is a known subclass of weakly chordal graphs, 2-clique-colouring is in $\mathcal{P}$. Indeed, chordal graphs are 2-clique-colourable and a 2-clique-colouring is computed in linear-time, as we shall see next. A graph is chordal if, and only if, it has a special order of its vertices called perfect elimination ordering [41]. The perfect elimination ordering is an ordering of the vertices of the graph such that, for each vertex v, v and its neighbours that occur after v in the order form a clique. A 2-clique-colouring of a chordal graph may be obtained by applying a greedy colouring algorithm to the vertices in the reverse of a perfect elimination ordering [82]. Consider the reverse of a perfect elimination ordering. Since a vertex u and its neighbours that occur before u form a clique, assign a colour to u that is distinct to the colour of another vertex of the clique. Then, 2 colours are enough to give a 2-clique-colouring. Finally, a perfect elimation ordering may be obtained in linear-time [88]. Hence, the time-complexity of a 2-clique-colouring algorithm for chordal graphs is linear. In particular, when restricted to split graphs – graphs in which the vertices can be partitioned into a clique and an stable set – which is a well known subclass of chordal graphs, the vertices of the clique (in every order) followed by the vertices of the stable set (in every order) form the reverse of a perfect elimination ordering. The greedy algorithm is enough to give a 2-clique-colouring. To be more precise, we assign *red* and *blue* to the clique and, for each vertex of the stable set, we assign colour *red* if, and only if, it has a neighbour with *blue* colour in the clique.

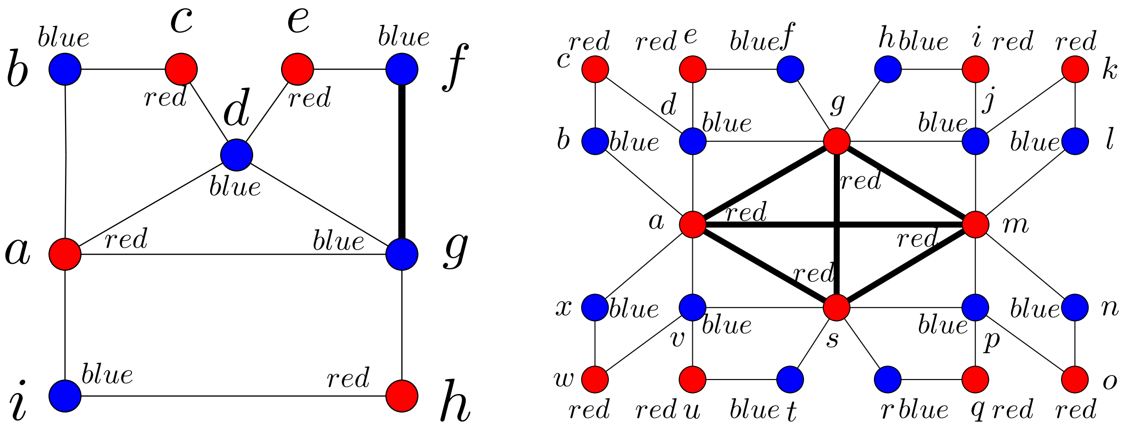
[1]The careful reader may think that Marx and Défossez's publishing dates are reversed in the references. We remark that Marx made available a manuscript of his work before 2009, so it was perfectly feasible to Défossez to strengthen Marx's work and publish it before 2011.

[2]Recall from Chapter 1 that the long-standing open problem of clique-colouring area is a conjecture of Duffus *et al.* [34]: *the class of perfect graphs is k-clique-colourable for some constant k.*

Weakly chordal graphs, in its turn, are not 2-clique-colourable, as we shall see.

Definition 4.2 (Bad cycle)**.** *Bad cycle* is a graph of odd order, at least 5, having a hamiltonian cycle, and no edge of that cycle lies in a triangle.

A bad cycle needs 3 colours to be clique-coloured. See Figure 4.1a for an example of a weakly chordal graph which is also a bad cycle. Notice that Figure 4.1a is also diamond-free. It is interesting to remark that diamond-free graphs are not 2-clique-colourable, but diamond-free graphs become 2-clique-colourable when bad cycles are forbidden [28]. Nevertheless, weakly chordal graphs *do not* become 2-clique-colourable when bad cycles are forbidden (see Figure 4.1b). We invite reader to check that both graphs in Figure 4.1 are not 2-clique-colourable.



(a) Diamond-free weakly chordal graph, which is a bad cycle

(b) Weakly chordal graph which is not a bad cycle

Figure 4.1: Weakly chordal graphs that are not 2-clique-colourable

A natural question is to classify the complexity of 2-clique-colouring weakly chordal graphs. In fact, we thought that 2-clique-colouring weakly chordal graphs would fall to the first level of the polynomial hierarchy because more induced graphs are forbidden, when compared to perfect graphs. Nevertheless, we show that 2-clique-colouring of weakly chordal graphs is a Σ_2^P-complete problem, improving the proof of Défossez [27] that 2-clique-colouring is a Σ_2^P-complete problem for perfect graphs. As a remark, Défossez [27] constructed a graph which is not a weakly chordal graph as long as it has chordless cycles with even number of vertices greater than 5 as induced subgraphs. Another related question is to show a subclass of weakly chordal and a superclass of split graphs in which 2-clique-colouring is neither a Σ_2^P-complete problem nor in $\mathcal{P}$. We shall see these results in Section 4.1. Then, we have determined a hierarchy of nested subclasses of weakly chordal graphs, namely weakly chordal graphs, (2, 1)-polar graphs and split graphs, whereby 2-clique-colouring each graph class is in a distinct complexity class, namely Σ_2^P-complete, $\mathcal{NP}$-complete, and $\mathcal{P}$.

Definition 4.3 ((α, β)-polar [19])**.** A graph is *(α, β)-polar* if there is a partition of its vertex set into two sets A and B such that all connected components of the subgraph induced by A and of the complementary subgraph induced by B are complete graphs. Moreover, the order of each connected component of the subgraph induced by A (resp. of the complementary subgraph induced by B) is upper bounded by α (resp. upper bounded by β).

A *satellite* of an (α, β)-polar graph is a connected component of the subgraph induced by A (see Figure 4.2a). In this chapter, we restrict ourselves to *(α, β)-polar* graphs with $\beta = 1$, so the subgraph induced by B is complete and the order of each satellite is upper bounded by α (see Figure 4.2b). An $(\alpha, 1)$-polar graph does not have a chordless cycle with a number of vertices greater than 4 as an induced subgraph. Indeed, every cycle of an $(\alpha, 1)$-polar graph of size at least 5 has a chord. An $(\alpha, 1)$-polar graph does not have the complement of a chordless cycle with odd order at least 5. Indeed, the complement of a chordless cycle with odd order at least 5 can be partitioned into two vertex-disjoint maximal complete sets plus a vertex with neighbours in both vertex-disjoint maximal complete sets. This configuration is not possible in an $(\alpha, 1)$-polar graph. Finally, $(\alpha, 1)$-polar graphs are a proper subclass of perfect graphs, since a complete bipartite graph with parts of size 2 and 3 is a perfect graph, but it is not an $(\alpha, 1)$-polar graph.

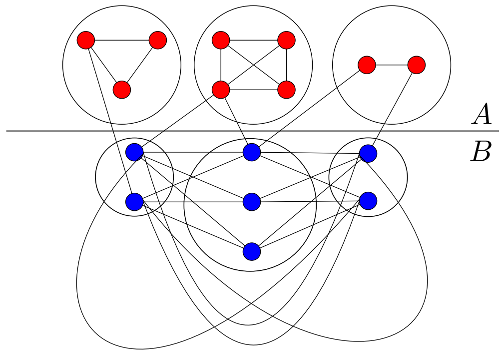
The class of $(\infty, 1)$-polar graphs forms an important class that plays an essential role in the areas of perfect graphs and clique-colouring.

Definition 4.4 (Generalized split [84])**.** A *generalized split graph* is a graph G such that either G or its complement is an $(\infty, 1)$-polar graph.

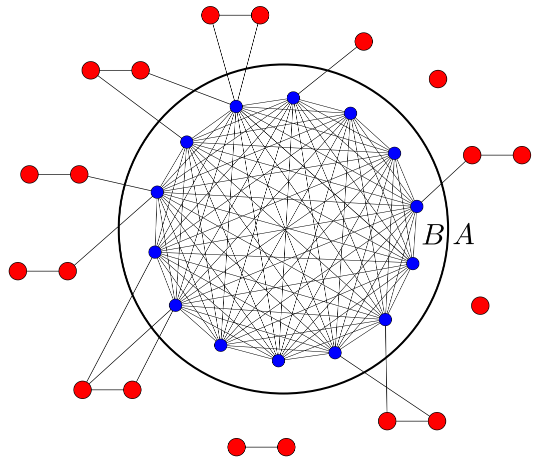
This class was introduced by Prömel and Steger in their probabilistic study of perfect graphs to show that almost all C_5-free graphs are generalized split graphs. See Figure 4.2b for an example of a generalized split graph, which is also a (2, 1)-polar graph. Notice that a generalized split graph is a subclass of perfect graphs. Indeed, $(\infty, 1)$-polar graphs are perfect graphs and the Weak Perfect Graph Theorem [65, 66] states that the complement of a perfect graph is a perfect graph. Prömel and Steger proved that strong perfect graph conjecture[3] was at least asymptotically true[4]. Approximately 14 years later, the strong perfect graph conjecture became the *Strong Perfect Graph Theorem* [22].

[3]The (now) celebrated *Strong Perfect Graph Theorem* of Chudnovsky *et al.* says that a graph is perfect if neither it nor its complement contain a chordless cycle with an odd number of vertices greater than 4.

[4]The strong perfect graph conjecture is asymptotically true because C_5-free graphs are a superclass of perfect graphs, which, in turn, are a superclass of generalized split graphs.



(a) $(4,3)$-polar graph



(b) Generalized split graph

Figure 4.2: (α, β)-polar graphs

Regarding clique-colouring, Bacsó *et al.* [5] proved that generalized split graphs are 3-clique-colourable[5] and concluded that almost all perfect graphs are 3-clique-colourable [5], since Prömel and Steger [84] proved that almost all perfect graphs are generalized split graphs). This conclusion supports the conjecture due to Duffus *et al.* [34]. In fact, there is no example of a perfect graph in which more than three colors would be necessary to clique-colour. Surprisingly, after more than 20 years, relatively little progress has been made on the conjecture.

The class of $(k, 1)$-polar graphs, for fixed $k \geq 3$, is incomparable to the class of weakly chordal graphs. Indeed, a chordless path with seven vertices P_7 and a complement of a chordless cycle with six vertices $\overline{C_6}$ are witnesses. Nevertheless, we shall see that the class of (2, 1)-polar graphs is comparable to the class of weakly chordal graphs.

Claim 4.1 (Inline, without proof, Appendix C)**.** *The (2, 1)-polar graphs are a proper subclass of weakly chordal graphs.*

Proof. A (2, 1)-polar graph does not have a chordless cycle with a number of vertices greater than 4 as an induced subgraph. Indeed, every cycle of a (2, 1)-polar graph of size at least 5 has a chord.

We now prove that (2, 1)-polar graph does not have the complement of a chordless cycle with a number of vertices greater than 4 as an induced subgraph. For the sake of contradiction, let $C = v_1, \ldots, v_{|C|}$ be the sequence of consecutive vertices of a chordless cycle with a number of vertices greater than 4, such that its complement

[5]Recall the greedy colouring algorithm to the vertices of a split graph. A naive algorithm to generalized split graphs follows similarly, considering the set B followed by the set A. Nevertheless, there are generalized split graphs that need a third colour. On the other hand, three colours are enough to give a clique-colouring. See Figure 4.1a for an example of a generalized split graph with clique-chromatic number 3.

is an induced subgraph of a (2, 1)-polar graph. If $|C| = 5$, then $\overline{C}$ is a hole, which is a contradiction. If $|C| = 6$, then $\overline{C}$ is a $\overline{C_6}$, which is *not* an induced subgraph of a $(k, 1)$-polar graph, for $k = 1, 2$. Now, consider $|C| \geq 7$. The maximum clique of $\overline{C}[v_3, v_4, v_5, v_6]$ has size 2. Then, at most 2 of its vertices are in part B and at least 2 of its vertices are in part A. Vertex v_1 is in part B, as otherwise, there is a P_3 or a K_3 in part A of G, which is a contradiction. Vertex v_2 is in part A, as otherwise, part B contains non-adjacent vertices v_1 and v_2, which is a contradiction. Vertices v_4 and v_6 are in part B. First, suppose v_4 and v_6 are in part A. Then, v_2, v_4, and v_6 induce a K_3 in part A, which is a contradiction. Last, suppose, by symmetry, that v_4 is in part A and v_6 is in part B. Then, v_5 is in part A. Otherwise, part B contains non-adjacent vertices v_5 and v_6, which is a contradiction. Moreover, v_2, v_4, and v_5 induce a P_3 in part A (contradiction). Finally, v_3 and v_5 are in part A. Otherwise, the maximum clique of $C[v_1, v_2, v_3, v_4, v_5, v_6]$ has size at least 4, which is a contradiction. Since v_2, v_3, and v_5 are in part A and they induce a P_3, we have a final contradiction.

Finally, (2, 1)-polar graphs are a proper subclass of weakly chordal graphs, since a complete bipartite graph with parts of size 2 and 3 is a weakly chordal graph, but it is not a (2, 1)-polar graph. □

The class of (1, 1)-polar graphs is precisely split graphs. It is interesting to recall that 2-clique-colouring of $(1, 1)$-polar graphs is in $\mathcal{P}$, since $(1, 1)$-polar graphs are a subclass of chordal graphs, which are 2-clique-colourable.

To address the question posed to show a subclass of weakly chordal graphs in which 2-clique-colouring is neither a Σ_2^P-complete problem nor in $\mathcal{P}$, we show that 2-clique-colouring (2, 1)-polar graphs is $\mathcal{NP}$-complete.

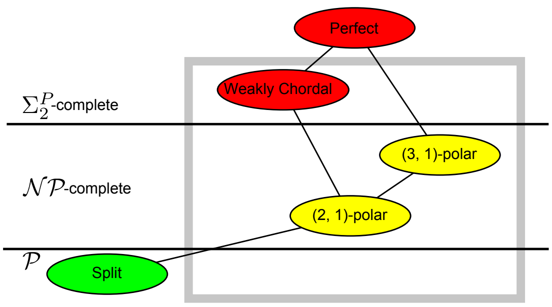
Last, but not least, consider the following three classes: chordal $\cap$ (2, 1)-polar, (2, 1)-polar, and weakly chordal. They form three nested classes with no other known classes lying in between them [25]. By our results, 2-clique-colouring problem is $\mathcal{P}$, $\mathcal{NP}$-complete, and Σ_2^P-complete, respectively. Intuitively, this tells us that either 2-clique-colouring gets really hard even when we consider few more graphs or there are big gaps between those considered classes.

Giving continuity to our results, we investigate an open problem left by Kratochvíl and Tuza to determine the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3. Restricting the size of the cliques to be at least 3, we first show that 2-clique-colouring is still $\mathcal{NP}$-complete for (3, 1)-polar graphs, even if it is restricted to weakly chordal graphs with all cliques having size at least 3. Subsequently, we prove that the 2-clique-colouring of (2, 1)-polar graphs becomes polynomial when all cliques have size at least 3. Recall that the 2-clique-colouring of (2, 1)-polar graphs is $\mathcal{NP}$-complete when there are no restrictions on the size of the cliques.

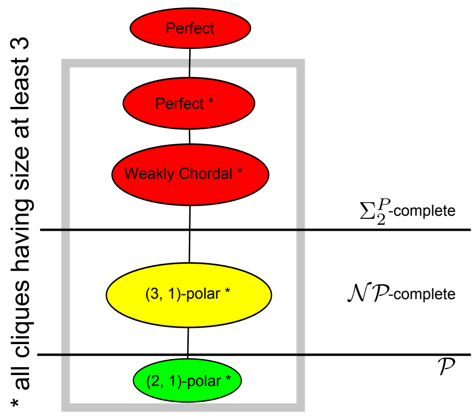
We finish the chapter answering the open problem about determining the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3 [57], by improving our proof that 2-clique-colouring is a Σ_2^P-complete problem for weakly chordal graphs. We replace each K_2 clique by a gadget with no clique of size 2, which forces distinct colours into two given vertices.

Last, but not least, consider the following three classes: (2, 1)-polar, (3, 1)-polar, and weakly chordal whereby each graph class has all cliques having size at least 3. They form three nested classes with no other known classes lying in between them. By our results, 2-clique-colouring problem is $\mathcal{P}$, $\mathcal{NP}$-complete, and Σ_2^P-complete, respectively. Again, intuitively, this tells us that either 2-clique-colouring gets really hard even when we consider few more graphs or there are big gaps between those considered classes.

This chapter is organized as follows. In Section 4.1, we show that 2-clique-colouring is still Σ_2^P-complete for weakly chordal graphs. Then, we determine a hierarchy of nested subclasses of weakly chordal graphs whereby 2-clique-colouring each graph class is in a distinct complexity class, namely Σ_2^P-complete, $\mathcal{NP}$-complete, and $\mathcal{P}$. See Figure 4.3a for all results established in Section 4.1. In Section 4.2, we determine the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3, answering the question of Kratochvíl and Tuza [57]. Then, we determine a hierarchy of nested subclasses of perfect graphs with all cliques having size at least 3 whereby 2-clique-colouring each graph class is in a distinct complexity class. See Figure 4.3b for all results established in Section 4.2. We refer reader to Table 4.1 for our results and related work about 2-clique-colouring complexity of perfect graphs. Shadowed cells indicate results established in this chapter.



(a) Results established in Section 4.1



(b) Results established in Section 4.2

Figure 4.3: Polynomial hierarchies of nested subclasses of perfect graphs

Table 4.1: 2-clique-colouring complexity of perfect graphs and subclasses

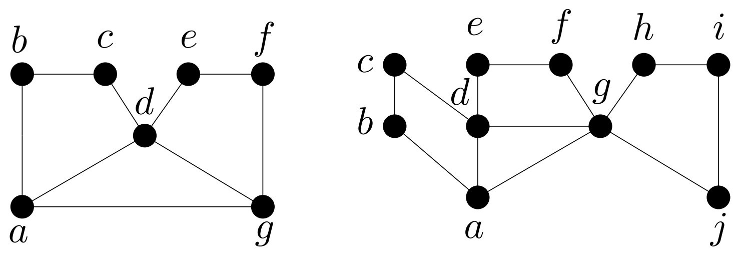
<table>
<tr><th colspan="3">Class</th><th>2-clique-colouring complexity</th></tr>
<tr><td rowspan="7">-</td><td rowspan="3">Perfect</td><td>-</td><td>Σ_2^P-complete [27]</td></tr>
<tr><td>K_4-free</td><td>$\mathcal{NP}$-complete [57]</td></tr>
<tr><td>K_3-free (Bipartite)</td><td>$\mathcal{P}$</td></tr>
<tr><td>Weakly chordal</td><td>-</td><td>Σ_2^P-complete</td></tr>
<tr><td>(3, 1)-polar</td><td>-</td><td rowspan="2">$\mathcal{NP}$-complete</td></tr>
<tr><td>(2, 1)-polar</td><td>-</td></tr>
<tr><td>Chordal (includes Split)</td><td>-</td><td>$\mathcal{P}$ [82]</td></tr>
<tr><td rowspan="4">All cliques having size at least 3</td><td>Perfect</td><td>-</td><td rowspan="2">Σ_2^P-complete</td></tr>
<tr><td rowspan="2">Weakly chordal</td><td>-</td></tr>
<tr><td>(3, 1)-polar</td><td>$\mathcal{NP}$-complete</td></tr>
<tr><td>(2, 1)-polar</td><td>-</td><td>$\mathcal{P}$</td></tr>
</table>

4.1 Hierarchical Complexity of 2-clique-colouring Weakly Chordal Graphs

Défossez proved that 2-clique-colouring of perfect graphs is a Σ_2^P-complete problem [27]. In this section, we strengthen this result by showing that it is still Σ_2^P-complete for weakly chordal graphs. We show a subclass of perfect graphs (resp. of weakly chordal graphs) in which 2-clique-colouring is neither a Σ_2^P-complete problem nor in $\mathcal{P}$, namely $(3,1)$-polar graphs (resp. $(2,1)$-polar graphs). Recall that 2-clique-colouring of $(1,1)$-polar graphs is in $\mathcal{P}$, since $(1,1)$-polar are a subclass of chordal graphs. Notice that weakly chordal graphs, $(2,1)$-polar graphs, and $(1,1)$-polar graphs are nested classes of graphs, as well as perfect graphs, $(3,1)$-polar graphs, and $(1,1)$-polar graphs.

Given a graph $G = (V, E)$ and adjacent vertices $a, g \in V$, we add to G a copy of an auxiliary graph $AK(a, g)$ of order 7 – depicted in Figure 4.4a – if we change the definition of G by doing the following: we first change the definition of V by adding to it copies of the five vertices $b, \ldots, f$ of the auxiliary graph $AK(a, g)$; then, we change the definition of E by adding to it copies of the eight edges (u, v) of $AK(a, g)$. Similarly, given a graph $G = (V, E)$ and non-adjacent vertices $a, j \in V$, we add to G a copy of an auxiliary graph $NAS(a, j)$ of order 10 – depicted in Figure 4.4b – if we change the definition of G by doing the following: we first change the definition of V by adding to it eight copies of the vertices $b, \ldots, i$ of the auxiliary graph $NAS(a, j)$; then, we change the definition of E by adding to it copies of the thirteen edges (u, v) of $NAS(a, j)$.

The auxiliary graph $AK(a, g)$ is constructed to force the same colour (in a 2-clique-colouring) to adjacent vertices a and g, while the auxiliary graph $NAS(a, j)$ is constructed to force distinct colours (in a 2-clique-colouring) to non-adjacent vertices a and j (see Lemma 4.2 and Lemma 4.3).



(a) $AK(a,g)$ (b) $NAS(a,j)$

Figure 4.4: Auxiliary graphs $AK(a,g)$ and $NAS(a,j)$

Lemma 4.2 (Lemma 1, Appendix C)**.** *Let G be a graph and a, g be adjacent vertices in G. If we add to G a copy of an auxiliary graph $AK(a,g)$, then in every 2-clique-colouring of the resulting graph, adjacent vertices a and g have the same colour.*

Lemma 4.3 (Lemma 2, Appendix C)**.** *Let G be a graph and a, j be non-adjacent vertices in G. If we add to G a copy of an auxiliary graph $NAS(a,j)$, then in every 2-clique-colouring of the resulting graph, non-adjacent vertices a and j have distinct colours.*

We improve the proof of Défossez [27] in order to determine the complexity of 2-clique-colouring for weakly chordal graphs. Consider QSAT2 problem, which is the Σ_2^P-complete canonical problem [81], as follows.

Problem 4.1. Quantified 2-Satisfiability (QSAT2)

Input: A formula $\Psi = (X, Y, D)$ composed of a disjunction D of implicants (that are conjunctions of literals) over two sets X and Y of variables.

Output: Is there a truth assignment for X such that for every truth assignment for Y the formula is true?

We prove that 2-clique-colouring weakly chordal graphs is Σ_2^P-complete by reducing the Σ_2^P-complete canonical problem QSAT2 to it. For a QSAT2 instance $\Psi = (X, Y, D)$, a weakly chordal graph G is constructed such that graph G is 2-clique-colourable if, and only if, there is a truth assignment of X, such that Ψ is true for every truth assignment of Y. We finish the proof showing that G is a weakly chordal graph. Now we are ready to state our main result of this section.

Theorem 4.4 (Theorem 3, Appendix C)**.** *The problem of 2-clique-colouring is Σ_2^P-complete for weakly chordal graphs.*

Now, our focus is on showing a subclass of weakly chordal graphs in which 2-clique-colouring is $\mathcal{NP}$-complete, namely (2, 1)-polar graphs.

Complements of bipartite graphs are a subclass of $(\infty, 1)$-polar graphs. Indeed, let $G = (V, E)$ be a complement of a bipartite graph, in which (A, B) is a partition of V into two disjoint complete sets. Clearly, G is an $(\infty, 1)$-polar graph. Défossez [27] showed that it is co$\mathcal{NP}$-complete to check whether a 2-colouring of a complement of a bipartite graph is a 2-clique-colouring [27]. Hence, it is co$\mathcal{NP}$-hard to check if a colouring of the vertices of an $(\infty, 1)$-polar graph is a 2-clique-colouring. On the other hand, we show next that, if k is fixed, listing all cliques of a $(k, 1)$-polar graph and checking if each clique is polychromatic can be done in polynomial-time, although the constant behind the big O notation is impraticable. The outline of the algorithm follows. We create a subroutine in which, given a satellite K of G, we check whether every clique of G containing a subset of K is polychromatic. Lemma 4.5 determines the complexity of the subroutine and proves its correctness. The algorithm runs the subroutine for each satellite of G. As a final step, we check whether part B is polychromatic if, and only if, part B is a clique of G. Theorem 4.6 determines the complexity of the algorithm.

Lemma 4.5 (Lemma 4, Appendix C)**.** *There is an $O(n)$-time algorithm to check whether every clique that contains a subset of a satellite S of a $(k, 1)$-polar graph, for a fixed $k \geq 1$,is polychromatic.*

As a remark, Algorithm 4.1 executes in at least $2^{k-1} \log k|B|$ steps even if we use dynamical programming.

Theorem 4.6 (Theorem 5, Appendix C)**.** *There is an $O(n^2)$-time algorithm to check whether a colouring of the vertices of a $(k, 1)$-polar graph, for a fixed $k \geq 1$, is a clique-colouring.*

Consider NAE-SAT problem, known to be $\mathcal{NP}$-complete [91].

> **Problem 4.2.** Not-all-equal satisfiability (NAE-SAT)
>
> Input: A set X of boolean variables and a collection C of clauses (set of literals over U), each clause containing at most three different literals.
>
> Output: Is there a truth assignment for X such that every clause contains at least one $true$ and at least one $false$ literal?

We first illustrate the framework of Kratochvíl and Tuza to argue that 2-clique-colouring is $\mathcal{NP}$-hard with a reduction from NAE-SAT, as follows. Consider an instance ϕ of NAE-SAT. We construct a graph G, as follows. For every variable x, add an edge between vertices x and $\overline{x}$. For every clause c, add a triangle on new vertices ℓ_c for all literals ℓ occurring in c. To finish the construction of G,

input : $G = (A, B)$, a $(k, 1)$-polar graph;
π, a 2-colouring of G;
A_i, a satellite of G.
output: *yes*, if every clique of G containing a subset of a satellite A_i is polychromatic.

begin
  if $|\pi(A_i)| \geq 2$ **then**
    for $i = 1$ **to** $|A_i|$ **do**
      $answer \longleftarrow$ *Algorithm* 4.1$(G, \pi, A_i \setminus \{x_i\})$;
      if $answer = no$ **then**
        return *no;*
    return *yes;*
  else
    $B_i \longleftarrow \bigcap_{v \in A_i} (N(v) \cap B)$;
    if $|\pi(A_i \cup B_i)| \geq 2$ **then**
      return *yes;*
    else
      return *no;*
end

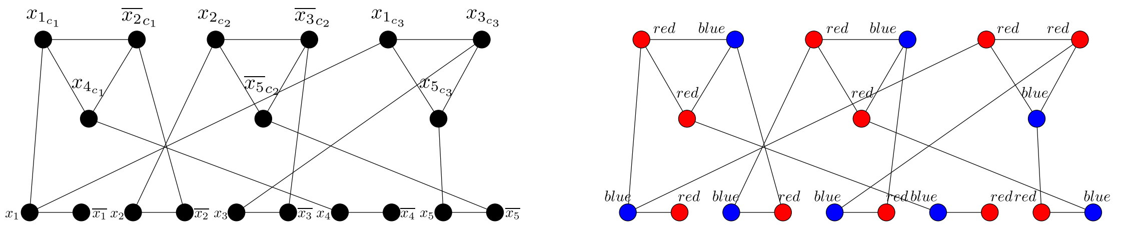
Algorithm 4.1: Subroutine of Algorithm 4.2

input : $G = (A, B)$, $(k, 1)$-polar graph;
π, a 2-colouring of G.
output: *yes*, if π is a 2-clique-colouring o G.

begin
  foreach *maximal complete set* $A' \in A$ **do**
    $answer \longleftarrow$ *Algorithm* 4.1(G, π, A');
    if $answer = no$ **then**
      return *no;*
  foreach *maximal complete set* $A' \in A$ **do**
    foreach $v \in A'$ **do**
      if $|N_B(v)| = |B|$ **then**
        return *yes;*
  if $|\pi(B)| \geq 2$ **then**
    return *yes;*
  else
    return *no;*
end

Algorithm 4.2: 2-clique-colouring $(k,\ 1)$-polar graphs, for fixed k, is in $\mathcal{NP}$

for every literal ℓ and for every clause c containing ℓ, add an edge between ℓ and ℓ_c. The (maximal) cliques of G are the edges $x\overline{x}$, $\ell\ell_c$, and triangles $\{\ell_c \mid \ell \in c\}$. Hence, G is 2-clique-colourable if, and only if, ϕ is not-all-equal satisfiable. Refer to Figure 4.5 for an example of such construction, given an NAE-SAT instance $\phi = (x_1 \vee \overline{x}_2 \vee y_2) \wedge (x_1 \vee x_3 \vee \overline{y}_2) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee y_1)$.



(a) Graph based on NAE-SAT instance $\phi = (x_1 \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge (x_1 \vee x_3 \vee x_5)$

(b) 2-clique-colouring based on a satisfying truth assignment ($x_1 = x_2 = x_3 = x_4 = \overline{x_5} = T$)

Figure 4.5: Graph construction following the framework of Kratochvíl and Tuza

A variation of this framework is used to prove that 2-clique-colouring is $\mathcal{NP}$-complete for K_4-free perfect graphs [57] and for graphs restricted to be of a maximum degree 3 [5]. Unfortunately, the above strategies create graphs that are not $(2,1)$-polar. We apply the ideas of the framework of Kratochvíl and Tuza to determine the complexity of 2-clique-colouring of (3, 1)-polar graphs. We prove that 2-clique-colouring of (3, 1)-polar graphs is $\mathcal{NP}$-complete by reducing from NAE-SAT problem to it. For a NAE-SAT instance ϕ, a (3, 1)-polar graph G is constructed such that graph G is 2-clique-colourable if, and only if, ϕ is not-all-equal satisfiable. We finish the proof showing that G is a (3, 1)-polar graph. This is an intermediary step to achieve the complexity of 2-clique-colouring of (2, 1)-polar graphs, which are a subclass of weakly chordal graphs.

Theorem 4.7 (Theorem 6, Appendix C)**.** *The problem of 2-clique-colouring is $\mathcal{NP}$-complete for (3, 1)-polar graphs.*

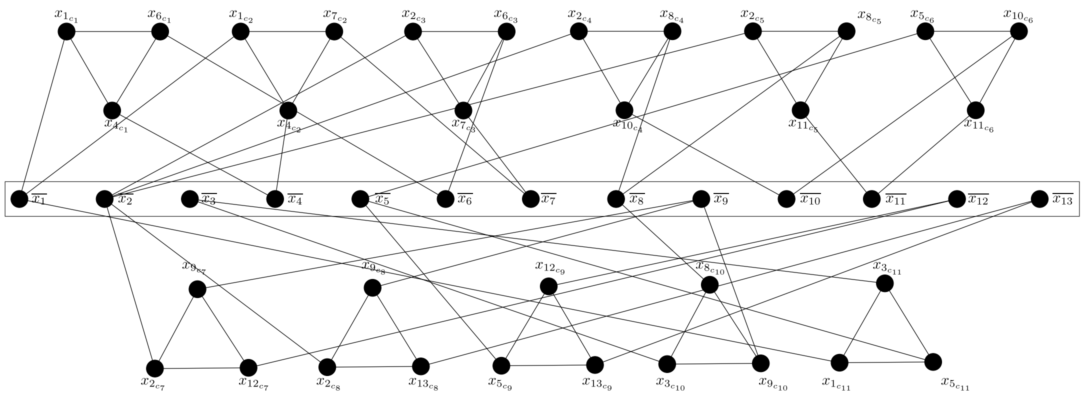
An additional requirement to NAE-SAT problem is that all variables must be positive (no negated variables). This defines the known variant Positive NAE-SAT [77, Chapter 7]. It is $\mathcal{NP}$-complete and the proof is by reduction from NAE-SAT. Replace every negated variable $\overline{l_i}$ by a fresh variable l_j, and add a new clause (l_i, l_j), to enforce the complement relationship. Notice that the new clause has only two literals. Hence, all you have to do follows.

- To duplicate a clause with a negated variable, say $\overline{l_i}$,
- To replace negated variable $\overline{l_i}$ by fresh variables l_j and l_k, respectively, and
- To add a new clause (l_i, l_j, l_k), to enforce the complement relationship.

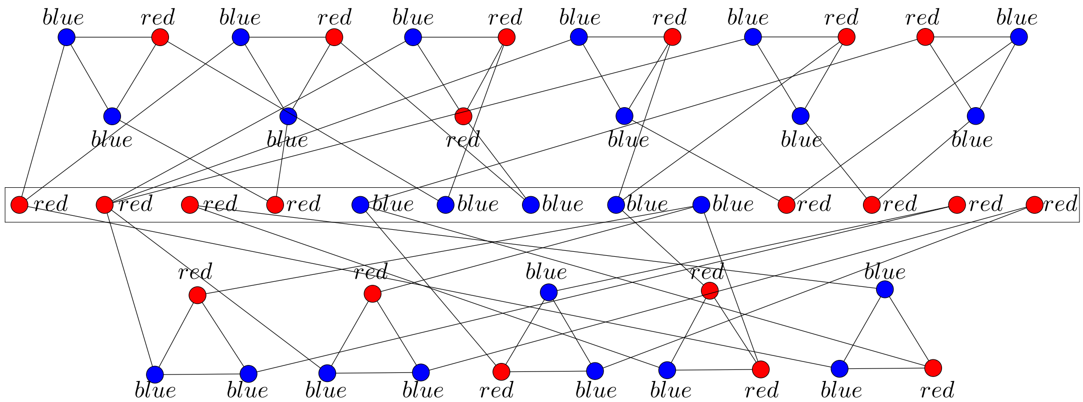
An alternative proof that 2-clique-colouring is $\mathcal{NP}$-complete for (3, 1)-polar graphs can be obtained by a reduction from Positive NAE-SAT. In contrast to

NAE-SAT, graph construction based on Positive NAE-SAT is as simple as the input. Moreover, the alternative proof follows similarly.

We construct graph G as follows. Create vertices and edges so that the set $\{\overline{x}_1, \ldots, \overline{x}_n\}$ induces a complete subgraph of G. For each clause c_j, $1 \leq j \leq m$, we create a complete set c_j with vertices corresponding to the (positive) literals of clause c_j. Moreover, we add an edge joinning each vertex of the complete set with the vertex corresponding to the negative literal of the same variable in the set $\{\overline{x}_1, \ldots, \overline{x}_n\}$. Refer to Figure 4.6 for an example of such construction, given Positive NAE-SAT instance $\phi = (x_1 \vee x_6 \vee x_4) \wedge (x_1 \vee x_7 \vee x_4) \wedge (x_2 \vee x_6 \vee x_7) \wedge (x_2 \vee x_8 \vee x_{10}) \wedge (x_2 \vee x_8 \vee x_{11}) \wedge (x_5 \vee x_{10} \vee x_{11}) \wedge (x_2 \vee x_9 \vee x_{12}) \wedge (x_2 \vee x_9 \vee x_{13}) \wedge (x_5 \vee x_{12} \vee x_{13}) \wedge (x_3 \vee x_8 \vee x_9) \wedge (x_1 \vee x_3 \vee x_5)$, which was obtained from the reduction from NAE-SAT to Positive NAE-SAT. The former NAE-SAT instance has been used to construct the graph of Theorem 4.7.



(a) Graph based on Positive NAE-SAT instance
$\phi = (x_1 \vee x_6 \vee x_4) \wedge (x_1 \vee x_7 \vee x_4) \wedge (x_2 \vee x_6 \vee x_7) \wedge (x_2 \vee x_8 \vee x_{10}) \wedge (x_2 \vee x_8 \vee x_{11}) \wedge (x_5 \vee x_{10} \vee x_{11}) \wedge (x_2 \vee x_9 \vee x_{12}) \wedge (x_2 \vee x_9 \vee x_{13}) \wedge (x_5 \vee x_{12} \vee x_{13}) \wedge (x_3 \vee x_8 \vee x_9) \wedge (x_1 \vee x_3 \vee x_5)$



(b) 2-clique-colouring based on a satisfying truth assignment
$(x_1 = x_2 = x_3 = x_4 = \overline{x_5} = \overline{x_6} = \overline{x_7} = \overline{x_8} = \overline{x_9} = x_{10} = x_{11} = x_{12} = x_{13} = T)$

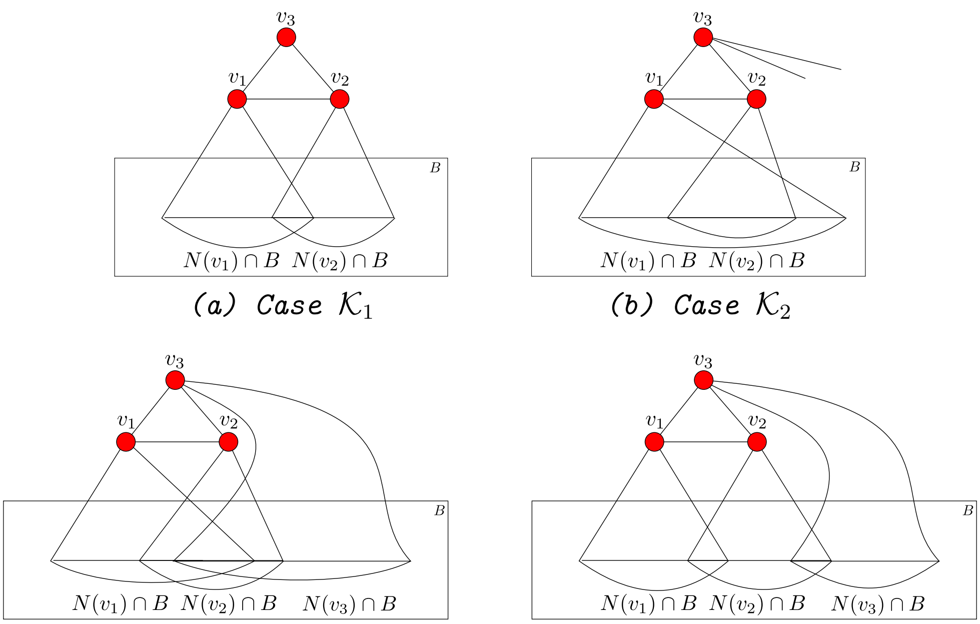
Figure 4.6: (3, 1)-polar graph construction of alternative proof of Theorem 4.7

In order to determine the complexity of 2-clique-colouring (2, 1)-polar graphs, we use a reduction from 2-clique-colouring (3, 1)-polar graphs. In what follows, we provide some notation to classify the structure of 2-clique-colouring of (2, 1)-polar graphs and of (3, 1)-polar graphs. We capture their similarities and make a reduction from 2-clique-colouring $(3,1)$-polar graphs to 2-clique-colouring (2, 1)-polar graphs feasible.

Let $G = (V, E)$ be a (3, 1)-polar graph. Let K be a satellite of G. Consider the following four cases: ($\mathcal{K}_1$) there is a vertex of K such that none of its neighbours is in part B; ($\mathcal{K}_2$) the complementary case of $\mathcal{K}_1$, in which there is a pair of vertices of K, such that the closed neighbourhood of one vertex of the pair is contained in the closed neighbourhood of the other vertex of the pair; ($\mathcal{K}_3$) the complementary case of $\mathcal{K}_2$, in which the intersection of the closed neighbourhood of the vertices of K is precisely K; and ($\mathcal{K}_4$) the complementary case of $\mathcal{K}_3$.

$$\mathcal{K}_1 : \exists u \in K, N_B(u) = \emptyset.$$
$$\mathcal{K}_2 : \nexists u \in K, N_B(u) = \emptyset; \quad \exists v, w \in K, v \neq w, N_B(v) \subseteq N_B(w).$$
$$\mathcal{K}_3 : \nexists u \in K, N_B(u) = \emptyset; \quad \nexists v, w \in K, v \neq w, N_B(v) \subseteq N_B(w); \bigcap_{z \in K} N_B(z) \neq \emptyset.$$
$$\mathcal{K}_4 : \nexists u \in K, N_B(u) = \emptyset; \quad \nexists v, w \in K, v \neq w, N_B(v) \subseteq N_B(w); \bigcap_{z \in K} N_B(z) = \emptyset.$$

Clearly, every satellite K is either in case $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, or $\mathcal{K}_4$. Refer to Figure 4.7 (resp. Figure 4.8) for an example of each case of a triangle (resp. edge) satellite.



(a) Case $\mathcal{K}_1$ (b) Case $\mathcal{K}_2$ (c) Case $\mathcal{K}_3$ (d) Case $\mathcal{K}_4$

Figure 4.7: Triangle satellite of an (α, β)-polar graph

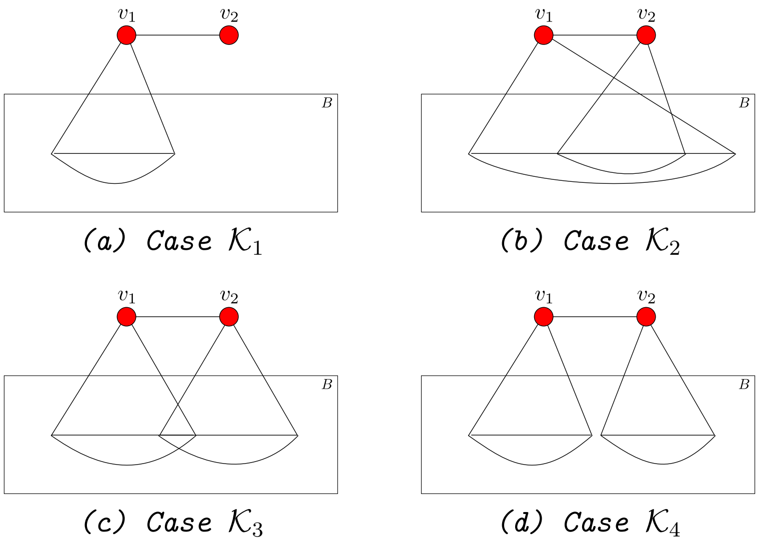


Figure 4.8: Edge satellite of an (α, β)-polar graph

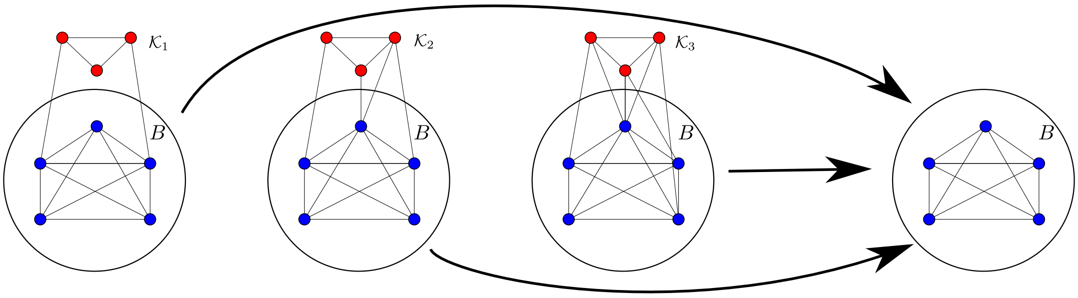
The following lemma is an important step to understand the role of triangles and edges that are either in case $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, or $\mathcal{K}_4$ in a 2-clique-colouring of (3, 1)-polar and of (2, 1)-polar graphs. The following lemma is also important towards the modification of a (3, 1)-polar graph into a (2, 1)-polar graph, which is closely related to Theorem 4.9.

Lemma 4.8 (Lemma 7, Appendix C)**.** *Let $G = (V, E)$ be a (3, 1)-polar graph, $\mathcal{K}$ be the set of satellites of G in case $\mathcal{K}_4$, and $K \in \mathcal{K}$.*

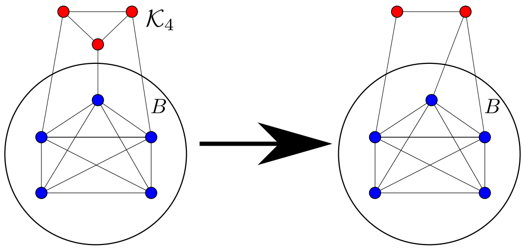
- *If G has a 2-clique-colouring, then $\bigcup_{v \in K} N_B(v)$ is polychromatic.*
- *If B has a 2-colouring that, for every $K' \in \mathcal{K}$, $\bigcup_{v \in K'} N_B(v)$ is polychromatic, then G is 2-clique-colourable.*

For a given (3, 1)-polar graph G, we proceed to get a (2, 1)-polar graph G' that is 2-clique-colourable if, and only if, G is 2-clique-colourable, as follows. For each triangle satellite, we have two cases. If K is not in case $\mathcal{K}_4$, the former case deletes K. If K is in case $\mathcal{K}_4$, the latter case replace K by an edge such that both complete sets have the same neighboorhod contained in B and the edge is also in case $\mathcal{K}_4$. See Figure 4.9 for examples. Such construction is done in polynomial-time and we depict it as Algorithm 4.3. See Figure 4.10 for an application of Algorithm 4.3 given a (3,1)-polar graph with clique-chromatic number 3. Algorithm 4.3 and Theorem 4.6 imply the following theorem.

Theorem 4.9 (Theorem 8, Appendix C)**.** *The problem of 2-clique-colouring is $\mathcal{NP}$-complete for (2, 1)-polar graphs.*



(a) Former case



(b) Latter case

Figure 4.9: One iteration of Algorithm 4.3

input : $G = (A, B)$, a (3, 1)-polar graph.
output: G', a (2, 1)-polar graph that is 2-clique-colourable if, and only if, G is 2-clique-colourable.

begin
  foreach *satellite* $K = \{v_1, v_2, v_3\}$ **do**
    $V' \longleftarrow \emptyset$;
    $E' \longleftarrow \emptyset$;
    if K *is in case* $\mathcal{K}_4$ **then**
      $V' \longleftarrow \{u_1, u_2\}$;
      $E' \longleftarrow \{(u_1, u_2)\}$;
      $E' \longleftarrow E' \cup \{(u_1, x) \mid x \in N_B(v_1)\}$;
      $E' \longleftarrow E' \cup \{(u_2, x) \mid x \in ((N_B(v_2) \cup N_B(v_3)) \setminus N_B(v_1))\}$;
    $G \longleftarrow G[V(G) \setminus K]$;
    $V(G) \longleftarrow V(G) \cup V'$;
    $E(G) \longleftarrow E(G) \cup E'$;
  return G;
end

Algorithm 4.3: Polynomial reduction of Theorem 4.9

As a remark, we noticed strong connections between hypergraph 2-colorability and 2-clique-colouring (2, 1)-polar graphs. Indeed, we have a simpler alternative proof showing that 2-clique-colouring (2, 1)-polar graphs is $\mathcal{NP}$-complete by a reduction from hypergraph 2-colouring. In contrast to graphs, it is $\mathcal{NP}$-complete to check whether a hypergraph is 2-colourable, even if all edges have cardinality at most 3 [67]. The property of hypergraph 2-colourability, known as *Property B*, is a central problem in combinatorics and it has strong connections with graph colouring and

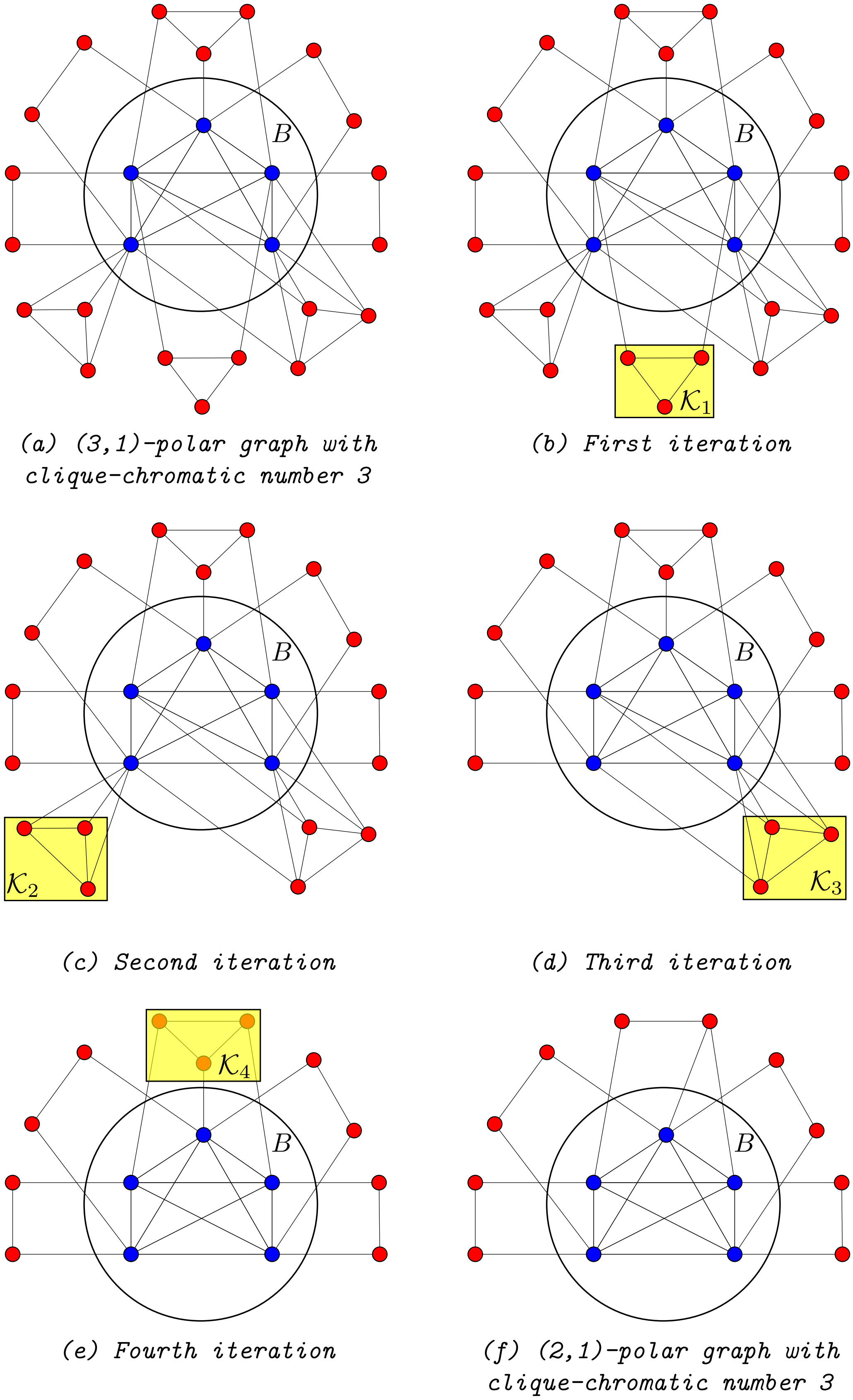


Figure 4.10: Application of Algorithm 4.3

satisfiability problems. Reader may ask why we did not exploit only the alternative proof that is quite shorter than the original proof. The reason is to be consistent with the next section, in which we show that even restricting the size of the cliques to be at least 3, the 2-clique-colouring of (3, 1)-polar graphs is still $\mathcal{NP}$-complete, while 2-clique-colouring of (2, 1)-polar graphs becomes a problem in $\mathcal{P}$.

Alternative proof of Theorem 4.9. First, 2-clique-colouring of (2, 1)-polar graphs is in $\mathcal{NP}$. Theorem 4.6 confirms that it is in $\mathcal{P}$ to check whether a colouring of a (2, 1)-polar graph is a 2-clique-colouring.

We prove that 2-clique-colouring of (2, 1)-polar graphs is $\mathcal{NP}$-hard by reducing hypergraph 2-colouring to it. The outline of the proof follows. For every hypergraph $\mathcal{H}$, a (2, 1)-polar graph G is constructed such that $\mathcal{H}$ is 2-colourable if, and only if, graph G is 2-clique-colourable. Let n (resp. m) be the number of hypervertices (resp. hyperedges) in hypergraph $\mathcal{H}$. We define graph G, as follows.

- For each hypervertex v_i, $1 \leq i \leq n$, we create a vertex v_i in G, so that the set $\{v_1, \ldots, x_n\}$ induces a complete subgraph of G, which is the part B of graph G;
- for each hyperedge $e_j = \{v_1, \ldots, v_l\}$, $1 \leq j \leq m$, we create two vertices u_{j_1} and u_{j_2}. Moreover, we create edges $u_{j_1}v_1, \ldots, u_{j_1}v_{l-1}$, and $u_{j_2}v_l$ so that $\{u_{j_1}u_{j_2}\}$ is a satellite in case $\mathcal{K}_4$.

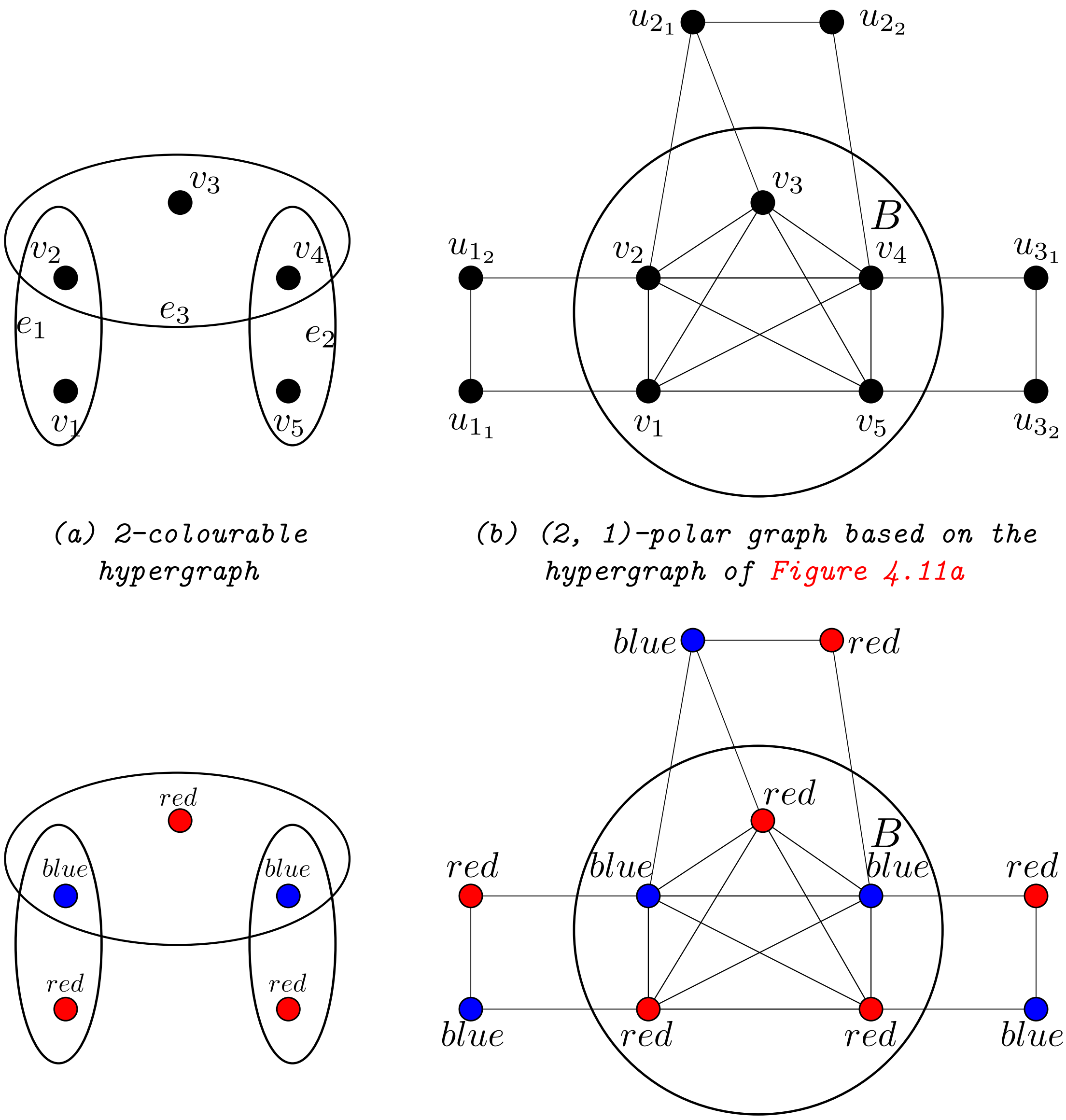
Clearly, G is a (2, 1)-polar graph and such construction is done in polynomial-time. Refer to Figure 4.11 for an example of such construction.

We claim that hypergraph $\mathcal{H}$ is 2-colourable if, and only if, graph G is 2-clique-colourable. Assume that there is a proper 2-colouring π of $\mathcal{H}$. We assign a colouring to graph G, as follows.

- Assign colour $\pi(v)$ for each v of part B, and
- extend the 2-clique-colouring for each clique $(\{u_{j_1}, u_{j_2}\})$ that is a satellite of G.

We still have to be prove that this is indeed a 2-clique-colouring. Consider the part $B = \{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$. Clearly, the above colouring assigns 2 colours to this set. Each satellite K of G is in case $\mathcal{K}_4$ and $\bigcup_{v \in K} N_B(v)$ is polychromatic, since $\bigcup_{v \in K} N_B(v) = e_j$. By Lemma 4.8, graph G is 2-clique-colourable.

For the converse, we now assume that G is 2-clique-colourable and we consider a 2-clique-colouring π' of G. We give a colouring to hypergraph $\mathcal{H}$, as follows. Assign colour $\pi'(v)$ for each hypervertex v. By Lemma 4.8, $\bigcup_{v \in K} N_B(v)$ is polychromatic for



(a) 2-colourable hypergraph

(b) (2, 1)-polar graph based on the hypergraph of Figure 4.11a

(c) 2-colouring of Figure 4.11a

(d) 2-clique-colouring based on 2-colouring of Figure 4.11c

Figure 4.11: (2, 1)-polar graph construction of alternative proof of Theorem 4.9

each satellite K of G. Then, hypergraph $\mathcal{H}$ is 2-colourable, since $\bigcup_{v \in K} N_B(v) = e_j$ for every hyperedge e_j. $\square$

We ask reader to contrast this alternative proof to the proofs obtained in the former (and much longer) path we took to determine that 2-clique-colouring of (2, 1)-polar graphs are $\mathcal{NP}$-complete. No longer restricting the size of the cliques, 2-clique-colouring (2, 1)-polar graphs become a (possibly harder) $\mathcal{NP}$-complete problem, while 2-clique-colouring of (3, 1)-polar graphs is still in the same complexity class.

4.2 Restricting the Size of the Cliques

Kratochvíl and Tuza are interested in determining the complexity of the problem 2-clique-colouring for perfect graphs with all cliques having size at least 3. We determine what happens with the complexity of 2-clique-colouring of (2, 1)-polar graphs, of (3, 1)-polar graphs, and of weakly chordal graphs, respectively, when all cliques are restricted to size at least 3. The latter result addresses Kratochvíl and Tuza's question.

In this direction, we show that 2-clique-colouring of weakly chordal (3, 1)-polar graphs and (2, 1)-polar graphs both with all cliques having size at least 3 are $\mathcal{NP}$-complete and polynomial, respectively. We ask reader to contrast these results to the results obtained in the former (and much longer) path we took to determine that 2-clique-colouring of (2, 1)-polar graphs is $\mathcal{NP}$-complete. First, we strengthen the result that 2-clique-colouring of (3, 1)-polar graphs is $\mathcal{NP}$-complete, even if it is restricted to be weakly chordal with all cliques having size at least 3. Second, if one restricts the size of the cliques to be at least 3, then 2-clique-colouring of (2, 1)-polar graphs becomes a polynomially solvable problem.

Given graph G and $b_1, b_2, b_3 \in V(G)$, we add to G a copy of an auxiliary graph $BP(b_1, b_2, b_3)$ of order 6 – depicted in Figure 4.12a – if we change the definition of G by doing the following: we first change the definition of V by adding to it copies of the vertices a_1, a_2, a_3 of the auxiliary graph $BP(b_1, b_2, b_3)$; second, we change the definition of E by adding to it copies of the edges (u, v) of $BP(b_1, b_2, b_3)$.

Similarly, given a graph G and $b_1, b_2 \in V(G)$, we add to G a copy of an auxiliary graph $BS(b_1, b_2)$ of order 17 – depicted in Figure 4.12b – if we change the definition of G by doing the following: we first change the definition of V by adding to it copies of the vertices b', b'', b''' of the auxiliary graph $BS(b_1, b_2)$; second, we change the definition of E by adding to it edges so that $B(G) \cup \{b_1, b_2, b', b'', b'''\}$ is a complete set; finally, we add copies of the auxiliary graphs $BP(b_1, b_2, b')$, $BP(b_1, b_2, b'')$, $BP(b_1, b_2, b''')$, $BP(b', b'', b''')$.

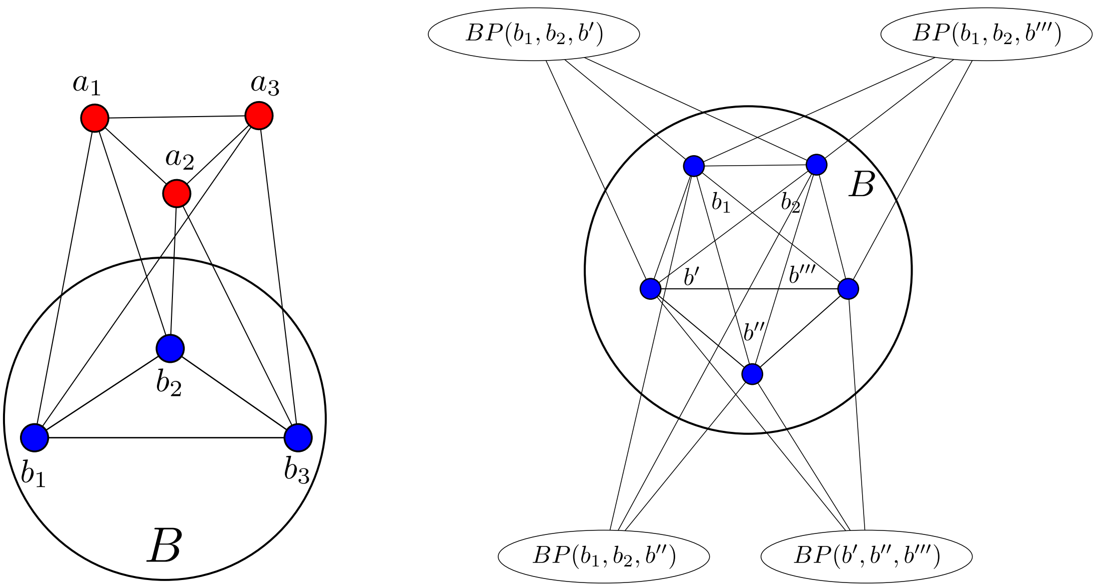
Lemma 4.10 (Lemma 9, Appendix C)**.** *Let G be a weakly chordal graph (resp. (3, 1)-polar graph) and $b_1, b_2, b_3 \in V(G)$ (resp. $b_1, b_2, b_3 \in B(G)$). If we add to G a copy of an auxiliary graph $BP(b_1, b_2, b_3)$, then the following assertions are true.*

- *The resulting graph G' is weakly chordal (resp. (3, 1)-polar).*
- *If all cliques of G have size at least 3, then all cliques of G' have size at least 3.*
- *Every 2-clique-colouring of G' assigns at least 2 colours to b_1, b_2, b_3.*
- *G is 2-clique-colourable if G' is 2-clique-colourable.*

- *G' is 2-clique-colourable if there is a 2-clique-colouring of G that assigns at least 2 colours to b_1, b_2, b_3.*

Lemma 4.11 (Lemma 10, Appendix C)**.** *Let G be a weakly chordal graph (resp. (3, 1)-polar graph) and $b_1, b_2 \in V(G)$ (resp. $b_1, b_2 \in B(G)$). If we add to G a copy of an auxiliary graph $BS(b_1, b_2)$, then the following assertions are true.*

- *The resulting graph G' is weakly chordal (resp. (3, 1)-polar).*
- *If all cliques of G have size at least 3, then all cliques of G' have size at least 3.*
- *Every 2-clique-colouring of G' assigns 2 colours to b_1 and b_2.*
- *G is 2-clique-colourable if G' is 2-clique-colourable.*
- *G' is 2-clique-colourable if there is a 2-clique-colouring of G that assigns 2 colours to b_1 and b_2.*



(a) Auxiliary graph $BP(b_1, b_2, b_3)$

(b) Auxiliary graph $BS(b_1, b_2)$

Figure 4.12: Auxiliary graphs $BP(b_1, b_2, b_3)$ and $BS(b_1, b_2)$

We strengthen the result that 2-clique-colouring of (3, 1)-polar graphs is $\mathcal{NP}$-complete, even restricting all cliques to have size at least 3.[6]

Theorem 4.12 (Theorem 11, Appendix C)**.** *The problem of 2-clique-colouring is $\mathcal{NP}$-complete for (weakly chordal) (3, 1)-polar graphs with all cliques having size at least 3.*

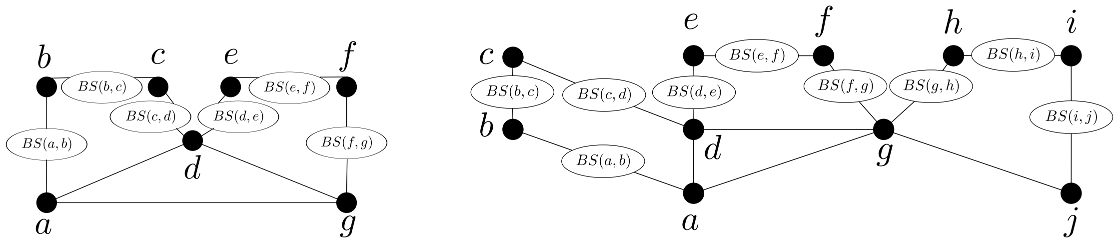
[6] (3, 1)-polar graphs having all cliques to have size at least 3 is a subclass of weakly chordal graphs.

As a remark, a shorter alternative proof that 2-clique-colouring is $\mathcal{NP}$-complete for weakly chordal (3, 1)-polar graphs with all cliques having size at least 3 can be obtained by a reduction from Positive NAE-SAT. The alternative proof follows analogously to the alternative proof of Theorem 4.7.

On the other hand, we prove that 2-clique-colouring (2, 1)-polar graphs becomes polynomial when all cliques have size at least 3.

Theorem 4.13 (Theorem 12, Appendix C)**.** *The problem of 2-clique-colouring is polynomial for (2, 1)-polar graphs with all cliques having size at least 3.*

In the proof that 2-clique-colouring of weakly chordal graphs is Σ_2^P-complete (Theorem 4.4), we constructed a weakly chordal graph with K_2 cliques to force distinct colours in their extremities (in a 2-clique-colouring). We can get a weakly chordal graph with no cliques of size 2 by adding copies of the auxiliary graph $BS(u, v)$, for every K_2 clique $\{u, v\}$. Auxiliary graphs AK and NAS become AK' and NAS', both depicted in Figure 4.13.



(a) $AK'(a, g)$ (b) $NAS'(a, j)$

Figure 4.13: Auxiliary graphs $AK'(a, g)$ and $NAS'(a, j)$

Finally, the weakly chordal graph constructed in Theorem 4.4 becomes a weakly chordal graph with no K_2 clique, depicted in Figure 4.14.

Such construction is done in polynomial-time. Notice that, in the constructed graph of Theorem 4.4, every K_2 clique $\{u, v\}$ has 2 distinct colours in a clique-colouring. Hence, one can check at Lemma 4.10 and Lemma 4.11 that the obtained graph is weakly chordal and it is 2-clique-colourable if, and only if, the constructed graph of Theorem 4.4 is 2-clique-colourable. This implies the following theorem.

Theorem 4.14 (Theorem 13, Appendix C)**.** *The problem of 2-clique-colouring is Σ_2^P-complete for weakly chordal graphs with all cliques having size at least 3.*

As a direct consequence of Theorem 4.14, 2-clique-colouring is Σ_2^P-complete for perfect graphs with all cliques having size at least 3.

Corollary 4.15 (Corollary 14, Appendix C)**.** *The problem of 2-clique-colouring is Σ_2^P-complete for perfect graphs with all cliques having size at least 3.*

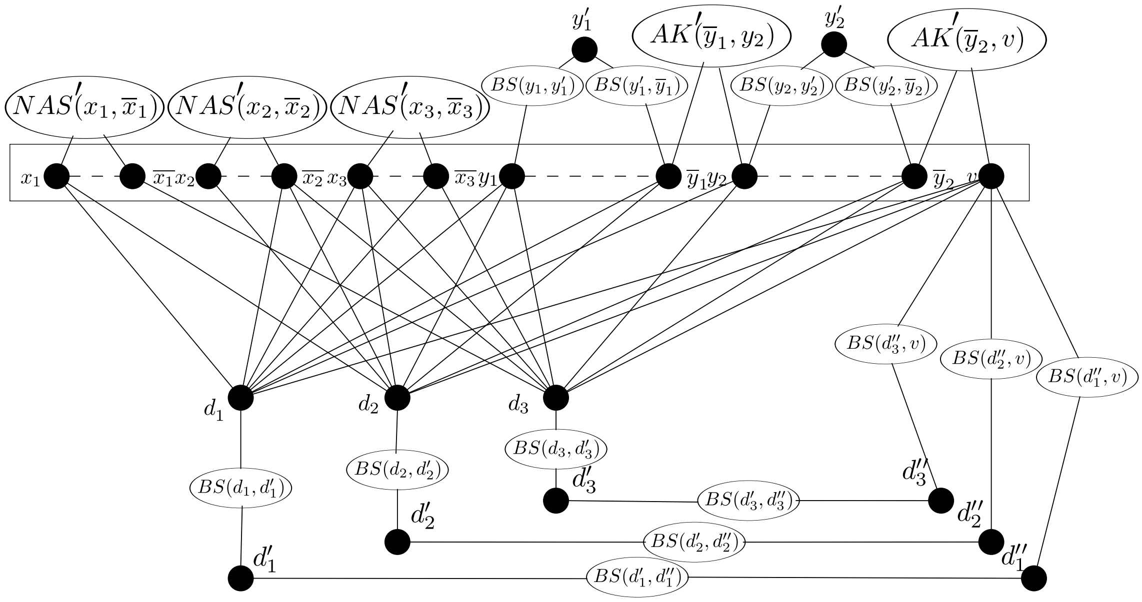


Figure 4.14: Graph construction of Theorem 4.14

4.3 Final Considerations

Marx [73] proved complexity results for k-clique-colouring, for fixed $k \geq 2$, and related problems that lie in between two distinct complexity classes, namely Σ_2^P-complete and Π_3^P-complete. Marx approaches the complexity of clique-colouring by fixing the graph class and diversifying the problem. In the present work, our point of view is the opposite: we rather fix the (2-clique-colouring) problem and classify the problem complexity according to the input graph class, which belongs to nested subclasses of weakly chordal graphs. We achieved complexities results lying in between three distinct complexity classes, namely Σ_2^P-complete, $\mathcal{NP}$-complete and $\mathcal{P}$. Figure 4.15 shows the relation of inclusion among the classes of graphs of Table 4.1. We highlight, for each class, 2-clique-colouring complexity.

Notice that the perfect graph subclasses for which the 2-clique-colouring problem is in $\mathcal{NP}$, mentioned so far in the present chapter, satisfy that the number of cliques is polynomial. We remark that the complement of a matching has an exponential number of cliques and yet the 2-clique-colouring problem is in $\mathcal{NP}$, since no such graph is 2-clique-colourable. Now, notice that the perfect graph subclasses for which the 2-clique-colouring problem is in $\mathcal{P}$, mentioned so far in the present chapter, satisfy that all graphs in the class are 2-clique-colourable. In Chapter 2, we have proved that unichord-free graphs are 3-clique-colourable, but a unichord-free graph is 2-clique-colourable if, and only if, it is perfect.

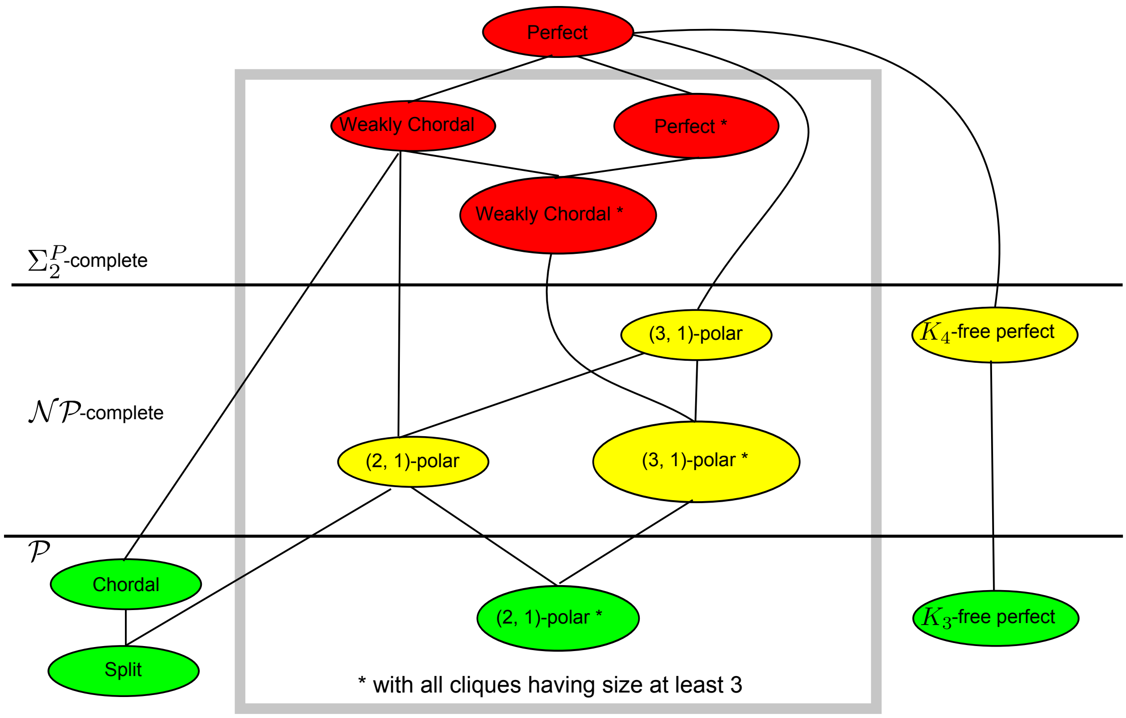


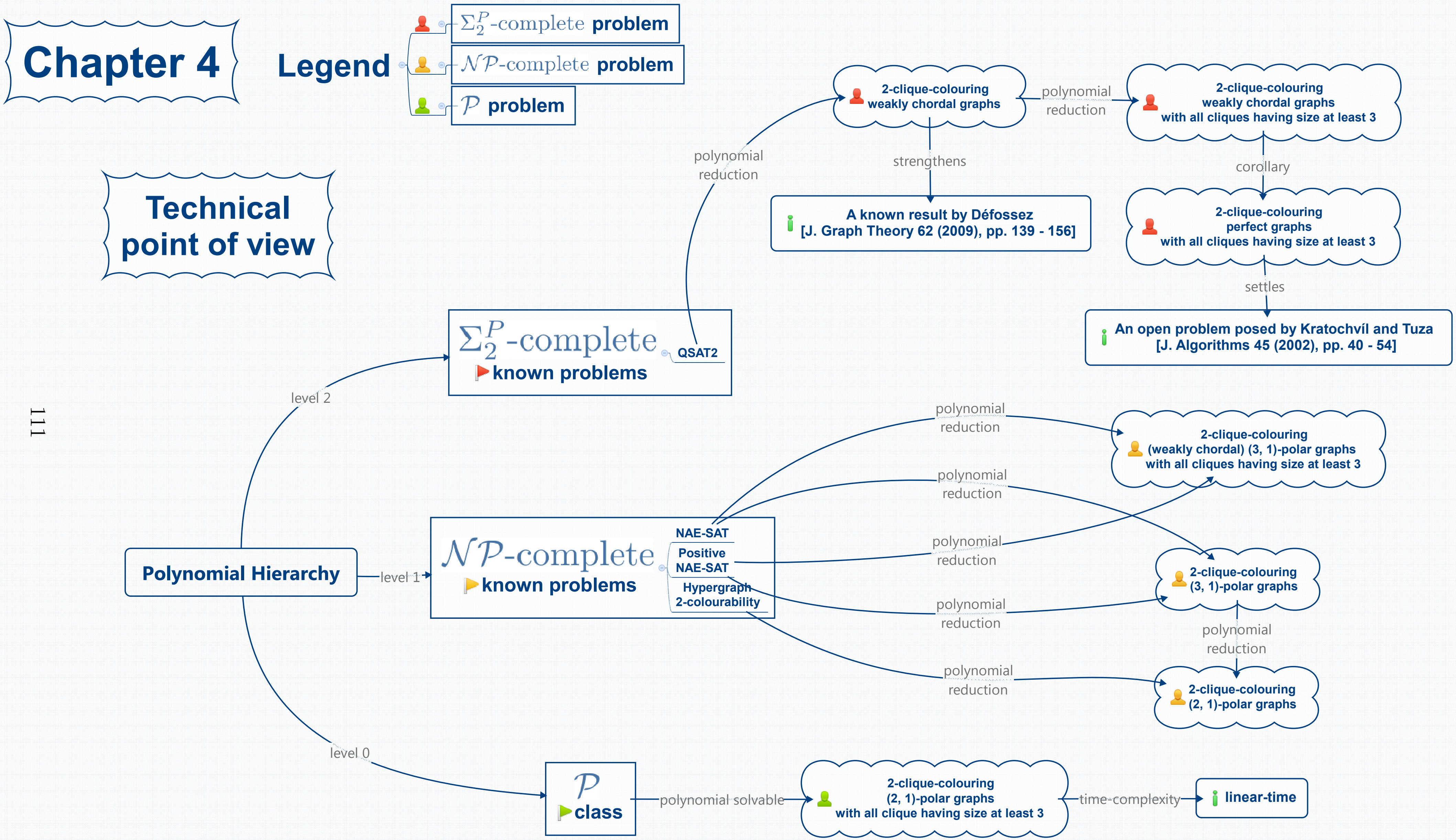
Figure 4.15: 2-clique-colouring complexity of perfect graphs and subclasses

4.3.1 Future Work

We aim to find subclasses of perfect graphs in which not all graphs are 2-clique-colourable and yet 2-clique-colouring problem is in $\mathcal{P}$ when restricted to the class. Another related question is to solve the conjecture of Duffus *et al.*, mentioned in the introduction of this thesis, for weakly chordal graphs, i.e. *weakly chordal graphs are k-clique-colourable for some constant k.*

Acknowledgment

We are grateful to Jayme Szwarcfiter for introducing us the class of (α, β)-polar graphs. Most of our results progressed after that.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polynomial reduction
polynomial reduction
polynomial reduction
2-clique-colouring (3, 1)-polar graphs
polynomial reduction
polynomial reduction
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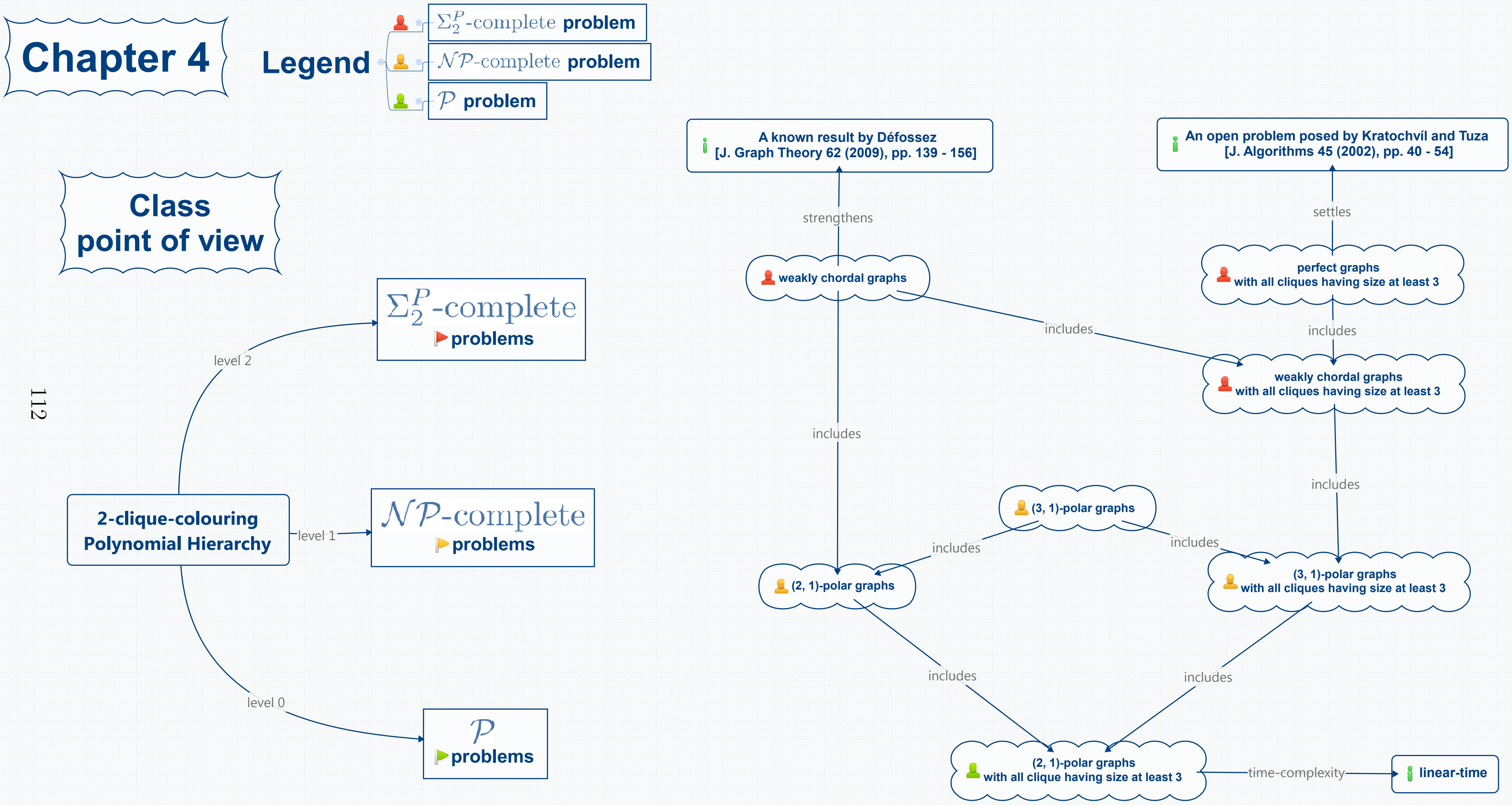
$\mathcal{P}$ class
polynomial solvable
2-clique-colouring (2, 1)-polar graphs with all clique having size at least 3
time-complexity
linear-time



Chapter 4
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(2, 1)-polar graphs
(3, 1)-polar graphs with all cliques having size at least 3
includes
includes
(2, 1)-polar graphs with all clique having size at least 3
time-complexity
linear-time

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Appendix A

Efficient algorithms for clique-colouring and biclique-colouring unichord-free graphs [*], [†]

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[†]This preprint has been submitted to Algorithmica.


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Efficient algorithms for clique-colouring and biclique-colouring unichord-free graphs ⋆

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Abstract. The class of unichord-free graphs was recently investigated in the context of vertex-colouring [J. Graph Theory 63 (2010) 31–67], edge-colouring [Theoret. Comput. Sci. 411 (2010) 1221–1234] and total-colouring [Discrete Appl. Math. 159 (2011) 1851–1864]. Unichord-free graphs proved to have a rich structure that can be used to obtain interesting results with respect to the study of the complexity of colouring problems. In particular, several surprising complexity dichotomies of colouring problems are found in subclasses of unichord-free graphs. In the present work, we investigate clique-colouring and biclique-colouring problems restricted to unichord-free graphs. We show that the clique-chromatic number of a unichord-free graph is at most 3, and that the 2-clique-colourable unichord-free graphs are precisely those that are perfect. Moreover, we describe an $O(nm)$-time algorithm that returns an optimal clique-colouring of a unichord-free graph input. We prove that the biclique-chromatic number of a unichord-free graph is either the increment of or exactly the size of a largest twin set. Moreover, we describe an $O(n^2m)$-time algorithm that returns an optimal biclique-colouring of a unichord-free graph input. The clique-chromatic and the biclique-chromatic numbers are not monotone with respect to induced subgraphs. The biclique-chromatic number presents an extra unexpected difficulty, as it is not the maximum over the biconnected components, which we overcome by considering additionally the star-biclique-chromatic number.
Keywords: unichord-free, decomposition, hypergraphs, clique-colouring, biclique-colouring.

1 Introduction

Let $G = (V, E)$ be a simple graph with $n = |V|$ vertices and $m = |E|$ edges. A *clique* of G is a maximal set of vertices that induces a complete subgraph of G with at least one edge. A *biclique* of G is a maximal set of vertices that induces

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a complete bipartite subgraph of G with at least one edge. A *clique-colouring* of G is a colouring of the vertices such that no clique is monochromatic. If the colouring uses at most k colours, then we say that it is a *k-clique-colouring.* A *biclique-colouring* of G is colouring of the vertices such that no biclique is monochromatic. If the colouring uses at most k colours, then we say that it is a *k-biclique-colouring.* The *clique-chromatic number* of G, denoted by $\kappa(G)$, is the least k for which G has a k-clique-colouring. The *biclique-chromatic number* of G, denoted by $\kappa_B(G)$, is the least k for which G has a k-biclique-colouring.

Both clique-colouring and biclique-colouring have a "hypergraph colouring version". Recall that a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is an ordered pair where V is a set of vertices and $\mathcal{E}$ is a set of hyperedges, each of which is a set of vertices. A colouring of hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a colouring of the vertices such that no hyperedge is monochromatic. Let $G = (V, E)$ be a graph and let $\mathcal{H}_C(G) = (V, \mathcal{E}_C)$ and $\mathcal{H}_B(G) = (V, \mathcal{E}_B)$ be the hypergraphs whose hyperedges are, respectively, $\mathcal{E}_C = \{K \subseteq V \mid K \text{ is a clique of } G\}$ and $\mathcal{E}_B = \{K \subseteq V \mid K \text{ is a biclique of } G\}$ — $\mathcal{H}_C(G)$ and $\mathcal{H}_B(G)$ are called, resp., the *clique-hypergraph* and the *biclique-hypergraph* of G. A clique-colouring of G is a colouring of its clique-hypergraph $\mathcal{H}_C(G)$; a biclique-colouring of G is a colouring of its biclique-hypergraph $\mathcal{H}_B(G)$.

Clique-colouring and biclique-colouring are analogous problems in the sense that they refer to the colouring of hypergraphs arising from graphs. In particular, the hyperedges are subsets of vertices that are clique (resp. biclique). The clique is a classical important structure in graphs, hence it is natural that the clique-colouring problem has been studied for a long time — see, for example, [1,3,12,17]. Bicliques, on the other hand, only recently started to be more extensively studied. Although complexity results for complete bipartite subgraph problems are mentioned in [6] and the (maximum) biclique problem is shown to be $\mathcal{NP}$-hard in [24], only in the last decade the (maximal) bicliques were rediscovered in the context of counting problems [7,21], enumeration problems [4,5,19,20], and intersection graphs [9]. For that reason, only recently the biclique-colouring problem started to be investigated [8] and it can be seen as "the state of the art" regarding the colouring of biclique-hypergraphs.

Clique-colouring and biclique-colouring have some similarities with usual vertex-colouring; in particular, any vertex-colouring is also a clique-colouring and a biclique-colouring. In other words, both the clique-chromatic number κ and the biclique-chromatic number κ_B are upper bounded by the vertex-chromatic number χ. Optimal vertex-colourings and clique-colourings coincide in the case of K_3-free graphs, while optimal vertex-colourings and biclique-colourings coincide in the (much more restricted) case of $K_{1,2}$-free graphs. Notice that the triangle K_3 is the minimal complete graph that includes the graph induced by one edge (K_2), while the $K_{1,2}$ is the minimal complete bipartite graph that includes the graph induced by one edge ($K_{1,1}$).

Clique-colouring and biclique-colouring share essential differences with respect to usual vertex-colouring. A clique-colouring (resp. biclique-colouring) may not be clique-colouring (resp. biclique-colouring) when restricted to a subgraph. Subgraphs may even have a larger clique-chromatic number (resp. larger biclique-

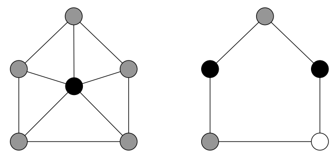


Fig. 1. Subgraphs may even have a larger clique-chromatic number.

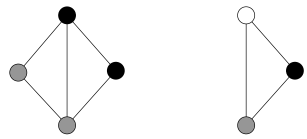


Fig. 2. Subgraphs may even have a larger biclique-chromatic number.

chromatic number). Indeed, an odd hole with six vertices C_5 and a wheel graph with five vertices W_6 are witnesses (resp. a triangle K_3 and a diamond $K_4 \setminus e$). See Fig. 1 (resp. see Fig. 2). Most remarkably, although $\kappa(G)$ is the maximum of the clique-chromatic numbers of the biconnected components, the parameter $\kappa_B(G)$, may not behave well under 1-cutset composition.

In the present work, we consider clique-colouring and biclique-colouring problems restricted to *unichord-free* graphs, which are graphs that do not contain a cycle with a unique chord as an induced subgraph. The class of unichord-free graphs has been investigated in the context of colouring problems — namely vertex-colouring [23], edge-colouring [15], and total-colouring [14]. Regarding the clique-colouring problem, we show that every unichord-free graph is 3-clique-colourable, and that the 2-clique-colourable unichord-free graphs are precisely those that are perfect. Moreover, we obtain an $O(nm)$-time algorithm that returns an optimal clique-colouring of a unichord-free graph input. The former result is interesting because perfect unichord-free graphs are a natural subclass of diamond-free perfect graphs, a class that attracted much attention in the context of clique-colouring — clique-colouring diamond-free perfect graphs is notably recognized as a difficult open problem [1,3]. As a remark, unichord-free and diamond-free graphs have a number of cliques linear in the number of vertices and edges, respectively. Every biconnected component of a unichord-free graph is a complete graph or a triangle-free graph (see Theorem 1), and every edge of a diamond-free graph is in exactly one clique (otherwise we have a diamond). It is also known that the problem of 2-clique-colouring is Σ_2^p-complete for perfect graphs and it is $\mathcal{NP}$-complete for $\{K_4$, diamond$\}$-free perfect graphs [3].

Regarding the biclique-colouring problem, we prove that the biclique-chromatic number of a unichord-free graph is the increment of or exactly the size of a largest set of mutually true twin vertices. Moreover, we describe an $O(n^2m)$-time algorithm that returns an optimal biclique-colouring by returning

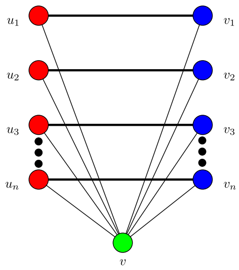


Fig. 3. Unichord-free graph with an exponential number of bicliques

an optimal biclique-colouring of a unichord-free graph input. Notice that even much restricted unichord-free graphs have a number of bicliques exponential in the number of vertices, e.g. every graph obtained by taking $t \geq 1$ copies of the complete graph K_3 with a vertex in common has $2^t + t$ bicliques (see Fig. 3).

Both clique-colouring and biclique-colouring algorithms developed in the present work follow the same general strategy that is frequently used to obtain vertex-colouring algorithms in classes defined by forbidden subgraphs: a specific structure F is chosen in such a way that one of the following cases holds.

1. a graph in the class does not contain F and so belongs to a more restricted subclass for which the solution is already known; or
2. a graph contains F and the presence of such structure entails a decomposition into smaller subgraphs in the same class.

The chosen structure for the clique-colouring algorithm is the triangle. If there exists a triangle in the unichord-free graph, we have a decomposition into two smaller graphs with a single vertex in common [23]. Otherwise, the graph is triangle-free and clique-colouring reduces to vertex-colouring. Based on an efficient algorithm for vertex-colouring unichord-free graphs [23], the construction of an efficient algorithm for clique-colouring unichord-free graphs is straightforward. Notice that vertex-colouring is $\mathcal{NP}$-hard when restricted to triangle-free graphs [16].

The biclique-colouring algorithm makes a deeper use of the decomposition results of Trotignon and Vušković [23]. The first chosen structure for the biclique-colouring algorithm is the triangle. The second chosen structure for the biclique-colouring algorithm is the square for {triangle, unichord}-free graphs. The third chosen structure is the 2-cutset in a particular setting (to be defined in the next section as a proper 2-cutset) for {square, triangle, unichord}-free graphs. Finally, an extremal decomposition — which is a decomposition whereby one of the biconnected components is undecomposable — is used to biclique-colour {triangle,square,unichord}-free graphs.

The composition of colourings along graphs decomposed when the triangle was the chosen structure is surprisingly tough in the context of biclique-colouring, while it is straightforward in the context of vertex-colouring and clique-colouring. In order to alleviate the composition, we introduce *star-colouring*, as follows. A *star* is a maximal set of vertices that induces a complete bipartite graph with a universal vertex and at least one edge. A *star-colouring* is a colouring of the vertices such that no star is monochromatic. If the colouring uses at most k colours, then we say that it is a *k-star-colouring*. The *star-chromatic number* of G, denoted by $\kappa_S(G)$, is the least k for which G has a k-star-colouring. A biclique-colouring which is also a star-colouring is the key to provide an $O(n^2 m)$-time algorithm that returns an optimal biclique-colouring by returning an optimal star-biclique-colouring of unichord-free graphs. We prove that the biclique-chromatic number and the star-biclique-chromatic number coincide for unichord-free graphs. We remark that by definition, clique-colouring and vertex-colouring coincide for triangle-free graphs, while biclique-colouring and star-biclique-colouring coincide for square-free graphs.

Table 1 highlights the computational complexity of colouring problems restricted to classes related to unichord-free graphs. We remark that by definition, clique-colouring and vertex-colouring coincide for triangle-free graphs, while biclique-colouring and star-colouring coincide for square-free graphs. It is interesting to note that the class of {square, unichord}-free provides: for total-colouring, the surprising example of a class for which total-colouring is Polynomial although edge-colouring is $\mathcal{NP}$-complete; while for biclique-colouring, since biclique-colouring and star-colouring coincide, the challenge is to colour the stars. For total-colouring, the tough decomposition is the proper 1-join, while for biclique-colouring, as we shall see, surprisingly, it is the 1-cutset. The biclique-colouring is a surprising vertex-colouring for which composition through 1-cutsets is not immediate.

Table 1. Computational complexity of colouring problems restricted to unichord-free and special subclasses — shadowed cells indicate results established in the paper.

Colouring Problem \ Class	general	unichord-free	{□,unichord}-free	{△,unichord}-free
vertex-col.	$\mathcal{NPC}$ [11]	$\mathcal{P}$ [23]	$\mathcal{P}$ [23]	$\mathcal{P}$ [23]
edge-col.	$\mathcal{NPC}$ [10]	$\mathcal{NPC}$ [15]	$\mathcal{NPC}$ [15]	$\mathcal{NPC}$ [15]
total-col.	$\mathcal{NPC}$ [18]	$\mathcal{NPC}$ [14]	$\mathcal{P}$ [13,14]	$\mathcal{NPC}$ [14]
clique-col.	$\Sigma_2^p\mathcal{C}$ [17]	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$ ($\kappa = \chi$)
biclique-col.	$\Sigma_2^p\mathcal{C}$ [8]	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$ ($\kappa_B = 2$)
star-col.	$\Sigma_2^p\mathcal{C}$ [8]	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$ ($\kappa_S = 2$)

Section 2 reviews the structure of unichord-free graphs according to the decomposition defined by Trotignon and Vušković [23], which is very useful towards clique-colouring and biclique-colouring unichord-free graphs throughout this paper. Section 3 contains the clique-colouring results for unichord-free graphs. Section 4 contains the biclique-colouring results for unichord-free graphs. Finally, Section 5 contains our concluding remarks.

2 Preliminary results

In the present section we review the structure of unichord-free graphs according to the decomposition defined by Trotignon and Vušković [23].

Given a graph F, we say that a graph G *contains* F if F is isomorphic to an induced subgraph of G. A graph G is F*-free* if it does not contain F. A chordless cycle is denoted by C_n, $n \geq 3$. A *hole* is a chordless cycle of length at least 4 and an ℓ*-hole* is a hole of length ℓ. A *triangle* is a cycle C_3 of length 3 and is a complete graph K_3 of order 3. A *square* is a chordless cycle C_4 of length 4 and a 4-hole.

The *Petersen graph* is the cubic graph on vertices $\{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ such that both $a_1a_2a_3a_4a_5a_1$ and $b_1b_2b_3b_4b_5b_1$ are chordless cycles, and such that the only edges between some a_i and some b_j are a_1b_1, a_2b_4, a_3b_2, a_4b_5, a_5b_3. Note that the Petersen graph contains a 5-hole. The *Heawood graph* is the cubic bipartite graph on vertices $\{a_1, \ldots, a_{14}\}$ such that $a_1a_2 \ldots a_{14}a_1$ is a cycle, and such that the only other edges are a_1a_{10}, a_2a_7, a_3a_{12}, a_4a_9, a_5a_{14}, a_6a_{11}, a_8a_{13}. Note that the Heawood graph contains a 6-hole. We invite the reader to check that both the Petersen graph and the Heawood graph are unichord-free.

A graph is *strongly 2-bipartite* if it is square-free and bipartite with bipartition (X, Y), where every vertex in X has degree 2 and every vertex in Y has degree at least 3. A strongly 2-bipartite graph is unichord-free because any chord of a cycle is an edge between two vertices of degree at least three, so that every cycle in a strongly 2-bipartite graph is chordless.

A graph G is called *basic* if it is a complete graph, a hole of length at least 7, a strongly 2-bipartite graph, or an induced subgraph (not necessarily proper) of the Petersen graph or of the Heawood graph. Note that every basic graph is square-free.

A *cutset* S of a connected graph G is a set of vertices or edges whose removal disconnects G.

A decomposition of a graph is the systematic removal of a cutset to obtain smaller graphs — called the *blocks of decomposition* — possibly adding some vertices and edges to connected components of $G \backslash S$, until obtaining a set of basic (indecomposable) graphs. The goal of decomposing a graph is to solve a problem on the original graph by combining the solutions on the blocks of decompositions. The following cutsets are used in the decomposition for unichord-free graphs.

- A *1-cutset* of a connected graph $G = (V, E)$ is a vertex v such that V can be partitioned into sets X, Y, and $\{v\}$, such that there is no edge between X and Y. We say that (X, Y, v) is a *split* of this 1-cutset.
- A *proper 2-cutset* of a connected graph $G = (V, E)$ is a pair of non-adjacent vertices a, b, both of degree at least three, such that V can be partitioned into sets X, Y, and $\{a, b\}$ such that: $|X| \geq 2$, $|Y| \geq 2$; there is no edge

between X and Y, and both $G[X \cup \{a, b\}]$ and $G[Y \cup \{a, b\}]$ contain a path $a \dots b$. We say that (X, Y, a, b) is a *split* of this proper 2-cutset.
- A *proper 1-join* of a graph $G = (V, E)$ is a partition of V into sets X and Y such that there exist sets $A \subseteq X$ and $B \subseteq Y$ such that: $|A| \geq 2$, $|B| \geq 2$; A and B are stable sets; there are all possible edges between A and B; there is no other edge between X and Y. We say that (X, Y, A, B) is a *split* of this proper 1-join.

The blocks of decomposition w.r.t. a 1-cutset, a proper 2-cutset, and a proper 1-join are defined precisely as follows. Moreover, all kinds of blocks of decomposition of a unichord-free graph were constructed in such a way that they remain unichord-free [23].

- The *block of decomposition* G_X (resp. G_Y) of a graph G w.r.t. a 1-cutset with split (X, Y, v) is $G[X \cup \{v\}]$ (resp. $G[Y \cup \{v\}]$).
- The *block of decomposition* G_X (resp. G_Y) of a graph G w.r.t. a proper 1-join with split (X, Y, A, B) is the graph obtained by taking $G[X]$ (resp. $G[Y]$) and adding a vertex y adjacent to every vertex of A (resp. x adjacent to every vertex of B). Vertices x, y are called *markers* of their respective blocks of decomposition.
- The *blocks of decomposition* G_X and G_Y of a graph G w.r.t. a proper 2-cutset with split (X, Y, a, b) are defined as follows. If there exists a vertex c of G such that $N_G(c) = \{a, b\}$, then let $G_X = G[X \cup \{a, b, c\}]$ and $G_Y = G[Y \cup \{a, b, c\}]$. Otherwise, block of decomposition G_X (resp. G_Y) is the graph obtained by taking $G[X \cup \{a, b\}]$ (resp. $G[Y \cup \{a, b\}]$) and adding a new vertex c adjacent to a, b. Vertex c is called the *marker* of the block of decomposition G_X (resp. G_Y).

A *decomposition tree* of a graph is a rooted tree in which each node corresponds to either G or to a block of decomposition of its parent. We strongly use a decomposition tree defined by Trotignon and Vušković [23], as follows. A *proper decomposition tree* of a connected unichord-free graph G is a rooted tree T_G such that the following hold:

1. G is the root of T_G.
2. Every node of T_G is a connected graph.
3. Every leaf of T_G is basic.
4. Every non-leaf node H of T_G is of one of the following types:
   - Type 1. The children of H in T_G are the blocks of decomposition w.r.t. a 1-cutset or a proper 1-join.
   - Type 2. H and all its descendants are {Petersen, triangle, square}-free and have no 1-cutset and no proper 1-join. Moreover, the children of H in T_G are the blocks of decomposition w.r.t. a proper 2-cutset and every non-leaf descendant of H is of type 2.
5. If a node of T_G is a triangle-free graph then all its descendants are triangle-free graphs.

We require another property on the type 2 non-leaf node H of T_G: (at least) one block of decomposition is basic. It is always possible, since Machado, de Figueiredo, and Vušković [15] proved that every non-basic biconnected {square,unichord}-free graph has the so-called *extremal* decomposition, which is a decomposition whereby every non-leaf has (at least) one basic block of decomposition. Such decomposition is suitable to extend the colouring of a non-leaf type 2 block of decomposition to the basic block of decomposition. This approach is useful to return an optimal biclique-colouring of {triangle, square, unichord}-free graphs.

Another decomposition result concerns the complete graphs. If a unichord-free graph G contains a triangle, then either G is a complete graph, or one vertex of the clique that contains this triangle is a 1-cutset of G [23]. Equivalently, we have the following statement, which is a useful tool towards clique-colouring and biclique-colouring unichord-free graphs throughout this paper.

Theorem 1 (Trotignon and Vušković [23]). *Every biconnected component of a unichord-free graph is a complete graph, or a triangle-free graph.*

Finally, Theorem 2 states an algorithm that computes an optimal vertex-colouring of a unichord-free graph. This algorithm is used as a black box to solve the triangle-free case on optimal clique-colouring.

Theorem 2 (Trotignon and Vušković [23]). *Let G be a unichord-free graph. The chromatic number of G is $\chi(G) \leq \max\{3, \omega(G)\}$. Moreover, there exists a $O(nm)$-time algorithm that computes an optimal vertex-colouring of any unichord-free graph.*

3 Clique-colouring unichord-free graphs

When a graph is triangle-free, clique-colouring reduces to vertex-colouring. Theorem 2 handles this case. If the unichord-free graph contains a triangle, we entail a decomposition by 1-cutsets given by Theorem 1. The following lemma states that an optimal clique-colouring of a graph can be obtained from optimal clique-colourings of its blocks of decomposition w.r.t. a 1-cutset G_X and G_Y. The key observation for its proof is that a subset of vertices of any graph G is a clique in G if, and only if, it is a clique either in G_X or in G_Y. We omit the proof due to its simplicity.

Lemma 1. *Let G be a graph. An optimal clique-colouring of G can be obtained from optimal clique-colourings of its blocks of decomposition w.r.t. a 1-cutset.*

A consequence of Lemma 1 is that the clique-chromatic number of a unichord-free graph is at most 3.

Theorem 3. *Every unichord-free graph is 3-clique-colourable.*

Proof. We argue by induction on the blocks of decomposition G_X and of G_Y w.r.t. 1-cutsets. If G does not contain a 1-cutset, then G is either a complete graph or a triangle-free graph. In the former case, any 2-colouring of its vertices is a clique-colouring, while in the latter case Theorem 2 assigns a 3-clique-colouring. If G contains a 1-cutset, we entail a decomposition by 1-cutset and apply the induction hypothesis on G_X and on G_Y, both graphs with less vertices than G. Hence, both blocks of decomposition have a clique-colouring that uses at most 3 colours. The proof of Lemma 1 combines clique-colourings of G_X and of G_Y to obtain a clique-colouring of G using at most the maximum number of colours from each clique-colouring, i.e. at most 3 colours. $\square$

Next, we obtain a characterization that the 2-clique-colourable unichord-free graphs are exactly those that are perfect.

Theorem 4. *A unichord-free graph is 2-clique-colourable if and only if it is perfect.*

Proof. Assume G is 2-clique-colourable. Let B be a biconnected component of G. If B is triangle-free, then a clique-colouring of B is also a vertex-colouring, such that B is 2-vertex-colourable (equivalently bipartite), hence perfect. If B has a triangle then, by Theorem 1, graph B is a complete graph, hence perfect. As a consequence, all blocks of decomposition of G are perfect and so is G.

For the converse, we first prove that G is unichord-free and perfect if and only if G is {unichord, odd-hole}-free and, second, we prove that every {unichord, odd-hole}-free graph is 2-clique-colourable.

Clearly, graph G is perfect only if G is odd-hole-free. Conversely, we claim that a {unichord, odd-hole} graph G is odd-antihole-free. Suppose G has an odd-antihole A and let $v_1, \ldots, v_{|A|}$ be the sequence of consecutive vertices of its complement $\overline{A}$. Indeed, if $|A| = 5$ then A is an odd-hole (contradiction). Otherwise, i.e. if $|A| \geq 7$, then $A[v_1, v_3, v_4, v_6]$ has a unichord v_1v_6 (contradiction). Therefore, G is {unichord,odd-hole,odd-antihole}-free which, by the Strong Perfect Graph Theorem [2], implies that G is unichord-free and perfect.

Now, we prove that a {unichord, odd-hole}-free graph is 2-clique-colourable. Let G be a {unichord, odd-hole}-free graph. Suppose G has an odd cycle. Since G is odd-hole-free, every odd cycle with order at least 5 has a chord, so it has a smallest even cycle and a smallest odd cycle. Hence, every biconnected component containing an odd cycle has a triangle. Since G is unichord-free, the biconnected component containing the triangle is a complete graph, and it is 2-clique-colourable. On the other hand, every triangle-free biconnected component has no odd-cycle and it is 2-vertex-colourable, hence 2-clique-colourable. Since every biconnected component is 2-clique-colourable, we conclude (with Lemma 1) that G is 2-clique-colourable. $\square$

3.1 Algorithmic aspects

Theorem 5 states the complexity to compute an optimal clique-colouring of a unichord-free graph.

Theorem 5. *There exists an $O(nm)$-time algorithm to compute an optimal clique-colouring of a unichord-free graph.*

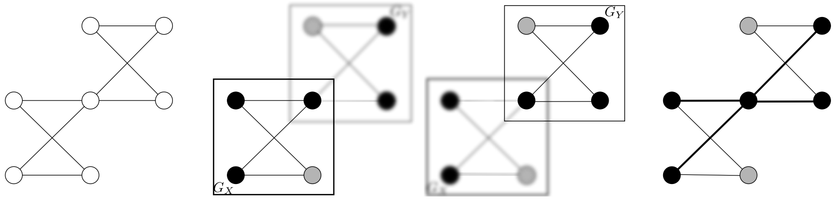
Proof. Let G be a unichord-free graph and (X, Y, v) be the split of a 1-cutset of G. We prove the statement by induction on the blocks of decomposition G_X and G_Y w.r.t. 1-cutset. If G does not contain a 1-cutset, then G is either a complete graph or a triangle-free graph. In the former case, any 2-colouring of its vertices is a clique-colouring, while in the latter case an optimal clique-colouring can be handled by Theorem 2. The given clique-colouring algorithms are respectively linear-time and $O(nm)$-time. If G contains a 1-cutset, we entail a decomposition by 1-cutset and apply the induction hypothesis on G_X and on G_Y, both graphs with less vertices than G. The proof of Lemma 1 gives a constant time algorithm to combine optimal clique-colourings of G_X and G_Y to obtain an optimal clique-colouring of G. Hence, we are left to prove that the overall time-complexity to give an optimal clique-colouring of G_X and of G_Y is $O(nm)$. Let $n_X = |V(G_X)|$, $n_Y = |V(G_Y)|$, $m_X = |E(G_X)|$, and $m_Y = |E(G_Y)|$.

Since blocks of decomposition w.r.t. a 1-cutset have only one vertex in common, we have $n = n_X + n_Y - 1$, $m = m_X + m_Y$. It follows that the overall time-complexity to give an optimal clique-colouring of G is $O(n_X m_X) + O(n_Y m_Y) + O(1) = O(nm)$ and we conclude our proof. □

4 Biclique-colouring unichord-free graphs

We now turn our attention to the biclique-colouring problem restricted to unichord-free graphs. In contrast to the case of clique-colouring, there exists no analogous of Lemma 1, for the case of biclique-colouring, to combine colourings along 1-cutsets. Indeed, an optimal biclique-colouring of the blocks of decomposition of a graph does not necessarily determine an optimal biclique-colouring of that graph. An example is illustrated in Fig. 4. A *star centered in v* is a star with universal vertex v. One can check that every star centered in a 1-cutset is also a biclique. The key idea of this section follows. We overcome monochromatic star bicliques when biclique-colourings are "glued" along 1-cutsets restricting the biclique-colouring to be also a star-colouring. So, we will always want both colourings: biclique-colouring and star-colouring. As a remark, notice that there are biclique-colourings that are not star-colourings and *vice-versa*. See Fig. 5 for instances. We call *star-biclique-colouring* the biclique-colouring that is also a star-colouring. If the star-biclique-colouring uses at most k colours, then we say that it is a *k-star-biclique-colouring*. See Fig. 6 for the corresponding star-biclique-colouring versions of the graphs of Fig. 5.

The *star-biclique-chromatic number* of G, denoted by $\kappa_{SB}(G)$, is the least k for which G has a k-star-biclique-colouring. Notice that one more restriction for biclique-colourings might impose the need for more than $\kappa_B(G)$ colours to biclique-colour the graph G. Fortunately, we show in Theorem 7 that the star-biclique-chromatic number and the biclique-chromatic number coincide for unichord-free graphs. It is quite interesting to notice that a further restriction



(a) Graph G (b) G_X, G_Y optimally biclique-coloured (c) Invalid colouring

Fig. 4. Unichord-free graph whose blocks of decomposition optimal biclique-colourings do not determine a biclique-colouring. Fig. 4a is a unichord-free graph, Fig. 4b is one optimal biclique-colouring of its blocks of decomposition, and Fig. 4c shows the existence of a monochromatic star biclique highlighted with bold edges.

(a) A star-colouring which is not a biclique-colouring (b) A biclique-colouring which is not a star-colouring

Fig. 5. There are biclique-colourings that are not star-colourings and *vice-versa*.

makes our lives easier, since we are free to glue biclique-colourings along 1-cutsets and only $\kappa_B(G)$ colours are still needed.

We divide this section into two parts. Subsection 4.1 starts with a 2-star-biclique-colouring algorithm for biconnected unichord-free graphs. This result is very important to start Subsection 4.2 with a constructive proof that star-biclique-chromatic number and the biclique-chromatic number coincide for unichord-free graphs. Subsection 4.2 then develops an optimal star-biclique-colouring algorithm for non-biconnected unichord-free graphs: Subsection 4.2 defines our proposed extremal decomposition tree; Subsection 4.2 establishes that the star-biclique-chromatic number of a unichord-free graph G is either $\beta(G)$ or $\beta(G) + 1$, where $\beta(G)$ is the maximum cardinality of a true twin set of graph G; Subsection 4.2 then describes the algorithm that decides between these two possible values.

4.1 Biconnected unichord-free graphs

From now on, we consider K_1 and K_2 biconnected components as in the case where the biconnected component is a complete graph and not in the case where the biconnected component is a triangle-free graph. This assumption helps us

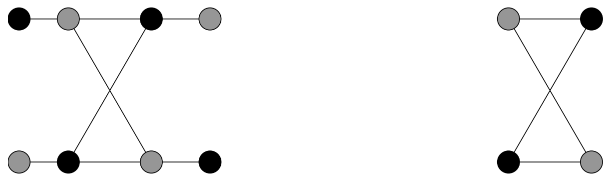


Fig. 6. Corresponding star-biclique-colouring versions for the graphs of Fig. 5.

with case analysis when we consider biconnected {triangle, unichord}-free graphs and cliques as two distinct cases.

In order to construct an algorithm to combine colourings along 1-cutsets, we start dealing with a biconnected unichord-free graph G, as follows. If G is a complete graph, then an optimal star-biclique-colouring uses $|V(G)|$ colours. Hence, we consider next biconnected {triangle, unichord}-free graphs with at least four vertices. An optimal star-biclique-colouring algorithm of a biconnected {triangle, unichord}-free graph strongly relies on the proper decomposition tree defined in Section 2, where (at least) one block of decomposition of the type 2 non-leaf node is basic. Recall that by definition biclique-colouring and star-colouring coincide for square-free graphs. Moreover, every leaf of a proper decomposition tree, a basic graph, is square-free, and that every type 2 non-leaf is square-free.

We hereby construct such optimal star-biclique-colouring algorithm of every biconnected {triangle, unichord}-free graph, as follows. We show in Lemma 2 that every basic graph has a 2-star-biclique-colouring. In fact, we give a slightly stronger result: a basic graph has a 2-star-biclique-colouring even if the colours of two arbitrary vertices at distance 2 are fixed. Such result is suitable to extend the star-biclique-colouring of a biconnected unichord-free graph to any basic graph. As a consequence of this strategy, we are able to show in Lemma 3 that every biconnected {triangle, square, unichord}-free graph has an optimal star-biclique-colouring obtained from optimal star-biclique-colourings of its blocks of decomposition w.r.t. a proper 2-cutset, when one of the blocks is basic.

Finally, we prove in Lemma 4 that every biconnected {triangle, unichord}-free graph has an optimal star-biclique-colouring obtained from optimal star-biclique-colourings of its blocks of decomposition w.r.t. a proper 1-join. It concludes our algorithm for biconnected {triangle, unichord}-free graphs.

Lemma 2. *Let G be a basic graph. Let M be a vertex of degree at least 2 and let a and b be two neighbors of M. There exists a 2-star-biclique-colouring of G where a and b have the same color (resp. have distinct colours).*

Proof. Let G be a basic graph. See Fig. 7 for a 2-star-biclique-colouring of G assuming the colours of two arbitrary vertices at distance 2 are fixed.

Now, we are left to exhibit a 2-star-biclique-colouring for induced subgraphs of the Petersen graph. Let P be the Petersen graph. Notice that the Petersen graph is vertex-transitive and suppose $a = a_1$, $M = a_2$, and $b = a_3$.

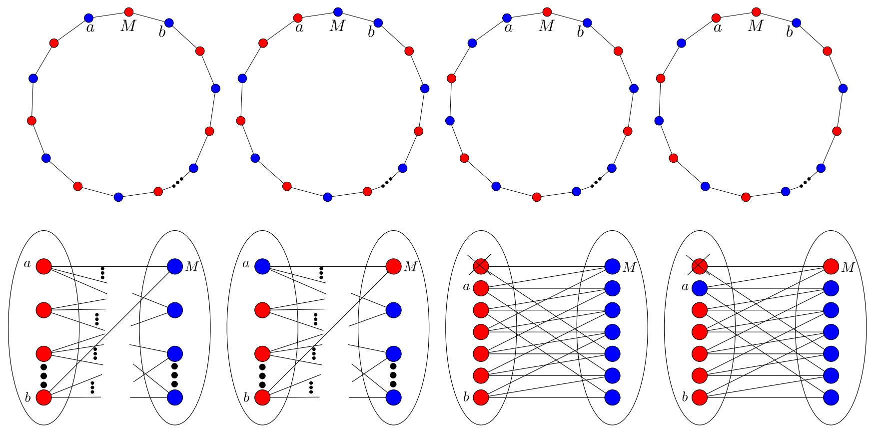


Fig. 7. A 2-star-biclique-colouring of an even hole, an odd hole, a strongly 2-bipartite, or an induced subgraph of a Heawood graph assuming the colours of two arbitrary vertices at distance 2 are fixed.

If a and b have same color, then consider the following colouring of P. Assign the same colour to all vertices of $V' = \{a_1, a_3, b_3, b_5\}$; assign another colour to all vertices of $V(P)\backslash V'$. Otherwise, consider the following colouring of P. Assign the same colour to all vertices of $V' = \{a_1, a_2, a_4, b_2, b_3, b_5\}$; assign another colour to all vertices of $V(P)\backslash V'$.

There is no C_4 in P, which implies that all bicliques are stars. Every star of G is polychromatic, since P is triangle-free and every vertex v of P has at most one neighbor with the same colour and at least one neighbor with another colour. Let G be a connected induced graph of P, such that G is neither isomorphic to K_1 nor K_2. Every star of G centered in a vertex v is polychromatic, since G is triangle-free and v has at most one neighbor with the same colour and at least one neighbor with another colour. Then, G is 2-star-biclique-colourable. □

Lemma 3. *Let G be a biconnected {triangle, square, unichord}-free graph. A 2-star-biclique-colouring of G can be obtained from 2-star-biclique-colourings of its blocks of decomposition w.r.t. a proper 2-cutset, when one of the blocks is basic.*

Proof. Let G be a {triangle, square, unichord}-free graph and let (X, Y, a, b) be a proper 2-cutset of G such that G_X is basic [15]. Recall that G_Y is a {triangle, square, unichord}-free graph. Assume, by hyphotesis, that G_Y has a 2-star-biclique-colouring. Vertices a and b of G_Y may have the same colour or distinct colours. In any case, by Lemma 2, we have a 2-star-biclique-colouring of G_X where vertices a, b, and c of G_X have the same colours of a, b, and c of G_Y. Clearly, the colours of the common vertices of G_X and G_Y match and it is a 2-colouring of G.

There is no C_4 in G, which implies that all bicliques are stars. We claim that every star of G is polychromatic. Clearly, every star of G contains a star in G_X or in G_Y. Every star of G_X and of G_Y is polychromatic. Then, every star of G is polychromatic. □

Lemma 4. *Let G be a biconnected {triangle, unichord}-free graph. A 2-star-biclique-colouring of G can be obtained from 2-star-biclique-colourings of its blocks of decomposition w.r.t. a proper 1-join.*

Proof. Let $G = (V, E)$ be a {triangle, unichord}-free graph and let (X, Y, A, B) be a proper 1-join of G. Assume that G_X and G_Y have 2-star-biclique-colourings with *red* and *blue* colours. Let vertices x, y be the markers of blocks of decomposition G_X and G_Y, respectively. Without loss of generality, suppose that vertex x has *red* colour. Vertice x and its neighbors in G_Y induce a star in G_Y and, by hypothesis, G_Y is 2-star-biclique-colourable. Hence, at least one of the neighbors of x in G_Y has *not* the *red* colour. If this neighbor has not the *blue* colour in G_Y, swap this colour with the *red* colour throughout all vertices of G_Y.

Now, without loss of generality, suppose that vertex y has *blue* colour in G_X. Analogously, at least one of the neighbors of y in G_X has *not* the *blue* colour. If this neighbor has not the *red* colour in G_X, swap this colour with the *red* colour throughout all vertices of G_X. Clearly, the colours of the common vertices of G_X and G_Y match and it is a 2-colouring of G.

We claim that every star and every biclique of G is polychromatic. Clearly, every biclique of G (resp. star of G) contains a biclique (resp. star) in G_X or in G_Y. Every biclique (resp. star) of G_X and of G_Y is polychromatic. Then, every biclique of G (resp. star of G) is polychromatic. □

A consequence of Lemmas 2, 3, and 4 is that the star-biclique-chromatic number of a {triangle, unichord}-free biconnected component is 2, stated next.

Theorem 6. *There exists a 2-star-biclique-colouring of a biconnected {triangle, unichord}-free graph.*

Proof. We prove the statement by induction on the blocks of decomposition G_X and G_Y w.r.t. proper 1-joins and proper 2-cutsets. If G does not contain a proper 1-join and a proper 2-cutset, then G is a basic graph. The proof of Lemma 2 gives us a recipe to assign a suitable 2-star-biclique-colouring to G. If G contains a proper 1-join, then we entail an extremal decomposition by proper 1-join and apply the induction hypothesis on G_X and on G_Y, both graphs with less vertices than G. The proof of Lemma 4 gives us a recipe to combine the 2-star-biclique-colourings of G_X and of G_Y to obtain a 2-star-biclique-colouring of G. If G does not contain a proper 1-join and contains a proper 2-cutset, then we entail a decomposition by proper 2-cutset and apply the induction hypothesis on basic G_X and on G_Y, both graphs with less vertices than G. Note that the hypothesis that G does not contain a proper 1-join implies that G is square-free. The proof of Lemma 3 gives us a recipe to combine 2-star-biclique-colourings of basic G_X and of G_Y to obtain a 2-star-biclique-colouring of G. This concludes our induction. □

4.2 Non-biconnected unichord-free graphs

The given 2-star-biclique-colouring for {triangle, unichord}-free biconnected components is the basis to provide a way to glue biclique-colourings along 1-cutsets. We show below that the star-biclique-colouring strategy also determines an optimal biclique-colouring of a unichord-free graph.

Theorem 7. *The biclique-chromatic number and the star-biclique-chromatic number coincide for unichord-free graphs.*

Proof. Consider that G is a biconnected unichord-free graph. If G is a complete graph, then the bicliques of G are precisely the edges of G and the bicliques of G are precisely the stars of G. Both optimal biclique-colouring and optimal star-biclique-colouring are actually optimal vertex-colouring and have $|V(G)|$ colours. Otherwise, i.e. G is a {triangle, unichord}-free and Theorem 6 states that the star-biclique-chromatic number is 2.

Now, consider that G is a non-biconnected graph. Suppose that G has an optimal biclique-colouring that is not a star-biclique-colouring, i.e. there is a monochromatic star, say S. Notice that S is a subset of vertices of a triangle-free biconnected component, say B. We give a new biclique-colouring to G with the same number of colours as before, such that (i) S is polychromatic and (ii) every biclique or star of G polychromatic before the new colouring of B is still polychromatic in the new biclique-colouring of G.

Assign a 2-star-biclique-colouring to B according to Theorem 6, in order to S be polychromatic.

For each 1-cutset v of G that is contained in B, proceed as follows. Let c_1 (resp. c_2) be the colour of v before (resp. after) the new 2-colouring of B. If $c_1 = c_2$, we are done. Let G' be the connected component obtained by all vertices of paths starting from v and not containing vertices of $B \setminus \{v\}$. Clearly, G' is a block of decomposition for some split (X, Y, v) of G. Assign colour c_2 to v and swap colours c_1 and c_2 in the block of decomposition G'. Clearly, since B has a 2-star-biclique-colouring, every star centered in v and every biclique or star properly contained in B are polychromatic. Moreover, clearly, every biclique or star properly contained in the block of decomposition G' is polychromatic (before and after the swap), and we are done.

Finally, if we repeat this process for all possible triangle-free biconnected components, we end with a star-biclique-colouring with the same number of colours of an optimal biclique-colouring. This concludes our proof. □

From now on, we are interested in determining an optimal star-biclique-colouring for unichord-free graphs.

Extremal decomposition tree for non-biconnected graphs Let B be a biconnected component of a (not necessarily unichord-free) graph G. If $|\mathcal{C}(B)| = 1$, then B is *type* $\mathcal{F}$. Now, denote by $\Gamma_G(B, v)$ the set of biconnected components of G that share vertex v with B. If every biconnected component in $\Gamma_G(B, v)$ is

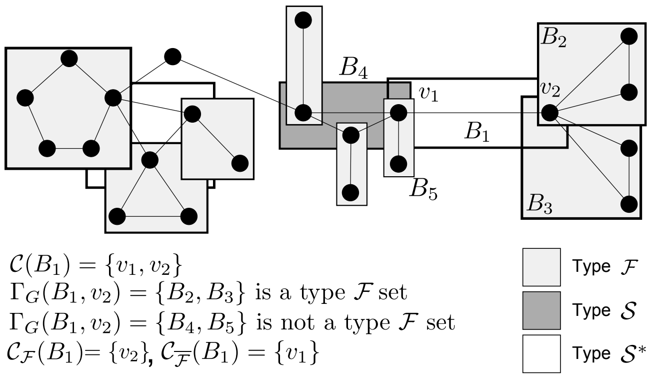


Fig. 8. Unichord-free graph enhancing the biconnected component types

type $\mathcal{F}$, then $\Gamma_G(B, v)$ is a *type* $\mathcal{F}$ *set*. Let $\mathcal{C}_{\mathcal{F}}(B)$ be the subset of $\mathcal{C}(B)$ where, for each vertex $v \in \mathcal{C}_{\mathcal{F}}(B)$, $\Gamma_G(B, v)$ is a type $\mathcal{F}$ set. If $|\mathcal{C}_{\mathcal{F}}(B)| \geq 1$, then B is *type* $\mathcal{S}$. Now, let $\mathcal{C}_{\overline{\mathcal{F}}}(B) = \mathcal{C}(B) \setminus \mathcal{C}_{\mathcal{F}}(B)$. If B is type $\mathcal{S}$ and $|\mathcal{C}_{\overline{\mathcal{F}}}(B)| \leq 1$, then B is *type* $\mathcal{S}^*$. Figure 8 shows examples of the terminology introduced, where the biconnected components not highlighted are not type $\mathcal{F}$, $\mathcal{S}$, nor $\mathcal{S}^*$.

Notice that the collections of biconnected components that are type $\mathcal{F}$ and type $\mathcal{S}$ are equal if, and only if, a connected graph has only one 1-cutset. Notice that a type $\mathcal{S}^*$ is type $\mathcal{S}$, but the converse is not always true.

For any non-biconnected graph, we prove that there always exist at least one type $\mathcal{F}$, one type $\mathcal{S}$, and one type $\mathcal{S}^*$ biconnected component.

Lemma 5. *Every non-biconnected graph has at least one type $\mathcal{F}$, one type $\mathcal{S}$, and one type $\mathcal{S}^*$ biconnected component.*

Proof. If we have one type $\mathcal{S}^*$, then we have one type $\mathcal{S}$ and one type $\mathcal{F}$. Nevertheless, our proof that there exists one type $\mathcal{S}^*$ needs the proof that there exists one type $\mathcal{S}$. The latter is proved in Proposition 1, while the former is proved in Proposition 2. □

Proposition 1. *Every non-biconnected graph has at least one type $\mathcal{S}$ biconnected component.*

Proof. Suppose, by contradiction, there is no type $\mathcal{S}$. Then, every biconnected component does not have vertices in common only with biconnected components that are type $\mathcal{F}$. Since we have at least two biconnected components, let B_{i+1} be such a non-type $\mathcal{F}$ and let B_i and B_{i+2} be two biconnected components that are non-type $\mathcal{F}$, each one containing a vertex in common with B_{i+1}, but B_i, B_{i+1}, and B_{i+2} do not have a vertex in common. Now, let $P_{B_{i+1}}$ be a path between the following two 1-cutsets of B_{i+1}: one contained in B_i and the other one contained in B_{i+2}. Finally, let C be a path obtained by concatenating paths $P_{B_i}, P_{B_{i+1}}, \ldots$ and so on. Since G is finite, $B_j = B_k$, for some $i \leq k < j$. Then, C contains a cycle, a contradiction. □

Proposition 2. *Every non-biconnected graph has at least one type $\mathcal{S}^*$ biconnected component.*

Proof. Suppose, by contradiction, there is no type $\mathcal{S}^*$. Then, every type $\mathcal{S}$ has at least two distinct vertices, each one contained in at least two distinct biconnected components that are non-type $\mathcal{F}$. Let B_{i+1} be type $\mathcal{S}$, guaranteed to exist by Proposition 1. Let B_i and B_{i+2} be biconnected components that are not type $\mathcal{F}$, each one containing a vertex in common with B_{i+1}, but B_i, B_{i+1}, and B_{i+2} do not have a vertex in common. We claim that at least two 1-cutsets of B_i (resp. B_{i+2}) are contained in at least two non-type $\mathcal{F}$ biconnected components. We already know that biconnected component B_i (resp. B_{i+2}) has at least two distinct 1-cutsets, where one of them is contained in B_{i+1}. Now, suppose that another 1-cutset of B_i (resp. B_{i+2}) is contained only in type $\mathcal{F}$ biconnected components, except B_i (resp. B_{i+2}). Thus, B_i (resp. B_{i+2}) is type $\mathcal{S}$ biconnected component and has at least two 1-cutsets, each one contained in at least two non-type $\mathcal{F}$ biconnected components. Now, let $P_{B_{i+1}}$ be a path between the following two 1-cutsets of B_{i+1}: one contained in B_i and the other one contained in B_{i+2}. Finally, let C be a path obtained by concatenating paths $P_{B_i}, P_{B_{i+1}}, \ldots$ and so on. Since G is finite, $B_j = B_k$, for some $i \leq k < j$. Then, C contains a cycle, a contradiction. $\square$

Now, we introduce an extremal decomposition for non-biconnected graphs via type $\mathcal{S}^*$ biconnected component, as follows. Consider a graph G with a type $\mathcal{S}^*$ biconnected component B^*. Graph G can be decomposed into subgraphs G_1 and G_2, such that $G_1 = B^* \cup \left(\bigcup_{v \in \mathcal{C}_{\mathcal{F}}(B^*)} \mathit{\Gamma}_G(B^*, v) \right)$ and $G_2 = G[V(G \backslash G_1) \cup V(B^*)]$. The decomposition algorithms of this section have the following general strategy. We first examine G_1 and based on it, we determine possible values for the number of colours needed in the vertices of B^* for a star-biclique-colouring of G. We modify G_2 in order to record the information and recursively decompose. If $G = B^* \cup \left(\bigcup_{v \in \mathcal{C}_{\mathcal{F}}(B^*)} \mathit{\Gamma}_G(B^*, v) \right)$, for a type $\mathcal{S}^*$ biconnected component B^*, then we call G a *prime graph*, a basic graph for the proposed extremal decomposition. Note that, if G is a prime graph, then $|\mathcal{C}_{\overline{\mathcal{F}}}(B^*)| = 0$, while if G is non-prime, then $\mathcal{C}_{\overline{\mathcal{F}}}(B^*) = \{v^*\}$.

Bounds for star-biclique-chromatic number of unichord-free graphs
Let u be a vertex of a graph G. The open neighbourhood of u is $N(u) = \{v \in V(G) \mid uv \in E(G)\}$, and the closed neighbourhood of u is $N[u] = N(u) \cup \{u\}$. Two distinct vertices u, v are *true twins* if $N[u] = N[v]$. This equivalence relation on the vertex set $V(G)$ of a graph defines a partition of $V(G)$ into *twin sets*. Note that a twin set induces a complete subgraph but a twin set is not necessarily a clique of G.

Let $\beta(G)$ be the size of a largest twin set. Clearly, any twin set — in particular, a largest one — requires distinct colours for each of its vertices in order to give a biclique-colouring.

Lemma 6. *The star-biclique-chromatic number of a graph G is at least $\beta(G)$.*

Proof. Consider a biclique-colouring for a graph G. Each edge of a twin set defines a biclique. Then, every edge of a twin set requires distinct colours for its extremities and every twin set T requires $|T|$ colours. In particular, a largest twin set requires $\beta(G)$ colours. Therefore, any biclique-colouring requires $\beta(G)$ colours, in particular the ones that are star-biclique-colouring. □

Let $\mathcal{C}(B)$ be the set of 1-cutsets of a graph G that are in a biconnected component B of G. Let $\overline{\mathcal{C}}(B) = V(B) \setminus \mathcal{C}(B)$. We determine precisely all twin sets of a unichord-free graph in the following lemma. We omit the proof due to its simplicity.

Lemma 7. *Let G be a unichord-free graph. Hence, the twin sets are precisely*

- *$\overline{\mathcal{C}}(B)$, for every complete biconnected component B;*
- *$\{v\}$, if v is a 1-cutset of G or a vertex of a triangle-free biconnected component.*

An example of the twin sets of a unichord-free graph is illustrated in Fig. 9. The size of the largest twin sets of Fig. 9 is 2.

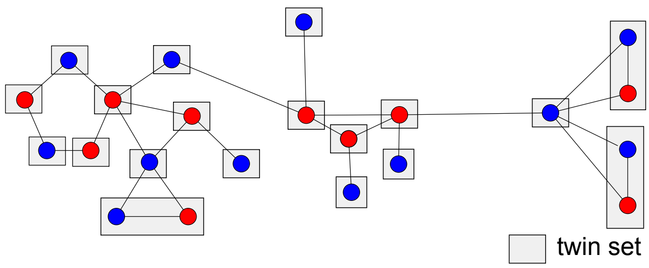


Fig. 9. Twin sets of a unichord-free graph G, with $\beta(G) = 2$ and $\kappa_B(G) = 2$

We prove that the star-biclique-chromatic number of a unichord-free graph G is at most $\beta(G)+1$. We first show that every non-biconnected unichord-free prime graph G has a $(\beta(G)+1)$-star-biclique-colouring. Second, we consider that G has a decomposition via type $\mathcal{S}^*$ biconnected component B^*. We assign a $(\beta(G)+1)$-star-biclique-colouring to the prime graph G_1. We modify G_2 to $\widetilde{G_2}$ by shrinking B^* to $\widetilde{B}$ isomorphic to a $K_{|V(B^*)\setminus\mathcal{C}(B^*)|+1}$, if B^* is complete, or to a K_2, if B^* is triangle-free. Note that v^*, the unique vertex in $\mathcal{C}_{\overline{\mathcal{F}}}(B^*)$, belongs to $\tilde{B}$. We assign recursively a $(\beta(\widetilde{G_2}) + 1)$-star-biclique-colouring to $\widetilde{G_2}$. Finally, we combine the colourings of G_1 and of $\widetilde{G_2}$ to obtain a $(\beta(G) + 1)$-star-biclique-colouring for G. This concludes the general idea of the proof of Theorem 8.

Theorem 8. *The star-biclique-chromatic number of a unichord-free graph G is at most $\beta(G)+1$.*

Proof. Suppose that G is a biconnected unichord-free graph. By Section 4.1, if G is complete, then $\kappa_{SB}(G) = |V(G)| = \beta(G)$, and if G is triangle-free, then $\kappa_{SB}(G) = 2 = \beta(G)+1$.

Now, consider that G is a non-biconnected unichord-free graph. We prove that the star-biclique-chromatic number of G is at most $\beta(G)+1$ by induction on the number of biconnected components of G. Let B^* be a type $\mathcal{S}^*$ biconnected component of G.

Suppose that G is a prime graph. Hence, $\mathcal{C}(B^*) = \mathcal{C}_{\mathcal{F}}(B^*)$. Let $F \in \bigcup_{v\in\mathcal{C}(B^*)} \Gamma_G(B^*,v)$. If F is complete, then we assign a $|V(F)|$-colouring to F, as follows. Assign colours $2,\ldots,|V(F)|$ to the vertices in $\overline{\mathcal{C}}(F)$ and assign colour 1 to the vertex of $\mathcal{C}(F)$. If F is triangle-free, we assign a 2-star-biclique-colouring (see Theorem 6) with colours 1 and 2, such that colour 1 is given to the vertex of $\mathcal{C}(F)$. Now, consider biconnected component B^*. If B^* is complete, then we assign a $(|\overline{\mathcal{C}}(B^*)|)$-colouring to B^*, as follows. We assign colours $1,\ldots,|\overline{\mathcal{C}}(B^*)|$ to the vertices of $\overline{\mathcal{C}}(B^*)$. If B^* is triangle-free, we assign a 2-star-biclique-colouring (see Theorem 6) with colours 1 and 2. For each $v\in\mathcal{C}(B^*)$, proceed as follows. If the colour assigned to v in B^* is 1, we are done. Otherwise, the colour assigned to v in B^* is 2 and we need to swap colours 1 and 2 in every $F\in\Gamma_G(B^*,v)$. Clearly, the given colouring assigns at most $\beta(G)+1$ colours and it is a star-biclique-colouring of G.

Now, consider that G is not a prime graph. Recall that a triangle-free biconnected component is 2-star-biclique-colourable. Hence, we use at least 2 colours in B^* in order to have a $(\beta(G)+1)$-star-biclique-colouring of G. Now, consider that B^* is complete. Notice that every pair of vertices in $\overline{\mathcal{C}}(B^*)$ induces a biclique of G. Then, all vertices in $\overline{\mathcal{C}}(B^*)$ should have distinct colours. Let v be a vertex of B^*, such that $v\in\mathcal{C}_{\mathcal{F}}(B^*)$. One can check that we can colour all biconnected components in $\Gamma_G(B^*,v)$ with at most $\beta(G)+1$ colours in order to make all stars and bicliques in $\Gamma_G(B^*,v)$ polychromatic. Then, v may have the same colour as that assigned to a vertex in $\overline{\mathcal{C}}(B^*)$. Moreover, all vertices of $\mathcal{C}_{\mathcal{F}}(B^*)$ may have the same colour. Hence, we use at least $|\overline{\mathcal{C}}(B^*)|$ colours in B^* in order to have a $(\beta(G)+1)$-star-biclique-colouring of G. In the light of the least number of colours we use in B^*, we modify G_2 to $\widetilde{G_2}$ by shrinking B^* to $\widetilde{B}$ isomorphic to K_2, if B^* is triangle-free, or isomorphic to $K_{|\overline{\mathcal{C}}(B^*)|+1}$, if B^* is complete, and we apply induction hyphotesis on $\widetilde{G_2}$. Consider a $(\beta(\widetilde{G_2})+1)$-star-biclique-colouring of $\widetilde{G_2}$. Clearly, $\beta(\widetilde{G_2})\le\beta(G)$. Let i be the colour assigned to the unique vertex v^* of $\mathcal{C}_{\overline{\mathcal{F}}}(B^*)$ in $\widetilde{G_2}$. Without loss of generality, consider $i = |\overline{\mathcal{C}}(B^*)|+1$. We extend the colouring of $\widetilde{G_2}$ to G, as follows. Let $F\in\bigcup_{v\in\mathcal{C}_{\mathcal{F}}(B^*)}\Gamma_G(B^*,v)$. If F is complete, then we assign a $|V(F)|$-colouring to F, as follows. Assign colours $2,\ldots,|V(F)|$ to the vertices in $\overline{\mathcal{C}}(F)$ and assign colour 1 to the vertex of $\mathcal{C}(F)$. If F is triangle-free, we assign a 2-star-biclique-colouring (see Theorem 6) with colours 1 and 2, such that colour 1 is assigned to the unique vertex of $\mathcal{C}(F)$. Now, consider biconnected component B^*. If B^* is complete, then we assign a $(|\overline{\mathcal{C}}(B^*)|+1)$-colouring to B^*, as follows. We assign colours $1, \ldots, |\overline{\mathcal{C}}(B^*)|$ to the vertices of $\overline{\mathcal{C}}(B^*)$. If B^* is triangle-free, we assign a 2-star-biclique-colouring (see Theorem 6) with colours 1 and i, such that colour i is assigned to the unique vertex v^* of $\mathcal{C}_{\overline{\mathcal{F}}}(B^*)$. For each $v \in \mathcal{C}_{\mathcal{F}}(B^*)$, proceed as follows. If the colour assigned to v in B^* is 1, we are done. Otherwise, the colour assigned to v in B^* is i and we need to swap colours 1 and i in every $F \in \Gamma_G(B^*, v)$. Clearly, the given colouring assigns at most $\beta(G)+1$ colours and it is a star-biclique-colouring of G. □

See Fig. 10 for an example of the $(\beta(G)+1)$-star-biclique-colouring algorithm given by the constructive proof of Theorem 8.

Notice that Theorem 8 yields an optimal star-biclique-colouring algorithm for any unichord-free graph G that is not $\beta(G)$-star-biclique-colourable — in particular, when $\beta(G)=1$. From now on, we consider $\beta(G) \geq 2$.

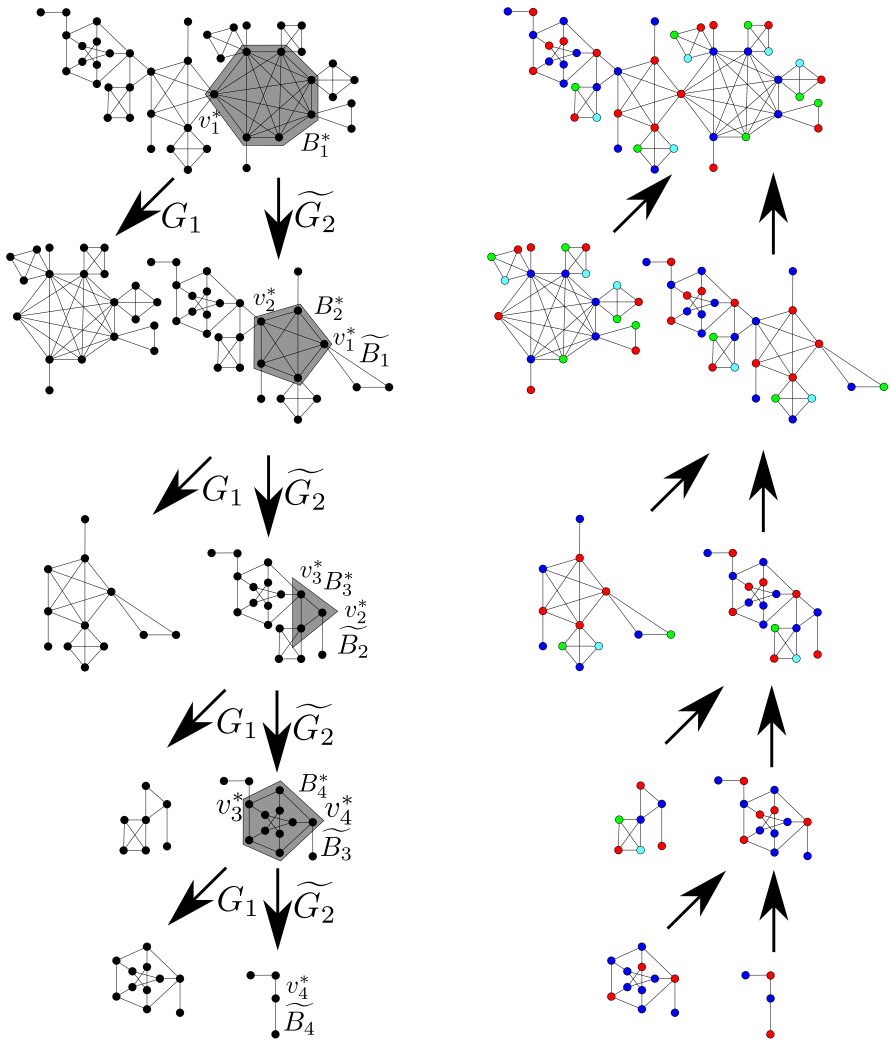
Optimal star-biclique-colouring algorithm In the previous section we showed how to star-biclique-colour a graph G using $\beta(G)+1$ colours. Now, we are going to try our best to use, when possible, only $\beta(G)$ colours, which is the lower bound for the biclique-chromatic number. Our strategy to optimally star-biclique-colour a graph G is to determine bounds on the number of colours needed in each biconnected component of G. More precisely, we look at the type $\mathcal{S}^*$ biconnected components looking for a certificate that a graph is **not** $\beta(G)$-star-biclique-colourable — otherwise, we decompose G and proceed the search in its decomposition blocks. The decomposition blocks are defined in such a way that, if no obstruction is found in any block of the decomposition tree, then the graph is $\beta(G)$-star-biclique-colourable. If a biconnected component is triangle-free, it can be star-biclique-coloured with 2 colours and is not an obstruction for a $\beta(G)$-star-biclique-colouring of G. The case of complete biconnected components is more difficult. As we shall see, a complete type $\mathcal{S}^*$ biconnected component may demand more than $\beta(G)$ colours even having only small (even unitary) twin classes. Let G be a non-biconnected unichord-free graph and B^* be a complete type $\mathcal{S}^*$ biconnected component. Consider the following partition of $\mathcal{C}_{\mathcal{F}}(B^*)$ into two disjoint sets.

- $\mathcal{T}(B^*) = \{v \in \mathcal{C}_{\mathcal{F}}(B^*) \mid \forall B \in \Gamma_G(B^*, v), B \simeq K_{\beta(G)+1}\}$;
- $\overline{\mathcal{T}}(B^*) = \mathcal{C}_{\mathcal{F}}(B^*) \setminus \mathcal{T}(B^*)$.

We have the following property about $\mathcal{T}(B^*)$.

Lemma 8. *Let G be a non-biconnected unichord-free graph, B^* be a complete type $\mathcal{S}^*$ biconnected component, and π be a $\beta(G)$-star-biclique-colouring of G. If $v \in \mathcal{T}(B^*)$, then $\pi(v) \neq \pi(u)$ for every $u \in B^*$, $u \neq v$.*

Proof. Let G be a non-biconnected unichord-free graph and B^* be a complete type $\mathcal{S}^*$ biconnected component. Let B be a complete biconnected component



(a) Graph decomposition (b) Colouring compositions

Fig. 10. $(\beta(G)+1)$-star-biclique-colouring algorithm for a unichord-free graph G

in $\Gamma_G(B^*, w)$, $w \in \mathcal{C}_{\mathcal{F}}(B^*)$, with order $\beta(G)+1$. Let π be a $\beta(G)$-star-biclique-colouring of G. $B \setminus \{w\}$ needs $\beta(G)$ colours because it is a twin set of size $\beta(G)$. So, one colour should be repeated at w. Clearly, if every biconnected component in $\Gamma_G(B^*, w)$ is isomorphic to $K_{\beta(G)+1}$, there exists a monochromatic star in $\Gamma_G(B^*, w)$ with the colour of w. Hence, $\pi(u) \neq \pi(w)$, for every $u \in B^*$, $u \neq w$, and for every $\beta(G)$-star-biclique-colouring π of G. $\square$

Let G be a unichord-free graph with $\kappa_{SB}(G) = \beta(G) \geq 2$. We denote by $f(B)$ the least number of colours used in a subgraph B of G for any $\beta(G)$-star-biclique-colouring of G. Let B^* be a type $\mathcal{S}^*$ biconnected component. The partition of the vertices of B^* into sets $\mathcal{C}_{\overline{\mathcal{F}}}(B^*)$, $\mathcal{T}(B^*)$, $\overline{\mathcal{T}}(B^*)$, and $\overline{\mathcal{C}}(B^*)$ allows us to determine in Lemma 9 bounds for the possible values of $f(B^*)$.

Lemma 9. *Let G be a non-prime $\beta(G)$-star-biclique-colourable non-biconnected unichord-free graph, and B^* be a type $\mathcal{S}^*$ biconnected component.*

- *If B^* is triangle-free, then $f(B^*) \leq 2$;*
- *If B^* is complete, $\overline{\mathcal{T}}(B^*) = \emptyset$, and $\overline{\mathcal{C}}(B^*) = \emptyset$ (implying $\mathcal{T}(B^*) \neq \emptyset$), then*

$$f(B^*) = |\mathcal{T}(B^*)| + 1;$$

- *If B^* is complete, $\overline{\mathcal{T}}(B^*) \neq \emptyset$, and $\overline{\mathcal{C}}(B^*) = \emptyset$, then*

$$|\mathcal{T}(B^*)| + 1 \leq f(B^*) \leq |\mathcal{T}(B^*)| + 2;$$

- *If B^* is complete and $\overline{\mathcal{C}}(B^*) \neq \emptyset$, then*

$$|\mathcal{T}(B^*)| + |\overline{\mathcal{C}}(B^*)| \leq f(B^*) \leq |\mathcal{T}(B^*)| + |\overline{\mathcal{C}}(B^*)| + 1.$$

Proof. If B^* is triangle-free then B^* is 2-star-biclique-colourable and $f(B^*) \leq 2$. Hence we consider that B^* is complete. Consider a $\beta(G)$-star-biclique-colouring of G. Let v be a vertex of B^*. If $v \in \overline{\mathcal{T}}(B^*)$, one can check that we can recolour all biconnected components in $\Gamma_G(B^*, v)$ in order to (1) keep G star-biclique-colourable with the same number of colours and (2) make all stars and bicliques in $\Gamma_G(B^*, v)$ polychromatic. Then, every $u \neq v$ in $\overline{\mathcal{T}}(B^*)$ may have the same colour as that assigned to v. In particular, we may assume that all vertices of $\overline{\mathcal{T}}(B^*)$ have the same colour. By Lemma 8, $f(B^*) \geq |\mathcal{T}(B^*)|+1$ if B^* is complete. Notice that v^* should have a distinct colour from those colours assigned to the vertices of $\mathcal{T}(B^*)$. Then, $f(B^*) = |\mathcal{T}(B^*)| + 1$ if B^* is complete, $\overline{\mathcal{T}}(B^*) = \emptyset$, and $\overline{\mathcal{C}}(B^*) = \emptyset$. On the other hand, notice that v^* may not have the same colour assigned to a vertex in $\overline{\mathcal{T}}(B^*)$. Then, $f(B^*) \leq |\mathcal{T}(B^*)| + 2$ if B^* is complete, $\overline{\mathcal{T}}(B^*) \neq \emptyset$, and $\overline{\mathcal{C}}(B^*) = \emptyset$. Now, notice that every pair of vertices in $\overline{\mathcal{C}}(B^*)$ induces a biclique of G. Then, all vertices in $\overline{\mathcal{C}}(B^*)$ have distinct colours. On the other hand, all vertices of $\overline{\mathcal{T}}(B^*)$ may have the same colour of a vertex in $\overline{\mathcal{C}}(B^*)$. Then, $f(B^*) \geq |\mathcal{T}(B^*)| + |\overline{\mathcal{C}}(B^*)|$ if B^* is complete and $\overline{\mathcal{C}}(B^*) \neq \emptyset$. Finally, notice that v^* may not have the same colour assigned to a vertex in $\overline{\mathcal{C}}(B^*)$. Then, $f(B^*) \leq |\mathcal{T}(B^*)| + |\overline{\mathcal{C}}(B^*)| + 1$ if B^* is complete and $\overline{\mathcal{C}}(B^*) \neq \emptyset$. □

Lemma 9 provides an easy necessary condition for a graph G to be $\beta(G)$-star-biclique-colourable, namely, that for any type $\mathcal{S}^*$ biconnected component B^*, the known lower bound for $f(B^*)$ does not exceed $\beta(G)$. For that reason, it is useful to define a new function $g(\cdot)$ that returns the lower bound for B^* in terms of $\mathcal{T}(B^*)$, $\overline{\mathcal{T}}(B^*)$, and $\overline{\mathcal{C}}(B^*)$:

- $g(B^*) = 1$, if B^* is triangle-free.

- $g(B^*) = |\mathcal{T}(B^*)| + 1$, if B^* is complete and $\overline{\mathcal{C}}(B^*) = \emptyset$.
- $g(B^*) = |\mathcal{T}(B^*)| + |\overline{\mathcal{C}}(B^*)|$, if B^* is complete and $\overline{\mathcal{C}}(B^*) \neq \emptyset$.

Note that $g(B^*)$ describes the exact number of colours that is needed in the vertices of $B^* \setminus v^*$, where v^* denotes the unique vertex in $\mathcal{C}_{\overline{\mathcal{F}}}(B^*)$. The range of two possible values for $f(B^*)$ in some of the cases of Lemma 9 is due to the fact that one cannot determine, only by looking to B^*, whether the vertex v^* demands a new colour unused in $B^* \setminus v^*$. In other words, the colour of v^* could possibly have the same colour as some vertex in $\overline{\mathcal{C}}(B^*)$ or in $\overline{\mathcal{T}}(B^*)$, as long as this does not create in G any monochromatic star centered in v^* — and this information will be given as soon as the recursion call returns.

In the case of prime graphs the inexistence of $v^* \in \mathcal{C}_{\overline{\mathcal{F}}}(B^*)$ allows the precise determination of $f(B^*)$ and the characterization of $\beta(G)$-star-biclique-colourable graphs. The proof is omitted due to its similarity with the proof of Lemma 9.

Lemma 10. *Let G be a prime non-biconnected unichord-free graph with $\beta(G) \geq 2$ and let B^* be a type $\mathcal{S}^*$ biconnected component of G. Graph G is $\beta(G)$-star-biclique-colourable if, and only if, (at least) one of the following condition holds.*

- *B^* is triangle-free;*
- *B^* is complete, $\overline{\mathcal{T}}(B^*) = \emptyset$, and $\overline{\mathcal{C}}(B^*) = \emptyset$ (implying $\mathcal{T}(B^*) \neq \emptyset$), and*
$$|\mathcal{T}(B^*)| \leq \beta(G);$$
- *If B^* is complete, $\overline{\mathcal{T}}(B^*) \neq \emptyset$, and $\overline{\mathcal{C}}(B^*) = \emptyset$, and*
$$|\mathcal{T}(B^*)| + 1 \leq \beta(G);$$
- *If B^* is complete and $\overline{\mathcal{C}}(B^*) \neq \emptyset$, and*
$$|\mathcal{T}(B^*)| + |\overline{\mathcal{C}}(B^*)| \leq \beta(G).$$

Consider that G has a decomposition via type $\mathcal{S}^*$ biconnected component B^*. In the light of Lemma 9, we modify G_2 to $\widehat{G_2}$ by possibly shrinking B^* to $\widehat{B}$ isomorphic to

- K_2 if B^* is triangle-free,
- $K_{|\mathcal{T}(B^*)|+1} = B^*$ if B^* is complete, $\overline{\mathcal{T}}(B^*) = \emptyset$, and $\overline{\mathcal{C}}(B^*) = \emptyset$,
- $K_{|\mathcal{T}(B^*)|+2}$ if B^* is complete, $\overline{\mathcal{T}}(B^*) \neq \emptyset$, and $\overline{\mathcal{C}}(B^*) = \emptyset$, or
- $K_{|\mathcal{T}(B^*)|+|\overline{\mathcal{C}}(B^*)|+1}$ if B^* is complete and $\overline{\mathcal{C}}(B^*) \neq \emptyset$.

Note that v^*, the unique vertex in $\mathcal{C}_{\overline{\mathcal{F}}}(B^*)$, belongs to $\widehat{B}$, and note that $\kappa_{SB}(\widehat{G_2}) \leq \kappa_{SB}(G)$. The following characterization of non-prime non-biconnected unichord-free graphs that are $\beta(G)$-star-biclique-colourable is the missing step to the proposed optimal star-biclique-colouring algorithm.

Theorem 9. *Let G be a non-prime non-biconnected unichord-free graph with $\beta(G) \geq 2$, B^* be a type $\mathcal{S}^*$ biconnected component and $\widehat{G_2}$ obtained by the decomposition via B^*. Graph G is $\beta(G)$-star-biclique-colourable if, and only if, $g(B^*) \leq \beta(G)$ and $\kappa_{SB}(\widehat{G_2}) \leq \beta(G)$.*

Proof. First, we prove the necessary condition. Notice that $g(B^*)$ determines a lower bound for the number of colours that appear in B^* in any $\beta(G)$-star-biclique-colouring of G. Then, G cannot be $\beta(G)$-star-biclique-colourable if $g(B^*) > \beta(G)$. Moreover, notice that a $\beta(G)$-star-biclique-colouring of G easily determines a $\beta(G)$-star-biclique-colouring of $\widehat{G_2}$. Assign to the vertices of $(V(\widehat{G_2}) \setminus V(\widehat{B})) \cup \{v^*\}$ the same colour they receive in G. Then, run a greedy colouring to the vertices of $V(\widehat{B}) \setminus \{v^*\}$.

For the converse, assume that $g(B^*) \leq \beta(G)$ and $\kappa_{SB}(\widehat{G_2}) \leq \beta(G)$. Consider a $\beta(G)$-star-biclique-colouring of $\widehat{G_2}$. We show how to extend this colouring to G. Let i be the colour in $\widehat{G_2}$ assigned to v^*, the unique vertex in $\mathcal{C}_{\overline{\mathcal{F}}}(B^*)$. Assign to each vertex in $(V(G) \setminus V(G_1)) \cup \{v^*\}$ the same colour as in the $\beta(G)$-star-biclique-colouring of $\widehat{G_2}$. Let $F \in \displaystyle\bigcup_{v \in \mathcal{C}_{\mathcal{F}}(B^*)} \mathit{\Gamma}_G(B^*, v)$. If F is complete, then we assign at most $\beta(G)$ colours to F, as follows. If $F \simeq K_{\beta(G)+1}$, a $\beta(G)$-colouring is given by assigning colours $1, \ldots, \beta(G)$ to the vertices of $\overline{\mathcal{C}}(F)$ and colour 1 to the unique vertex of $\mathcal{C}(F)$. If $F \not\simeq K_{\beta(G)+1}$, a $|V(F)|$-colouring is given by assigning colours $2, \ldots, |V(F)|$ to vertices of $\overline{\mathcal{C}}(F)$ and colour 1 to the unique vertex of $\mathcal{C}(F)$. If F is triangle-free, we assign a 2-star-biclique-colouring (see Theorem 6) with colours 1 and 2, such that colour 1 is given to the vertex of $\mathcal{C}(F)$.

Now, consider biconnected component B^*. If B^* is complete, then consider the following colouring.

- Assign colours $1, \ldots, |\mathcal{T}(B^*)|$ to vertices of $\mathcal{T}(B^*)$;
- Assign colour $|\mathcal{T}(B^*)| + 1$ to every vertex in $\overline{\mathcal{T}}(B^*)$;
- Assign colours $|\mathcal{T}(B^*)| + 1, \ldots, |\mathcal{T}(B^*)| + |\overline{\mathcal{C}}(B^*)|$ to vertices of $\overline{\mathcal{C}}(B^*)$;
- If v^* is the unique vertex of $\widehat{B}$ with colour i, then v^* is assigned colour
  - $|\mathcal{T}(B^*)| + 1$, if $\overline{\mathcal{T}}(B^*) = \emptyset$, and $\overline{\mathcal{C}}(B^*) = \emptyset$, or
  - $|\mathcal{T}(B^*)| + 2$, if $\overline{\mathcal{T}}(B^*) \neq \emptyset$, and $\overline{\mathcal{C}}(B^*) = \emptyset$, or
  - $|\mathcal{T}(B^*)| + |\overline{\mathcal{C}}(B^*)| + 1$, if $\overline{\mathcal{C}}(B^*) \neq \emptyset$.

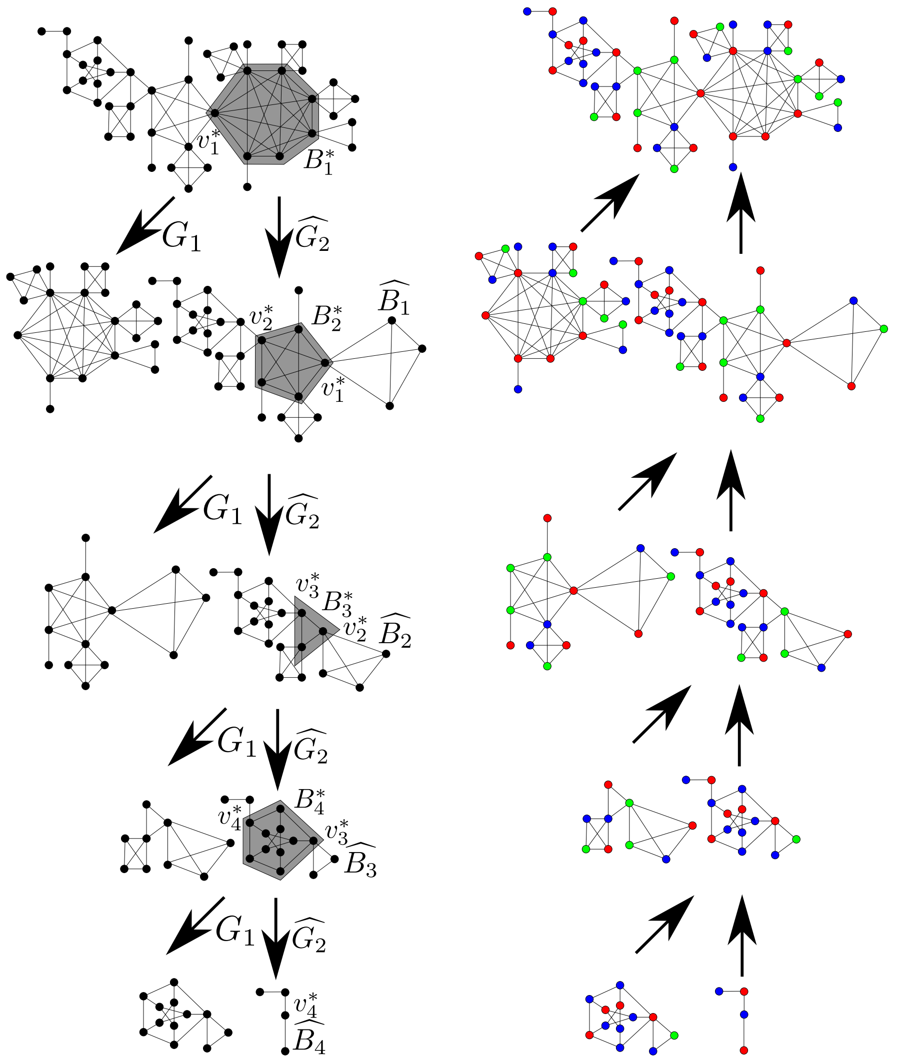
  Otherwise, v^* is assigned colour
  - $|\mathcal{T}(B^*)| + 1$, if $\overline{\mathcal{T}}(B^*) = \emptyset$, and $\overline{\mathcal{C}}(B^*) = \emptyset$, or
  - $|\mathcal{T}(B^*)| + 1$, if $\overline{\mathcal{T}}(B^*) \neq \emptyset$, and $\overline{\mathcal{C}}(B^*) = \emptyset$, or
  - $|\mathcal{T}(B^*)| + |\overline{\mathcal{C}}(B^*)|$, if $\overline{\mathcal{C}}(B^*) \neq \emptyset$.

If B^* is triangle-free, we assign a 2-star-biclique-colouring (see Theorem 6) with colours 1 and 2.

For each $v \in \mathcal{C}_{\mathcal{F}}(B^*)$, proceed as follows. If the colour assigned to v in B^* is 1, we are done. Otherwise, we need to swap colours 1 and the colour assigned to v in B^* in every $F \in \mathit{\Gamma}_G(B^*, v)$. Finally, if the colour j assigned to v^* in B^* is equal to i, we are done. Otherwise, we need to swap colours i and j in every

$$B \in \left(\bigcup_{v \in \mathcal{C}_{\mathcal{F}}(B^*)} \mathit{\Gamma}_G(B^*, v) \right) \cup B^*. \qquad \square$$

See Fig. 11 for an example of the $\beta(G)$-star-biclique-colouring algorithm given by the constructive proof of Theorem 9, where $\kappa_{SB}(G) = \beta(G) = 3$.



(a) Graph decomposition (b) Colouring compositions

Fig. 11. $\beta(G)$-star-biclique-colouring algorithm for a $\beta(G)$-star-biclique-colourable unichord-free graph G

4.3 Algorithmic aspects

Let G be a biconnected unichord-free graph. If G is a complete graph, then an optimal star-biclique-colouring uses $|V(G)|$ colours. Now, we consider G as a biconnected {triangle, unichord}-free graph. An optimal star-biclique-colouring algorithm of G strongly relies on the proper decomposition tree defined in Section 2, where (at least) one block of decomposition of the type 2 non-leaf node is basic.

We construct a proper decomposition tree T_G of the input graph G in time $O(n^2m)$. It is shown by Trotignon and Vušković [23] an $O(nm)$-time algorithm to output a proper decomposition tree of a unichord-free graph. In their algorithm, we replace the $O(n+m)$-time algorithm to find a proper 2-cutset (if any) in a unichord-free graph with no 1-cutset and no proper 1-join by the following $O(nm)$-time algorithm. Consider all possible 2-cutset decompositions of G and pick a proper 2-cutset S that has a block of decomposition B whose size is smallest possible. Machado, Figueiredo, and Vušković [15] showed that B must be basic. All proper 2-cutsets (and its blocks of decomposition orders) can be found in $O(nm)$-time. Indeed, for any vertex v, find all 1-cutsets and all blocks of decomposition of $G \setminus \{v\}$ with depth-first search. For any such block of decomposition, check whether the corresponding 1-cutset u is such that $\{u, v\}$ is a proper 2-cutset. Keep in memory the size of its blocks of decomposition and choose a proper 2-cutset with a block of decomposition of minimum size among them. We now have an algorithm to output a proper decomposition tree such that every proper 2-cutset subtree decomposition is extremal. On the other hand, it raised to $O(n^2m)$ the time-complexity of the algorithm to output such proper decomposition tree, because we replaced an $O(n+m)$-time algorithm to find a proper 2-cutset by an $O(nm)$-time algorithm.

Now, we discuss the time-complexity to combine solutions of the blocks of decomposition of a given node of T_G. We have three cases.

- Non-leaf node H of type 1 of T_G. Lemma 4 shows how to proceed (in constant time) to find a star-biclique-colouring of H by asking recursively for appropriately chosen star-biclique-colouring of its children.
- Non-leaf node H of type 2 of T_G. Lemma 3 shows how to proceed (in constant time) to find a star-biclique-colouring of H in two steps. First, by asking recursively for appropriately chosen star-biclique-colouring of its non-basic child, if exists such non-basic child, or one basic child, otherwise. Second, the given star-biclique-colouring is extended to the other block of decomposition by asking recursively for an appropriately chosen star-biclique-colouring of the non-basic child. Note that since H is of type 2, every non-leaf descendant of H is of type 2. Hence, all leaves under H will have a 2-star-biclique-colouring computed by Lemma 2.
- Leaf node H of T_G. Lemma 2 shows how to proceed in linear-time to find a 2-star-biclique-colouring of H.

So the time-complexity to combine solutions of the blocks of decomposition at each non-leaf node of T_G is $O(1)$. It is proved that T_G is $O(n)$ [23] and the sum of time-complexity to give a 2-star-biclique-colouring at the leaves of T_G is $O(n+m)$, which means that the time-complexity to process the tree is $O(n+m)$.

Notice that the bottleneck of this algorithm is to construct the proper decomposition tree, i.e. the time-complexity is $O(n^2m)$.

Lemma 11. *There exists an $O(n^2m)$-time algorithm to compute a 2-star-biclique-colouring of a {triangle, unichord}-free graph.*

Now, we determine the largest size of a twin set and how to find a $\mathcal{S}^*$ biconnected component. This will be useful to compute the optimal star-biclique-colouring for unichord-free graphs. The proposed algorithms rely on Tarjan's linear-time algorithm to determine all biconnected components of a graph [22]. We add two integers for each vertex v of the input graph. Integer i_1 stores the number $|\Gamma_G(B, v) \cup B|$ of biconnected components containing v and we can easily modify Tarjan's linear-time algorithm to compute it. Integer i_2 stores the number of biconnected components containing v that are type $\mathcal{F}$. We identify every biconnected component that is type $\mathcal{F}$ in the graph searching for those biconnected components that contain exactly one vertex with $i_1 \geq 2$.

We identify every biconnected component that is type $\mathcal{S}^*$ in a unichord-free graph, as follows. Search for the biconnected components that contain exactly one vertex with $i_1 = i_2 \geq 2$, for the biconnected components that are type $\mathcal{F}$, or contain at least one vertex with $i_1 = i_2 + 1 \geq 2$ and at most one vertex with $i_1 \geq i_2 + 2$, for biconnected components that are non-type $\mathcal{F}$.

Lemma 12. *There exists a linear-time algorithm to find all $\mathcal{S}^*$ biconnected component of a unichord-free graph.*

We invite the reader to check that to obtain all $\mathcal{S}^*$ biconnected components of a decomposed unichord-free graph is proportional to the size of the last type $\mathcal{S}^*$ biconnected component decomposition. Hence, the overall-time complexity to find all type $\mathcal{S}^*$ decompositions of a graph is done in linear-time.

We compute the largest size of a twin set of a graph G, as follows. By Lemma 7, we have the following twin sets: each vertex of a triangle-free biconnected component, each vertex with $i_1 \geq 2$, and each maximal subset of vertices with $i_1 = 1$ belonging to the same complete biconnected component. Keep in memory the size of each twin set and choose a largest one. Clearly, it is a linear-time algorithm.

Lemma 13. *There exists a linear-time algorithm to compute the largest size of a twin set of a graph.*

Now, we discuss the constructive proof of Theorem 8, which yields an $O(n^2m)$-time algorithm to compute a $(\beta(G) + 1)$-star-biclique-colouring of a unichord-free graph. We already discussed in Lemma 11 when G is a biconnected component. Now, consider that G is a non-biconnected unichord-free graph. Suppose that G is a prime graph. Theorem 8 yields a linear-time algorithm, if G does not contain a triangle-free biconnected component, or an $O(n^2m)$-time algorithm, otherwise, to compute a $(\beta(G) + 1)$-star-biclique-colouring of G. Now, suppose that G is not a prime graph. We entail a decomposition via type $\mathcal{S}^*$ biconnected components, say B^*. Theorem 8 applies recursion on $\widetilde{G_2}$ to assign a $(\beta(\widetilde{G_2}) + 1)$-star-biclique-colouring, where $\beta(\widetilde{G_2}) \leq \beta(G)$. The proof of Theorem 8 gives an algorithm to extend the star-biclique-colouring of $\widetilde{G_2}$ to G, which is a $(\beta(G)+1)$-star-biclique-colouring of G_1. Graphs G_1 and $\widetilde{G_2}$ have at least K_2 and at most B^* in common. Hence, we have

- $|V(G_1)| + |V(\widetilde{G_2})| - |V(B^*)| \leq n \leq |V(G_1)| + |V(\widetilde{G_2})| - 2$ and
- $|E(G_1)| + |E(\widetilde{G_2})| - 1 \leq m \leq |E(G_1)| + |E(\widetilde{G_2})| - |E(B^*)|$.

It follows that the overall time-complexity to give a $(\beta(G)+1)$-star-biclique-colouring of G_1 and of G_2 is $O(n^2m)$.

Lemma 14. *There exists an $O(n^2m)$-time algorithm to compute a $(\beta(G)+1)$-star-biclique-colouring of a given unichord-free graph G.*

The constructive proof of Theorem 9 yields an $O(n^2m)$-time algorithm to compute a $(\beta(G))$-star-biclique-colouring of a unichord-free graph, if such colouring is possible. The time-complexity case analysis is very similar to that of Lemma 14.

Lemma 15. *There exists an $O(n^2m)$-time algorithm to compute a $\beta(G)$-star-biclique-colouring of a given unichord-free graph G, if such colouring exists.*

In order to check if the unichord-free graph input is indeed $\beta(G)$-star-biclique-colourable, we need to check if $g(B^*)$ is at most $\beta(G)$ and $\widehat{G_2}$ is $\beta(G)$-star-biclique-colourable. If it is not the case, Lemma 14 yields the optimal star-biclique-colouring to G. Otherwise, Lemma 15 yields the optimal star-biclique-colouring to G.

Theorem 10. *There exists an $O(n^2m)$-time algorithm to assign an optimal star-biclique-colouring to a unichord-free graph G.*

5 Conclusion

In our extended abstract presented at LATIN 2012, we showed that the biclique-chromatic number of a unichord-free graph is at most its clique-number. Unfortunately, this upper bound may be very large compared to the actual biclique-chromatic number. Let H_n be the graph on vertices $\{a_1, \ldots, a_{\frac{n}{2}}, b_1, \ldots, b_{\frac{n}{2}}\}$ such that $\{a_1, \ldots, a_{\frac{n}{2}}\}$ is a complete set, $\{b_1, \ldots, b_{\frac{n}{2}}\}$ is a stable set, and such that the only edges between some a_i and some b_j are a_ib_i, for each $1 \leq i \leq \frac{n}{2}$. Since H_n is square-free, every biclique is a star, and H is clearly 2-biclique-colourable, but the clique-number of H_n is $\frac{n}{2}$. In the present work, we strengthen the bounds by showing that the biclique-chromatic number of a unichord-free graph is the increment of or exactly the size of a largest twin set. Note that $\beta(H_n) = 1$.

The graph H_6 is called a *net.* A *block graph* is a graph in which every biconnected component is a clique. Groshaus, Soulignac, and Terlisky [8] gave an optimal star-biclique-colouring algorithm for net-free block graphs. A *cactus graph* is a graph in which every nontrivial biconnected component is a cycle. In our previous extended abstract, we gave an optimal biclique-colouring algorithm for cacti graphs. The class of net-free block graphs is incomparable to the class of cacti graphs. Indeed, a complete graph with four vertices K_4 and a chordless cycle with four vertices C_4 are witnesses. Nevertheless, both classes are unichord-free subclasses. In the present work, we give an optimal biclique-colouring algorithm for unichord-free graphs.

Finally, we show that $\{K_3, unichord\}$-free graphs are polynomial-time 2-star-biclique-colourable. On one hand, the obtained 2-star-biclique-colouring leads to an optimal biclique-colouring polynomial-time algorithm for unichord-free graphs. On the other hand, it is an open problem to determine the biclique-colouring complexity for K_3-free graphs [8]. We remark that Groshaus, Soulignac, and Terlisky [8] gave a polynomial-time 2-star-colouring algorithm for K_3-free graphs.

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Appendix B

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Pós-Graduação e Pesquisa de Engenharia
COPPE UFRJ

Biclique-colouring verification complexity and biclique-colouring power graphs [*], [†], [‡]

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Biclique-colouring verification complexity and biclique-colouring power graphs ☆

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Biclique-colouring

ABSTRACT

Biclique-colouring is a colouring of the vertices of a graph in such a way that no maximal complete bipartite subgraph with at least one edge is monochromatic. We show that it is co$\mathcal{NP}$-complete to check whether a given function that associates a colour to each vertex is a biclique-colouring, a result that justifies the search for structured classes where the biclique-colouring problem could be efficiently solved. We consider biclique-colouring restricted to powers of paths and powers of cycles. We determine the biclique-chromatic number of powers of paths and powers of cycles. The biclique-chromatic number of a power of a path P_n^k is $\max(2k+2-n, 2)$ if $n \geq k+1$ and exactly n otherwise. The biclique-chromatic number of a power of a cycle C_n^k is at most 3 if $n \geq 2k+2$ and exactly n otherwise; we additionally determine the powers of cycles that are 2-biclique-colourable. All proofs are algorithmic and provide polynomial-time biclique-colouring algorithms for graphs in the investigated classes.

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1. Introduction

Let $G = (V, E)$ be a simple graph with order $n = |V|$ vertices and $m = |E|$ edges. A *clique* of G is a maximal set of vertices of size at least 2 that induces a complete subgraph of G. A *biclique* of G is a maximal set of vertices that induces a complete bipartite subgraph of G with at least one edge. A *clique-colouring* of G is a function π that associates a colour to each vertex such that no clique is monochromatic. If the function uses at most c colours we say that π is a c-*clique-colouring*. A *biclique-colouring* of G is a function π that associates a colour to each vertex such that no biclique is monochromatic. If the function π uses at most c colours we say that π is a c-*biclique-colouring*. The *clique-chromatic number* of G, denoted by $\kappa(G)$, is the least c for which G has a c-clique-colouring. The *biclique-chromatic number* of G, denoted by $\kappa_B(G)$, is the least c for which G has a c-biclique-colouring.

Both clique-colouring and biclique-colouring have a "hypergraph colouring version". Recall that a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is an ordered pair where V is a set of vertices and $\mathcal{E}$ is a set of hyperedges, each of which is a set of vertices. A colouring

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of hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a function that associates a colour to each vertex such that no hyperedge is monochromatic. Let $G = (V, E)$ be a graph and let $\mathcal{H}_C(G) = (V, \mathcal{E}_C)$ and $\mathcal{H}_B(G) = (V, \mathcal{E}_B)$ be the hypergraphs in which hyperedges are, respectively, $\mathcal{E}_C = \{K \subseteq V \mid K$ is a clique of $G\}$ and $\mathcal{E}_B = \{K \subseteq V \mid K$ is a biclique of $G\}$—hypergraphs $\mathcal{H}_C(G)$ and $\mathcal{H}_B(G)$ are called, resp., the *clique-hypergraph* and the *biclique-hypergraph* of G. A clique-colouring of G is a colouring of its clique-hypergraph $\mathcal{H}_C(G)$; a biclique-colouring of G is a colouring of its biclique-hypergraph $\mathcal{H}_B(G)$.

Clique-colouring and biclique-colouring are analogous problems in the sense that they refer to the colouring of hypergraphs arising from graphs. In particular, the hyperedges are subsets of vertices that are clique (resp. biclique). The clique is a classical important structure in graphs, hence it is natural that the clique-colouring problem has been studied for a long time—see [1,13,21,25]. It is co $\mathcal{NP}$-complete [1] to check whether a given function that associates a colour to each vertex is a clique-colouring, and it is Σ_2^P-complete [25] to determine whether a graph is k-clique-colourable, for $k \geq 2$. The biclique-colouring problem, on the other hand, only recently started to be investigated by Terlisky et al. [18], who determined the complexity of the problem – it is Σ_2^P-complete to determine whether a graph is k-biclique-colourable, for $k \geq 2$ – and studied the problem in several graph classes, including H-free graphs for almost all H on three vertices, graphs with restricted diamonds, split graphs, threshold graphs, and net-free block graphs.

Many other problems, initially stated for cliques, have their version for bicliques [3,20], such as *Ramsey number* and *Turán's theorem*. The combinatorial game called on-line Ramsey number also has a version for bicliques [12]. Although complexity results for complete bipartite subgraph problems are mentioned in [16] and the (maximum) biclique problem is shown to be $\mathcal{NP}$-hard in [32], only in the last decade the (maximal) bicliques were rediscovered in the context of counting problems [17,28], enumeration problems [14,27], and intersection graphs [19].

Clique-colouring and biclique-colouring have similarities with usual vertex-colouring. A proper vertex-colouring is also a clique-colouring and a biclique-colouring—in other words, both the clique-chromatic number and the biclique-chromatic number are bounded above by the vertex-chromatic number. Optimal vertex-colourings and clique-colourings coincide in the case of K_3-free graphs, while optimal vertex-colourings and biclique-colourings coincide in the (much more restricted) case of $K_{1,2}$-free graphs—notice that the triangle K_3 is the minimal complete graph that includes the graph induced by one edge (K_2), while the $K_{1,2}$ is the minimal complete bipartite graph that includes the graph induced by one edge ($K_{1,1}$). But there are also essential differences. Most remarkably, it is possible that a graph has a clique-colouring (resp. biclique-colouring), which is not a clique-colouring (resp. biclique-colouring) when restricted to one of its subgraphs. Subgraphs may even have a larger clique-chromatic number (resp. biclique-chromatic number) than the original graph.

Clique-colouring and biclique-colouring also have similarities on complexity issues. It is known that it is co $\mathcal{NP}$-complete to check whether a given function that associates a colour to each vertex is a clique-colouring by a reduction from 3DM [1]. Later, an alternative co $\mathcal{NP}$-completeness proof was obtained by a reduction from a variation of 3SAT, in order to construct the complement of a bipartite graph [13]. Based on the latter reduction, we open this paper providing a corresponding result regarding the biclique-colouring problem: it is co $\mathcal{NP}$-complete to check whether a given function that associates a colour to each vertex is a biclique-colouring. The co $\mathcal{NP}$-completeness holds even when the input is a $\{C_4, K_4\}$-free graph.

We select two structured classes for which we provide linear-time biclique-colouring algorithms: powers of paths and powers of cycles. The choice of those classes has also a strong motivation since they have been recently investigated in the context of well studied variations of colouring problems. For instance, for a power of a path P_n^k, its b-chromatic number is n, if $n \leq k+1$; $k+1+\lfloor \frac{n-k-1}{3} \rfloor$, if $k+2 \leq n \leq 4k+1$; or $2k+1$, if $n \geq 4k+2$; whereas, for a power of a cycle C_n^k, its b-chromatic number is n, if $n \leq 2k+1$; $k+1$, if $n = 2k+2$; at least $\min(n-k-1, k+1+\lfloor \frac{n-k-1}{3} \rfloor)$, if $2k+3 \leq n \leq 3k$; $k+1+\lfloor \frac{n-k-1}{3} \rfloor$, if $3k+1 \leq n \leq 4k$; or $2k+1$, if $n \geq 4k+1$ [15]. Moreover, other well studied variations of colouring problems when restricted to powers of cycles have been investigated: chromatic index [26], total chromatic number [9], choice number [29], and clique-chromatic number [8]. It is known, for a power of a cycle C_n^k, that the chromatic number and the choice number are both $k+1+\lceil r/q \rceil$, where $n = q(k+1)+t$ with $q \geq 1$, $0 \leq t \leq k$ and $n \geq 2k+1$, that the chromatic index is the maximum degree of C_n^k if, and only if, n is even, that the total chromatic number is at most the maximum degree of C_n^k plus 2, when n is even and $n \geq 2k+1$, and that the clique-chromatic number is 3 for odd cycles with size at least 5 and, otherwise, it is 2. Note that total colouring is an open and difficult problem and remains unsolved for powers of cycles [9]. Other significant works have been done in power graphs [7,10]. In particular, works in powers of paths and powers of cycles [5,6,22–24,31].

2. Complexity of biclique-colouring

The biclique-colouring problem is a variation of the clique-colouring problem. Hence, it is natural to investigate the complexity of biclique-colouring based on the tools that were developed to determine the complexity of clique-colouring. As in the case of clique-colouring, it is known that it is Σ_2^P-complete to determine whether a graph is k-biclique-colourable, for $k \geq 2$ [18]. The complexities of clique-choosability and biclique-choosability are even "higher" in the polynomial hierarchy: it is Π_3^P-complete to determine if a graph is k-clique-choosable [25] (resp. k-biclique-choosable [18]), for $k \geq 2$. Despite the existence of several complexity results for variations of clique-colouring and biclique-colouring problems, the "natural" verification problem has been investigated only in the context of clique-colouring, leaving a gap in the table of complexity results for these problems (see Table 1).

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Table 1
The computational complexity of clique-colouring and biclique-colouring problems.

Problem	Verification	k-colouring	k-choosability
Clique-colouring	co $\mathcal{NPC}$ [1,13]	$\Sigma_2^p\mathcal{C}$ [25]	$\Pi_3^p\mathcal{C}$ [25]
Biclique-colouring	co $\mathcal{NPC}$	$\Sigma_2^p\mathcal{C}$ [18]	$\Pi_3^p\mathcal{C}$ [18]

It is worth mentioning two important aspects about the motivation for investigating the complexity of biclique-colouring verification problem.[1] First, the fact that k-biclique-colouring, for $k \geq 2$, is Σ_2^P-complete [18] is a strong evidence that biclique-colouring verification is co $\mathcal{NP}$-complete—it holds unless the Polynomial hierarchy collapses. However, it was still open the question of whether to check the "natural certificate" for the biclique-colouring problem is co $\mathcal{NP}$-complete. Second, it is possible to modify the Σ_2^P-completeness proof of Terlisky et al. [18] so as to obtain a proof of the co $\mathcal{NP}$-completeness of the biclique verification problem. Nevertheless, we believe that the present proof is much simpler, inspired by Defossez' construction [13] to reduce 3-SAT to 2-clique-colouring, and explicitly describes the complexity of the biclique verification problem.

In the present section, we give a proof of the co $\mathcal{NP}$-completeness of the biclique verification problem. In order to achieve a result in this direction, we prove the $\mathcal{NP}$-completeness of the following problem: of deciding whether there exists a biclique of a graph G contained in a given subset of vertices of G. Indeed, a function that associates a colour to each vertex of a given graph G is a biclique-colouring if, and only if, there is *no* biclique of G contained in a subset of the vertices of G associated with the same colour.

We call Biclique Containment the problem that decides whether there exists a biclique of a graph G contained in a given subset of vertices of G.

Problem 2.1. Biclique Containment
Input: *Graph $G = (V, E)$ and $V' \subset V$*
Question: *Is there a biclique B of G such that $B \subseteq V'$?*

In order to show that Biclique Containment is $\mathcal{NP}$-complete, we use in Theorem 1 a reduction from 3SAT problem.

Theorem 1. *The* Biclique Containment *problem is $\mathcal{NP}$-complete, even if the input graph is $\{K_4, C_4\}$-free.*

Proof. Deciding whether a graph has a biclique in a given subset of vertices is in $\mathcal{NP}$: a biclique is a certificate and verifying this certificate is trivially polynomial.

We prove that Biclique Containment problem is $\mathcal{NP}$-hard by reducing 3SAT to it. The proof is outlined as follows. For every formula ϕ, a graph G is constructed with a subset of vertices denoted by V', such that ϕ is satisfiable if, and only if, there exists a biclique B of G such that $B \subseteq V'$.

Let n (resp. m) be the number of variables (resp. clauses) in formula ϕ. We define the graph G as follows.

- For each variable x_i, $1 \leq i \leq n$, there exist two adjacent vertices x_i and $\overline{x_i}$. Let $L = \{x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$.
- For each clause c_j, $1 \leq j \leq m$, there exists a vertex c_j. Moreover, each c_j, $1 \leq j \leq m$, is adjacent to a vertex $l \in \{x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$ if, and only if, the literal corresponding to l is in the clause corresponding to vertex c_j. Let $C = \{c_1, \ldots, c_m\}$.
- There exists a universal vertex u adjacent to all $x_i, \overline{x_i}$, $1 \leq i \leq n$, and to all c_j, $1 \leq j \leq m$.

We define the subset of vertices V' as $\{u, x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$. Refer to Fig. 1 for an example of such construction given a formula $\phi = (x_1 \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge (x_1 \vee x_3 \vee x_5)$.

We claim that formula ϕ is satisfiable if, and only if, there exists a biclique of $G[V']$ that is also a biclique of G.

Each biclique B of $G[V']$ containing vertex u corresponds to a choice of precisely one vertex of $\{x_i, \overline{x_i}\}$, for each $1 \leq i \leq n$, and so B corresponds to a truth assignment v_B that gives true value to variable x_i if, and only if, the corresponding vertex $x_i \in B$.

Notice that we may assume three properties on the 3SAT instance.

- A variable and its negation do not appear in the same clause. Else, any assignment of values (true or false) to such a variable satisfies the clause.
- A variable appears in at least one clause. Else, any assignment of values (true or false) to such a variable is indifferent to formula ϕ.
- Two distinct clauses have at most one literal in common. Else, we can modify the instance as follows. For each clause (l_i, l_j, l_k), we replace it by clauses (l_i, x'_1, x'_2), $(l_j, x'_1, \overline{x'_2})$, $(l_j, \overline{x'_1}, x'_3)$, and $(l_k, \overline{x'_1}, \overline{x'_3})$ with variables x'_1, x'_2, and x'_3. Clearly, the number of variables and clauses created is upper bounded by 7 times the number of clauses in the original instance. Moreover, the original formula is satisfiable if, and only if, the new formula is satisfiable.

1 We thank one of the referees for pointing out the need for this discussion.

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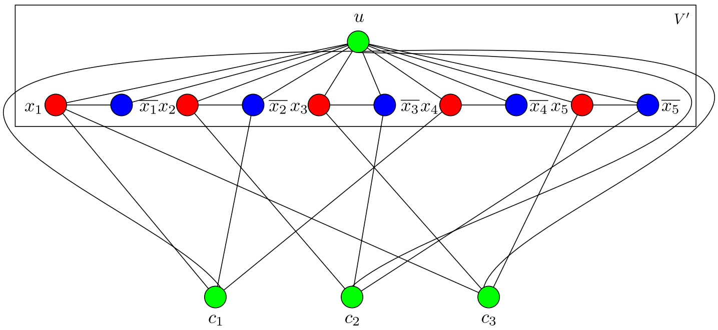

Fig. 1. Example for $\phi = (x_1 \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge (x_1 \vee x_3 \vee x_5)$.

We consider the bicliques of $G[V']$ according to two cases.

1. Biclique B does not contain vertex u. Then, the biclique is precisely formed by a pair of vertices, say x_i and $\overline{x_i}$, where $1 \le i \le n$. Now, our assumption says that there exists a c_j adjacent to one precise vertex in $\{x_i, \overline{x_i}\}$ which implies that B is not a biclique of G.
2. Biclique B contains vertex u. Then, the biclique is precisely formed by vertex u and one vertex of $\{x_i, \overline{x_i}\}$, for each $1 \le i \le n$. B is a biclique of G if, and only if, for each $1 \le j \le m$, there exists a vertex $l \in L \cap B$ such that c_j is adjacent to l, which in turn occurs if, and only if, the truth assignment v_B satisfies ϕ. Therefore, B is a biclique of G if, and only if, v_B satisfies ϕ.

Now, we still have to prove that G is $\{K_4, C_4\}$-free.

For the sake of contradiction, suppose that there exists a K_4 in G, say K. There are no two distinct vertices of C in K, since C is an independent set. There are no three distinct vertices of L in K, since there is a non-edge between two of these three vertices. Hence, K precisely contains vertex u, one vertex of C, and two vertices of L. Since K is a complete set, the two vertices in $L \cap K$ are adjacent and the vertex of $C \cap K$ is adjacent to both vertices of $L \cap K$. This contradicts our assumption that a variable and its negation do not appear in the same clause.

For the sake of contradiction, suppose there exists a C_4 in G, say H. The universal vertex u cannot belong to H. Since C is an independent set, H contains at most two vertices of C. Now, if H contains two vertices of C, then the other two vertices of H must be two literals, which contradicts our assumption that two distinct clauses have at most one literal in common. Since L induces a matching, H is not contained in L. Therefore, H contains one vertex of C and three vertices of L, which by the construction of G gives the final contradiction.

Corollary 2. *Let G be a $\{C_4, K_4\}$-free graph. It is co$\mathcal{NP}$-complete to check if a colouring of the vertices of G is a biclique-colouring.*

3. Powers of cycles and of paths

A *power of a cycle* C_n^k, for $n, k \ge 1$, is a simple graph on n vertices with $V(G) = \{v_0, \dots, v_{n-1}\}$ and $\{v_i, v_j\} \in E(G)$ if, and only if, $\min\{(j-i) \mod n, (i-j) \mod n\} \le k$. Note that C_n^1 is the induced cycle C_n, and C_n^k with $n \le 2k+1$ is the complete graph K_n. In a power of a cycle C_n^k, we take $(v_0, \dots, v_{n-1})$ to be a *cyclic order* on the vertex set and we always perform arithmetic modulo n on vertex indices. The *reach* of an edge $\{v_i, v_j\}$ is $\min\{(i-j) \mod n, (j-i) \mod n\}$. Note that the reach of any edge of a power of a cycle C_n^k is at most k. A *power of a path* P_n^k, for $k \ge 1$, is a simple graph on n vertices with $V(G) = \{v_0, \dots, v_{n-1}\}$ and $\{v_i, v_j\} \in E(G)$ if, and only if, $|i-j| \le k$. Note that P_n^1 is the induced path P_n, and P_n^k with $n \le k+1$ is the complete graph K_n. In a power of a path P_n^k, we take $(v_0, \dots, v_{n-1})$ to be a *linear order* on the vertex set, and the *reach* of an edge $\{v_i, v_j\}$ is $|i-j|$. The definition of reach is extended to an induced path to be the sum of the reach of its edges. Once more, that the reach of any edge of a power of a path P_n^k is at most k. A *block* is a maximal set of consecutive vertices w.r.t. the linear ordering. The *size* of a block is the number of vertices in the block. Note that a power of a path P_n^k can be seen as the subgraph of C_{n+3k}^k, induced by $\{v_0, \dots, v_{n-1}\}$.

We say that a biclique of size 2 is a P_2 biclique and that a biclique of size 3 is a P_3 biclique. We say that vertices x and y are *true twins* if their closed neighbourhood coincide, i.e., $N[x] = N[y]$. In a general graph a pair of vertices is a P_2 biclique if and only if these two vertices are true twins. In particular, every pair of universal vertices defines a P_2 biclique, and every biclique which is not a P_2 biclique contains a P_3 as an induced subgraph.

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We identify next a key property about the bicliques of powers of cycles and of paths.

Claim 3. *Every P_2 biclique of a power of a cycle or of a power of a path is a pair of universal vertices.*

Proof. We prove the contrapositive. Let G be a power of a path P_n^k. Consider a pair of adjacent vertices $v_i, v_j, i < j$, such that v_i and v_j are *not both* universal vertices. Clearly, P_n^k is not a complete graph. Moreover, v_0 is not adjacent to v_j or v_{n-1} is not adjacent to v_i. By symmetry, consider that v_i and v_{n-1} are not adjacent. Vertex v_i is not adjacent to v_{i+k+1} and so $v_{i+k+1} \neq v_j$. Moreover, $j \geq i+1$ and so v_j, v_{i+k+1} is a pair of adjacent vertices. This concludes our proof that $\{v_i, v_j, v_{i+k+1}\}$ is a P_3 biclique.

Now, let G be a power of a cycle C_n^k. Consider a pair of adjacent vertices $v_i, v_j, \ i < j$, such that v_i and v_j are *not both* universal vertices. Clearly, C_n^k is not a complete graph. Vertex v_i is not adjacent to v_{i+k+1} and so $v_{i+k+1} \neq v_j$. Moreover, $j \geq i+1$ and so v_j, v_{i+k+1} is a pair of adjacent vertices. This concludes our proof that $\{v_i, v_j, v_{i+k+1}\}$ is a P_3 biclique.

As a consequence, a non-complete power of a cycle has no P_2 bicliques, as well as a power of a path P_n^k with $n \geq 2k+1$. Considering that powers of cycles and powers of paths are $K_{1,3}$-free, and that powers of paths are C_4-free, we remark that the bicliques of non-complete powers of paths and non-complete powers of cycles are very restricted, as follows.

P_n^k: The bicliques of a non-complete power of a path P_n^k are precisely
- P_2 bicliques and P_3 bicliques, if $k+2 \leq n \leq 2k$; and
- P_3 bicliques if $n \geq 2k+1$.

C_n^k: The bicliques of a non-complete power of a cycle C_n^k are precisely
- C_4 bicliques, if $2k+2 \leq n \leq 3k+1$;
- P_3 bicliques and C_4 bicliques, if $3k+2 \leq n \leq 4k$; and
- P_3 bicliques, if $n \geq 4k+1$.

4. Determining the biclique-chromatic number of powers of paths and powers of cycles

In the present section, we determine the biclique-chromatic number of powers of paths and powers of cycles. Lemma 5 is a key result that is applied to determine the biclique-chromatic number of powers of paths and also provides an upper bound on the biclique-chromatic number of powers of cycles.

We address connections with number theory and the so-called division algorithm is explicitly used in proposed biclique-colouring algorithms. The division algorithm says that any natural number a can be expressed using the equation $a = bq+t$, with a requirement that $0 \leq t < b$. We shall use the following version where b is even and $0 \leq t < 2k$.

Theorem 4 (*Division Algorithm*)**.** *Given two natural numbers n and k, with $n \geq 2k$, there exist unique natural numbers a and t such that $n = ak + t$, $a \geq 2$ is even, and $0 \leq t < 2k$.*

Given a non-complete power of a cycle, Lemma 5 shows that there exists a 3-colouring of its vertices such that no P_3 is monochromatic. As we shall see, Lemma 5 provides an upper bound of 3 for the biclique-chromatic number of a power of a cycle—the proof of Lemma 5 additionally yields an efficient 3-biclique-colouring algorithm using the version of the division algorithm stated in Theorem 4. Moreover, this upper bound of 3 to the biclique-chromatic number is tight. Please refer to Fig. 2 for an example of a graph not 2-biclique-colourable.

Lemma 5. *Let G be a non-complete power of a cycle C_n^k. Then, G admits a* 3*-colouring of its vertices such that*

1. *G has* no *monochromatic P_3.*
2. *v_{k-1} and v_k have distinct colours.*
3. *One colour class is contained in $\{v_{-1}, \ldots, v_{-3k}\}$.*

Proof. Let G be a power of a cycle C_n^k with $n \geq 2k+2$. Theorem 4 says that $n = ak + t$ for natural numbers a and t, $a \geq 2$ is even, and $0 \leq t < 2k$. If $0 \leq t \leq k$, we define $\pi : V(G) \to \{blue, red, green\}$ as follows. Starting at v_0, an even number a of monochromatic-blocks of size k switching colours *red* and *blue* alternately, followed by a monochromatic-block of size t with colour *green*. Otherwise, i.e. $k < t < 2k$, we define $\pi : V(G) \to \{blue, red, green\}$ as follows. An odd number $a-1$ of monochromatic-blocks of size k switching colours *red* and *blue* alternately, followed by a monochromatic-block of size k with colour *green*, a monochromatic-block of size k with colour *blue*, and a monochromatic-block of size $t-k$ with colour *green*. We refer to Fig. 3(a) to illustrate the former 3-biclique-colouring and to Fig. 3(b) to illustrate the latter 3-biclique-colouring.

Consider any three vertices v_i, v_j and v_ℓ with the same colour. Then, either they are in the same monochromatic-block – and induce a triangle – or two of them are not in consecutive monochromatic-blocks – and induce a disconnected graph. In both cases, v_i, v_j and v_ℓ do not induce a P_3.

Finally, notice that v_{k-1} and v_k have distinct colours and all green vertices belong to $\{v_{-1}, \ldots, v_{-3k}\}$.

We then turn to applications of Lemma 5, starting with powers of paths.

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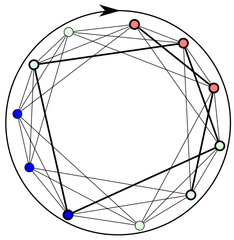
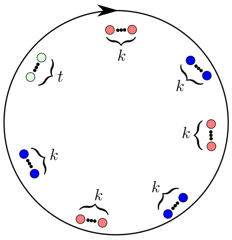
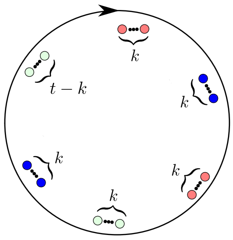


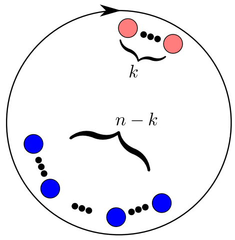
Fig. 2. Power of a cycle C_{11}^3 with biclique-chromatic number 3. We highlight in bold a P_3 biclique of reach 4 and a C_4 biclique. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)



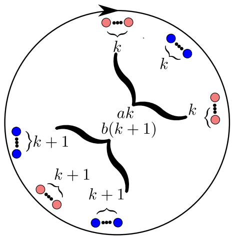
(a) 3-biclique-colouring, when $n \geq 2k+2$ and $0 \leq t \leq k$.



(b) 3-biclique-colouring, when $n \geq 2k+2$ and $k < t < 2k$.



(c) 2-biclique-colouring when $2k+2 \leq n \leq 3k+1$.



(d) 2-biclique-colouring of a 2-biclique-colourable graph, when $n \geq 3k+2$.

Fig. 3. Biclique-colouring of powers of cycles. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

4.1. Determining the biclique-chromatic number of P_n^k

A power of a path P_n^k with $n \leq k+1$ is the complete graph K_n, which implies a biclique-chromatic number of n. Lemma 5 has some interesting consequences on biclique-colouring powers of paths, showing that there exists a 2-colouring of its vertices such that no P_3 is monochromatic.

Lemma 6. *Let G be a non-complete power of a path P_n^k. Then, G admits a 2-colouring of its vertices such that*

1. *G has* no *monochromatic P_3.*
2. *v_{k-1} and v_k have distinct colours.*

Proof. Recall that a power of a path $G = P_n^k$ can be seen as the induced subgraph of $C_{n'}^k$ in which $n' = n + 3k$ and $\{v_0, \ldots, v_{n-1}\}$ are their vertices in common. Hence, by Lemma 5, G admits a 2-colouring – one colour set of Lemma 5 is not used anymore – with no monochromatic induced P_3, and v_{k-1} and v_k still have distinct colours.

As corollary of Lemma 6, we now settle the case of non-complete powers of paths.

Corollary 7. *A non-complete power of a path P_n^k has biclique-chromatic number* $\max(2, 2k+2-n)$.

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Proof. Let G be a power of a path P_n^k with $k+2 \le n \le 2k$. Then, vertices $v_{n-1-k}, \dots, v_k$ are all the $2k+2-n$ universal vertices of G, which implies a biclique-chromatic number of at least $2k+2-n$. Now, we first define $\pi : V(G) \to \{blue, red\}$ as the 2-colouring of Lemma 6, so that all P_3 are polychromatic. Recall that v_{k-1} and v_k have distinct colours. Then, we assign new and distinct colours for vertices $v_{n-1-k}, \dots, v_{k-2}$ that are all the $2k-n$ remaining universal vertices. Clearly, every P_2 biclique and every P_3 are polychromatic, which implies that now G has a $(2k+2-n)$-biclique-colouring.

Now, let G be a power of a path P_n^k with $n \ge 2k+1$. Notice that G has no pair of universal vertices. Hence, by Claim 3, there is no P_2 biclique, and every biclique of G contains a P_3. By Lemma 6, G has a 2-biclique-colouring.

4.2. Determining the biclique-chromatic number of C_n^k

A power of a cycle C_n^k with $n \le 2k+1$ is the complete graph K_n, which implies a biclique-chromatic number of n. On the other hand, a non-complete power of a cycle C_n^k with $n \ge 2k+2$ has no P_2 biclique (see Claim 3) and does have a 3-colouring with no monochromatic P_3 (see Lemma 5), which establishes an upper bound of 3 to its biclique-chromatic number.

As a consequence, every non-complete power of a cycle has biclique-chromatic number 2 or 3, and it is a natural question how to decide between the two values. We now consider two cases of powers of cycles, besides the densest case: the less dense case $n \in [2k+2, 3k+1]$ and the sparse case $n \in [3k+2, \infty)$. We first settle the less dense case $n \in [2k+2, 3k+1]$. We show that all powers of cycles in the less dense case $n \in [2k+2, 3k+1]$ are 2-biclique-colourable. Notice that the proof of Theorem 8 does not rely on Lemma 5, and yields an efficient 2-biclique-colouring algorithm.

Theorem 8. *A power of a cycle C_n^k, when $2k+2 \le n \le 3k+1$, has biclique-chromatic number 2.*

Proof. Let G be a power of a cycle C_n^k with $2k+2 \le n \le 3k+1$. We define $\pi : V(G) \to \{blue, red\}$ as follows. A monochromatic-block of size k with colour *red* followed by a monochromatic-block of size $n-k$ with colour *blue*. We refer to Fig. 3(c) to illustrate the given 2-biclique-colouring.

Notice that every biclique of G is a C_4 biclique. For the sake of contradiction, suppose that there exists a monochromatic set H of four vertices. If H is contained in the block of size k, then H induces a K_4 and cannot be a C_4. Otherwise, H is contained in the block of size $n-k \le 2k+1$ and there exists a subset of H which induces a triangle, so that H cannot be a C_4 biclique.

The sparse case $n \ge 3k+2$ is more tricky. For a power of a cycle in the sparse case, there could exist monochromatic P_3 as long as these induced subgraphs are not bicliques. Nevertheless, we prove that G has biclique-chromatic number 2 if, and only if, there exists a 2-colouring of G such that *no* P_3 is monochromatic, which happens exactly when there exists a 2-colouring of G where every monochromatic-block has size k or $k+1$.

Lemma 9. *Let G be a power of a cycle C_n^k, where $n \ge 2k+2$, and consider a 2-colouring of its vertices. If every monochromatic-block has size k or $k+1$, then G has* no *monochromatic P_3. Otherwise, i.e. if* not *every monochromatic-block has size k or $k+1$, then G has a monochromatic P_3 with reach $k+1$ or $k+2$. In particular, when $n = 3k+2$, G has a monochromatic P_3 with reach $k+1$ or G has a monochromatic C_4.*

Proof. Let G be a power of a cycle C_n^k with $n \ge 2k+2$. Consider a 2-colouring π of the vertices of G such that every monochromatic-block has size k or $k+1$.

Consider any three vertices v_i, v_j and v_ℓ with the same colour. Then, either they are in the same monochromatic-block—and induce a triangle—or two of them have indices that differ by at least $k+1$ with respect to the third vertex—and the three vertices induce a disconnected graph. In both cases, v_i, v_j and v_ℓ do not induce a P_3. Hence, no P_3 is monochromatic.

Now, consider a 2-colouring π of the vertices of G such that there exists a monochromatic-block of size $x \neq k, k+1$. Consider a monochromatic-block of size $p \ge k+2$ with vertices $v_i, v_{i+1}, v_{i+2}, \dots, v_{i+k+1}, \dots$, and v_{i+p-1}. Notice that vertices v_i, v_{i+1}, and v_{i+k+1} induce a P_3. So, we may assume that there exists a monochromatic-block with vertices $v_i, v_{i+1}, v_{i+2}, \dots, v_{i+k+1}, \dots, v_{i+k-x-1}$, where $x > 0$. By symmetry, consider that v_i has blue colour. Notice that vertices v_{i-1} and v_{i+k-x} are adjacent and with red colour. Please refer to Fig. 4. Suppose that vertex v_{i+k} has red colour. Then, vertices v_{i-1}, v_{i+k-x}, and v_{i+k} induce a monochromatic P_3 with reach $k+1$ (see Fig. 4(a)). Now, consider vertex v_{i+k} has blue colour. Suppose that vertex v_{i+k+1} has blue colour, then vertices v_{i+k}, v_{i+k+1}, and v_i induce a monochromatic P_3 with reach $k+1$ (see Fig. 4(b)). Now, consider vertex v_{i+k+1} has red colour and vertices v_{i-1}, v_{i+k-x}, and v_{i+k+1} induce a monochromatic P_3 with reach $k+2$ (see Fig. 4(c)).

Now, consider the case $n = 3k+2$. We know that G has a monochromatic P_3 of reach $k+1$ or $k+2$. In the first case, we are done, so we assume that G has a monochromatic P_3 v_{i-1}, v_{i+k-x}, and v_{i+k+1} of red colour. Moreover, vertex v_i (resp. vertex v_{i+k}) has blue colour, otherwise vertices v_i, v_{i+k-x}, and v_{i+k+1} (resp. vertices v_{i-1}, v_{i+k-x}, and v_{i+k}) would induce a monochromatic P_3 with reach $k+1$. Vertices v_{i-1}, v_{i+k-x}, v_{i+k+1}, and v_{i+2k+1} induce the unique C_4 that includes vertices v_{i-1}, v_{i+k-x}, and v_{i+k+1}. Please refer to Fig. 5. Suppose vertex v_{i+2k+1} has red colour, then vertices v_{i-1}, v_{i+k-x}, v_{i+k+1}, and v_{i+2k+1} induce a monochromatic C_4 (see Fig. 5(a)). Now, consider vertex v_{i+2k+1} has blue colour. Suppose that vertex v_{i+2k} (resp. v_{i+2k+2}) has blue colour, then vertices v_{i+k}, v_{i+2k}, and v_{i+2k+1} (resp. v_{i+2k+1}, v_{i+2k+2}, and v_{i+3k+2}) induce a monochromatic P_3 with reach $k+1$ (see Fig. 5(b)). Now, consider vertices v_{i+2k} and v_{i+2k+2} have red colour. Vertices v_{i+k+1}, v_{i+2k}, and v_{i+2k+2} induce a monochromatic P_3 with reach $k+1$ (see Fig. 5(c)).

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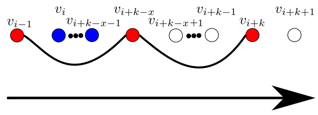
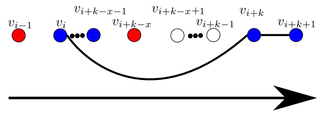
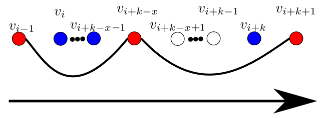


(a) Vertices v_{i-1}, v_{i+k-x}, and v_{i+k} induce a monochromatic P_3 with reach $k+1$.

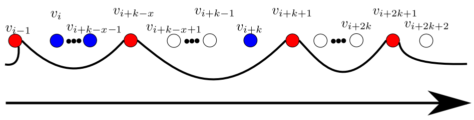


(b) Vertices v_i, v_{i+k}, and v_{i+k+1} induce a monochromatic P_3 with reach $k+1$.

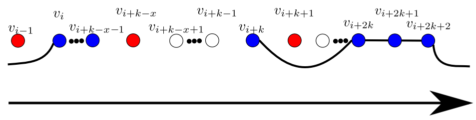


(c) Vertices v_{i-1}, v_{i+k-x}, and v_{i+k+1} induce a monochromatic P_3 with reach $k+2$.

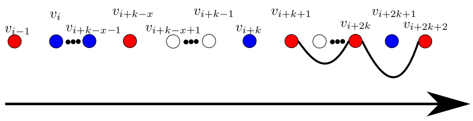
Fig. 4. A monochromatic-block of size $x \neq k, k+1$ in a power of a cycle C_n^k, with $n \geq 2k+2$, implies a monochromatic P_3 with reach $k+1$ or $k+2$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)



(a) Vertices v_{i-1}, v_{i+k-x}, v_{i+k+1}, and v_{i+2k+1} induce a monochromatic C_4.



(b) Vertices v_{i+k}, v_{i+2k}, and v_{i+2k+1} (resp. v_{i+2k+1}, v_{i+2k+2}, and v_i) induce a monochromatic P_3 with reach $k+1$.



(c) Vertices v_{i+k+1}, v_{i+2k}, and v_{i+2k+2} induce a monochromatic P_3 with reach $k+1$.

Fig. 5. A monochromatic-block of size $x \neq k, k+1$ in a power of a cycle C_n^k, with $n = 3k+2$, implies a monochromatic P_3 with reach $k+1$ or a monochromatic C_4. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Theorem 10. *A power of a cycle C_n^k, when $n \geq 3k+2$, has biclique-chromatic number 2 if, and only if, there exist natural numbers a and b, such that $n = ak + b(k+1)$ and $a + b \geq 2$ is even.*

Proof. Let G be a power of a cycle C_n^k with $n \geq 3k+2$. First, consider natural numbers a and b, such that $n = ak + b(k+1)$ and $a + b \geq 2$ is even. Then, there exists a 2-colouring π such that every monochromatic-block has size k or $k+1$. Lemma 9 says that G has *no* monochromatic P_3 and therefore π is a 2-biclique-colouring. We refer to Fig. 3(d) to illustrate such 2-biclique-colouring.

For the converse, suppose that there are no such a and b, which implies that any 2-colouring π' of the vertices of G is such that there exists a monochromatic-block of size $x \neq k, k+1$. Consider $n = 3k+2$. Lemma 9 says that such 2-colouring of the vertices of G has a monochromatic P_3 with reach $k+1$ or a monochromatic C_4. Every P_3 with reach $k+1$ is a biclique and every C_4 is a biclique, which implies that π' is not a 2-biclique-colouring, which is a contradiction. Now, consider $n > 3k+2$. Lemma 9 says that such 2-colouring of the vertices of G has a monochromatic P_3 with reach $k+1$ or $k+2$. Every P_3 with reach $k+1$ or $k+2$ is a P_3 biclique, which implies that π' is not a 2-biclique-colouring, which is a contradiction.

There exists an efficient algorithm that verifies if the system of equations of Theorem 10 has a solution. If so, it also computes values of a and b–the proof of Theorem 11 yields Algorithm 1 to determine if the biclique-chromatic number is 2 or 3 and also computes values of a and b. When the biclique-chromatic number is 2, we define a 2-biclique-colouring $\pi : V(G) \rightarrow \{blue, red\}$ as follows. A number a of monochromatic-blocks of size k plus a number b of monochromatic-blocks of size $k+1$ switching colours *red* and *blue* alternately. We refer to Fig. 3(d) to illustrate the given 2-biclique-colouring.

Theorem 11. *There exists an efficient algorithm that computes the biclique-chromatic number of a power of a cycle C_n^k, when $n \geq 3k+2$.*

Proof. Recall that the biclique-chromatic number of a power of a cycle C_n^k is at most 3 and Theorem 10 states that a power of a cycle C_n^k with $n \geq 3k+2$ has biclique-chromatic number 2 if, and only if, there exist natural numbers a and b, such that $n = ak + b(k+1)$ and $a + b \geq 2$ is even.

Algorithm 1: Biclique-chromatic number of a power of a cycle C_n^k with $n \geq 3k+2$

input : C_n^k, a power of a cycle with $n \geq 3k+2$
output: $\kappa_B(C_n^k)$, the biclique-chromatic number of C_n^k.

begin
  $c \longleftarrow \left\lfloor \frac{n}{k} \right\rfloor$;
  $b \longleftarrow n - ck$;
  if $c \bmod 2 = 0$ *and* $c \geq b$ **then**
    return 2;
  else
    $c \longleftarrow \left\lfloor \frac{n}{k} \right\rfloor - 1$;
    $b \longleftarrow n - ck$;
    if $c \bmod 2 = 0$ *and* $c \geq b$ **then**
      return 2;
    else
      return 3;
end

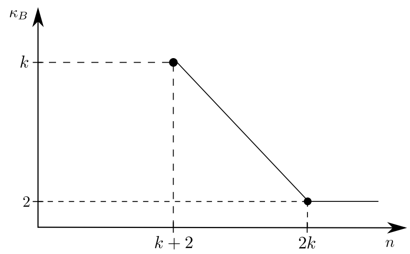


Fig. 6. The biclique-chromatic number of a non-complete power of a path for a fixed value of k and an increasing n.

Let $c = a + b$. We show that there exist natural numbers b and c, such that $n = ck + b$, $b \leq c$, and c is even if, and only if, natural numbers $c_0 = \left\lfloor \frac{n}{k} \right\rfloor$ and $b_0 = n - c_0 k$ have the following properties: c_0 is even and $b_0 \leq c_0$; or natural numbers $c_1 = \left\lfloor \frac{n}{k} \right\rfloor - 1$ and $b_1 = n - c_1 k$ have the following properties: c_1 is even and $b_1 \leq c_1$.

Clearly, b_0 and c_0 (resp. b_1 and c_1) are natural numbers such that $n = c_0 k + b_0$ (resp. $n = c_1 k + b_1$), $b_0 \leq c_0$ (resp. $b_1 \leq c_1$), c_0 (resp. c_1) is even, and $c_0 \geq 2$ (resp. $c_1 \geq 2$) since $n \geq 2k + 2$.

For the converse, suppose that there exist natural numbers a and b, such that $n = ck + b$ and c is even. Let $b' = b$ and $c' = c$. While $b' \geq 2k$, do $c' := c' + 2$ and $b' := b' - 2k$. Clearly, in the end of the loop, we have c' even, $b' \geq 0$, and $c' \geq b'$. Moreover, we consider two cases.

- $b' < k$ in the end of the loop. Then, $c' = \left\lfloor \frac{n}{k} \right\rfloor$ and $b' = n - c'k$.
- $k \leq b' < 2k$ in the end of the loop. Then, $c' = \left\lfloor \frac{n}{k} \right\rfloor - 1$ and $b' = n - c'k$.

As a remark, in Theorem 11, we let $c = a + b$ and rewrite the equation $n = ak + b(k + 1)$ as $n = ck + b$, very similar to the Division Algorithm formula. Nevertheless, there is a rather subtle difference: in the Division Algorithm formula, the choice for the value of the remainder is bounded by the value of the divisor, while in the equation $n = ck + b$, the choice for the value of the remainder is bounded by the choice for the value of the quotient (recall $b \leq c$). This subtle difference may change drastically the behaviour of the equation. More precisely, given two natural numbers n and k, with $n \geq 2k + 2$, it is not necessarily true that there exist natural numbers b and c such that $n = ck + b$, $c \geq 2$ is even, and $b \leq c$. For instance, there do not exist natural numbers b and c such that $11 = 3c + b$, $c \geq 2$ is even, and $b \leq c$.

5. Final considerations

Table 2 highlights the exact values for the biclique-chromatic number of the power graphs settled in this work. In Figs. 6 and 7, we illustrate the biclique-chromatic number for a fixed value of k and an increasing n of powers of paths and powers of cycles, respectively.

As a corollary of Theorem 10, every non-complete power of a cycle C_n^k with $n \geq 2k^2$ has biclique-chromatic number 2. Thus, the biclique-chromatic number of a power of a cycle C_n^k, for a fixed value of k and an increasing $n \geq 3k + 2$, does not oscillate forever.

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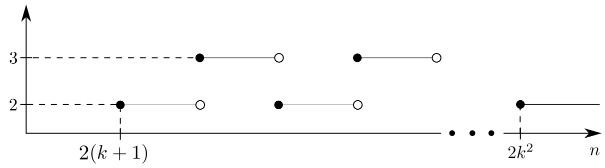

Fig. 7. The biclique-chromatic number of a non-complete power of a cycle for a fixed value of k and an increasing n.

Table 2
Biclique- and star-chromatic numbers of powers of cycles and paths.

Graph G	Range of n	$\kappa_B(G)$	$\kappa_S(G)$
P_n^k	$[1, k+1]$	n	n
	$[k+2, 2k]$	$2k+2-n$	$2k+2-n$
	$[2k+1, \infty[$	2	2
C_n^k	$[1, 2k+1]$	n	n
	$[2k+2, 3k+1]$	2	2, if there exist natural numbers a and b, such that $n = ak + b(k+1)$ and $a+b \geq 2$ is even; 3, otherwise.
	$[3k+2, 2k^2[$	2, if there exist natural numbers a and b, such that $n = ak + b(k+1)$ and $a+b \geq 2$ is even; 3, otherwise.	2, if there exist natural numbers a and b, such that $n = ak + b(k+1)$ and $a+b \geq 2$ is even; 3, otherwise.
	$[2k^2, \infty[$	2	2

Corollary 12. *A non-complete power of a cycle C_n^k with $n \geq 2k^2$ has biclique-chromatic number* 2.

Proof. Theorem 4 says that $n = a'k + t$ for natural numbers a' and t, $a' \geq 2$ is even, and $0 \leq t < 2k$. If we can rewrite $n = ak + b(k+1)$ with natural numbers a and b, such that $a+b \geq 2$ is even, then Theorem 10 says that a power of a cycle C_n^k with $n \geq 2k^2$ has biclique-chromatic number 2. Since $0 \leq t \leq 2k$, $n \geq 2k^2$, and a' is an even natural number, we have

$$n = a'k + t \geq 2k^2$$
$$a'k \geq 2k^2 - 2k + 1$$
$$a' \geq 2k - 1$$
$$a' \geq 2k.$$

Let $a = a' - t$ and $b = t$. Clearly, a and b are natural numbers. Moreover, $a + b \geq 2$ is even.

Groshaus, Soulignac, and Terlisky have recently proposed a related hypergraph colouring, called *star-colouring* [18], defined as follows. A *star* is a maximal set of vertices that induces a complete bipartite graph with a universal vertex and at least one edge. The definition of star-colouring follows the same line as clique-colouring and biclique-colouring: a *star-colouring* of a graph G is a function that associates a colour to each vertex such that no star is monochromatic. The *star-chromatic number* of a graph G, denoted by $\kappa_S(G)$, is the least number of colours c for which G has a star-colouring with at most c colours. Many of the results of biclique-colouring achieved in the present work are naturally extended to star-colouring. Since the constructed graph of Corollary 2 is C_4-free and the bicliques in a C_4-free graph are precisely the stars of the graph, we can restate Corollary 2 as follows.

Corollary 13. *Let G be a $\{C_4, K_4\}$-free graph. It is co$\mathcal{NP}$-complete to check if a colouring of the vertices of G is a star-colouring.*

About star-colouring and the investigated classes of power graphs, we also have some few remarks. On one hand, the bicliques of a power of a path P_n^k are the stars of the graph and, consequently, all results obtained for biclique-colouring powers of paths hold to star-colouring powers of paths. On the other hand, a power of a cycle C_n^k is not necessarily C_4-free, and there are examples of powers of cycles with P_3 stars that are not bicliques due to the fact that such P_3 stars are contained in C_4 bicliques of the graph. This happens for instance in the case $n \in [2k+2, 3k+1]$ and one such example is graph C_{11}^4 exhibited in Fig. 8. Notice that the highlighted vertices form a monochromatic P_3 star, so that the colouring is not a 2-star-colouring. The three highlighted vertices together with vertex u, on the other hand, form a polychromatic C_4 biclique—indeed, the exhibited colouring is a 2-biclique-colouring. We summarize the results about star-colouring powers of paths and powers of cycles in the following theorems and also in Table 2. Please refer to the line of the table where we consider a power of a cycle with $n \in [2k+2, 3k+1]$ to check the difference between the biclique-chromatic number (which is always 2) and the star-chromatic number (which depends on n and k).

Theorem 14. *For any power of a path, the star-chromatic number is equal to the biclique-chromatic number.*

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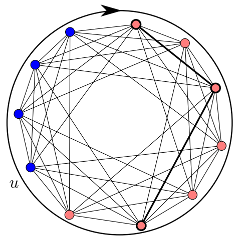

Fig. 8. Power of a cycle C_{11}^4 with a 2-biclique-colouring which is not a 2-star-colouring. Notice that there exists a monochromatic P_3 star highlighted in bold. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Theorem 15. *A power of a cycle C_n^k, when $n \leq 2k+1$ or $n \geq 3k+2$, has star-chromatic number equal to the biclique-chromatic number. If $2k+2 \leq n \leq 3k+1$, then C_n^k has star-chromatic number 2 if, and only if, there exist natural numbers a and b, such that $n = ak + b(k+1)$ and $a + b \geq 2$ is even. If there does not exist such natural numbers, it has star-chromatic number 3.*

Future work

A superclass of powers of paths that is natural to be considered in the context of biclique-colouring is the class of proper interval graphs. We observe that any proper interval graph is an induced subgraph of a power of a path, so that Lemma 6 implies that any proper interval graph with no twins – hence no P_2 biclique – is 2-biclique-colourable. The open question is whether there is an efficient algorithm to optimally biclique-colour proper interval graphs that have twin vertices. It is interesting to note that, while the size $\beta(G)$ of the largest set of mutually true twin vertices is certainly a lower bound for the biclique-chromatic number of any graph G, if one considers a proper interval graph G', it is easy to obtain a biclique-colouring that uses twice that value, so that $\beta(G') \leq \kappa_B(G') \leq 2\beta(G')$. Even then, it does not appear easy to develop optimal biclique-colouring algorithms for proper interval graphs. Indeed, this same scenario occurs for several other colouring problems, such as edge-colouring, total-colouring, and clique-colouring, whose difficulty motivates research in restricted subclasses, such as powers of paths.

A superclass of powers of cycles that appear amenable to the use of techniques developed in the present work is the one of circulant graphs, as we describe in the following. A *distance graph* $P_n(d_1, \ldots, d_k)$ is a simple graph with $V(G) = \{v_0, \ldots, v_{n-1}\}$ and $E(G) = E^{d_1} \cup \cdots \cup E^{d_k}$, such that $\{v_i, v_j\} \in E^{d_\ell}$ if, and only if, it has reach – in the context of a power of a path – d_ℓ. Notice that a distance graph $P_n(d_1, \ldots, d_k)$ is a power of a path if $d_1 = 1$, $d_i = d_{i-1} + 1$, and $d_k < n - 1$. A *circulant graph* $C_n(d_1, \ldots, d_k)$ has the same definition as the distance graph, except by the reach, which, in turn, is in the context of a power of a cycle. Notice that a circulant graph $C_n(d_1, \ldots, d_k)$ is a power of a cycle if $d_1 = 1$, $d_i = d_{i-1} + 1$, and $d_k < \lfloor \frac{n}{2} \rfloor$. Circulant graphs have been proposed for various practical applications [4]. We suggest, as a future work, to biclique colour the classes of distance graphs and circulant graphs, since colouring problems for distance graphs and for circulant graphs have been extensively investigated [2,30,33]. Moreover, some results of intractability have been obtained, e.g. determining the chromatic number of circulant graphs in general is an $\mathcal{NP}$-hard problem [11].

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The authors would like to thank Renan Henrique Finder for the discussions on the algorithm to compute the biclique-chromatic number of a power of a cycle C_n^k, when $n \geq 3k+2$; and to thank Vinícius Gusmão Pereira de Sá and Guilherme Dias da Fonseca for discussions on the complexity of numerical problems. We thank Vanessa Cavalcante for the careful proofreading of earlier versions of this paper. We are deeply indebted to a referee whose careful reading and several insightful suggestions contributed to a much better presentation and understanding of our results.

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Appendix C

Hierarchical complexity of 2-clique-colouring weakly chordal graphs and perfect graphs having cliques of size at least 3 [*], [†], [‡]

Co-authors:

Celina Figueiredo

Raphael Machado

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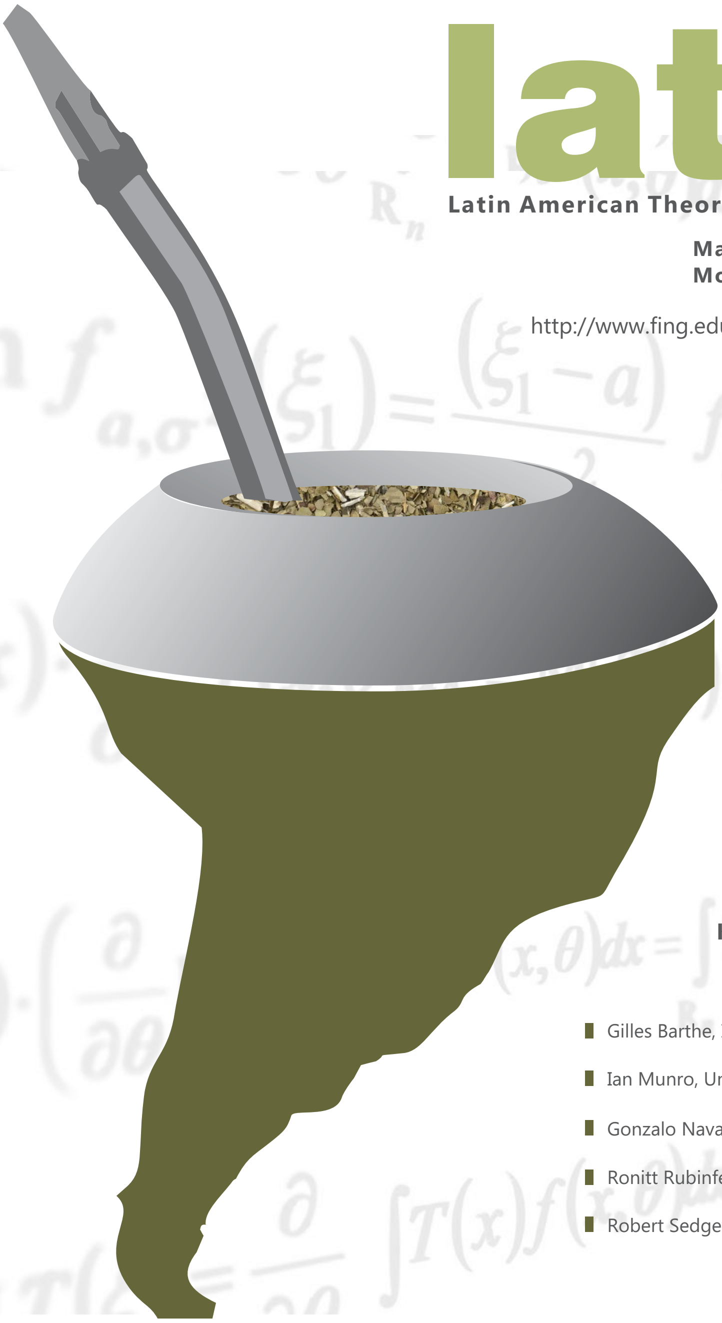
[†] This preprint has been submitted to Theoretical Computer Science.

[‡] Available online: <http://arxiv.org/abs/1312.2086>.



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Hierarchical complexity of 2-clique-colouring weakly chordal graphs and perfect graphs having cliques of size at least 3[☆,☆☆]

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Abstract

A clique of a graph is a maximal set of vertices of size at least 2 that induces a complete graph. A k-clique-colouring of a graph is a colouring of the vertices with at most k colours such that no clique is monochromatic. Défossez proved that the 2-clique-colouring of perfect graphs is a Σ_2^P-complete problem [J. Graph Theory 62 (2009) 139–156]. We strengthen this result by showing that it is still Σ_2^P-complete for weakly chordal graphs. We then determine a hierarchy of nested subclasses of weakly chordal graphs whereby each graph class is in a distinct complexity class, namely Σ_2^P-complete, $\mathcal{NP}$-complete, and $\mathcal{P}$. We solve an open problem posed by Kratochvíl and Tuza to determine the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3 [J. Algorithms 45 (2002), 40–54], proving that it is a Σ_2^P-complete problem. We then determine a hierarchy of nested subclasses of perfect graphs with all cliques having size at least 3 whereby each graph class is in a distinct complexity class, namely Σ_2^P-complete, $\mathcal{NP}$-complete, and $\mathcal{P}$.

Keywords: clique-colouring, hierarchical complexity, perfect graphs, weakly chordal graphs, (α, β)-polar graphs.

1. Introduction

Let $G = (V, E)$ be a simple graph with $n = |V|$ vertices and $m = |E|$ edges. A *clique* of G is a maximal set of vertices of size at least 2 that induces a complete graph. A *k-clique-colouring* of a graph is a colouring of the vertices with at most k colours such that no clique is monochromatic. Any undefined notation concerning complexity classes follows that of Marx [9].

A *cycle* is a sequence of vertices starting and ending at the same vertex, with each two consecutive vertices in the sequence adjacent to each other in the

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graph. A *chord* of a cycle is an edge joining two nodes that are not consecutive in the cycle.

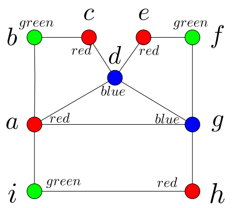
The *clique-number* $\omega(G)$ of a graph G is the number of vertices of a clique with the largest possible size in G. A *perfect graph* is a graph in which every induced subgraph H needs exactly $\omega(H)$ colours in its vertices such that no K_2 (not necessarily clique) is monochromatic. The celebrated *Strong Perfect Graph Theorem* of Chudnovsky et al. [3] says that a graph is perfect if neither it nor its complement contains a chordless cycle with an odd number of vertices greater than 4. A graph is *chordal* if it does not contain a chordless cycle with a number of vertices greater than 3, and a graph is *weakly chordal* if neither it nor its complement contains a chordless cycle with a number of vertices greater than 4.

Both clique-colouring and perfect graphs have attracted much attention due to a conjecture posed by Duffus et al. [5] that *perfect graphs are k-clique-colourable for some constant k*. This conjecture has not yet been proved. Following the chronological order, Kratochvíl and Tuza gave a framework to argue that 2-clique-colouring is $\mathcal{NP}$-hard and proved that 2-clique-colouring is $\mathcal{NP}$-complete for K_4-free perfect graphs [7]. Notice that K_3-free perfect graphs are bipartite graphs, which are clearly 2-clique-colourable. Moreover, 2-clique-colouring is in Σ_2^P, since it is co$\mathcal{NP}$ to check that a colouring of the vertices is a clique-colouring. A few years later, the 2-clique-colouring problem was proved to be a Σ_2^P-complete problem by Marx [9], a major breakthrough in the clique-colouring area. Défossez [4] proved later that 2-clique-colouring of perfect graphs remained a Σ_2^P-complete problem.

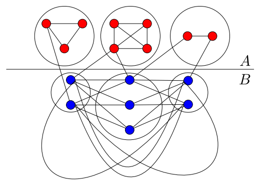
When restricted to chordal graphs, 2-clique-colouring is in $\mathcal{P}$, since all chordal graphs are 2-clique-colourable [11]. Notice that chordal graphs are a subclass of weakly chordal graphs, while perfect graphs are a superclass of weakly chordal graphs. In constrast to chordal graphs, not all weakly chordal graphs are 2-clique-colourable (see Fig. 1a).

We show that 2-clique-colouring of weakly chordal graphs is a Σ_2^P-complete problem, improving the proof of Défossez [4] that 2-clique-colouring is a Σ_2^P-complete problem for perfect graphs. As a remark, Défossez [4] constructed a graph which is not a weakly chordal graph as long as it has chordless cycles with even number of vertices greater than 5 as induced subgraphs. We determine a hierarchy of nested subclasses of weakly chordal graphs whereby each graph class is in a distinct complexity class, namely Σ_2^P-complete, $\mathcal{NP}$-complete, and $\mathcal{P}$.

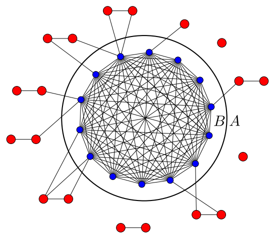
A graph is *(α,β)-polar* if there exists a partition of its vertex set into two sets A and B such that all connected components of the subgraph induced by A and of the complementary subgraph induced by B are complete graphs. Moreover, the order of each connected component of the subgraph induced by A (resp. of the complementary subgraph induced by B) is upper bounded by α (resp. upper bounded by β) [2]. A *satellite* of an (α,β)-polar graph is a connected component of the subgraph induced by A (see Fig. 1b). In this work, we restrict ourselves to the *(α,β)-polar* graphs with $\beta = 1$, so the subgraph induced by B is complete and the order of each satellite is upper bounded by α (see Fig. 1c). Clearly,



(a) A weakly chordal graph with an optimal 3-clique-colouring



(b) A $(4, 3)$-polar graph



(c) A generalized split graph, which is a $(k, 1)$-polar graph, for fixed $k \geq 2$

Figure 1: Examples of (α, β)-polar graphs

$(\alpha, 1)$-polar graphs are perfect, since they do not contain chordless cycles with an odd number of vertices greater than 4 nor their complements.

A *generalized split graph* is a graph G such that G or its complement is an $(\infty, 1)$-polar graph [12]. See Fig. 1c for an example of a generalized split graph, which is a (2, 1)-polar graph. The class of generalized split graphs plays an important role in the areas of perfect graphs and clique-colouring. This class was introduced by Prömel and Steger [12] to show that the strong perfect graph conjecture is at least asymptotically true by proving that almost all C_5-free graphs are generalized split graphs. Approximately 14 years later the strong perfect graph conjecture became the *Strong Perfect Graph Theorem* by Chudnovsky et al. [3]. Regarding clique-colouring, Bacsó et al. [1] proved that generalized split graphs are 3-clique-colourable and concluded that almost all perfect graphs are 3-clique-colourable [1]. This conclusion supports the conjecture due to Duffus et al. [5]. In fact, there is no example of a perfect graph where more than three colors would be necessary to clique-colour. Surprisingly, after more than 20 years, relatively little progress has been made on the conjecture.

The class of $(k, 1)$-polar graphs, for fixed $k \geq 3$, is incomparable to the class of weakly chordal graphs. Indeed, a chordless path with seven vertices P_7 and a complement of a chordless cycle with six vertices $\overline{C_6}$ are witnesses. Nevertheless, (2, 1)-polar graphs are a subclass of weakly chordal graphs, since they do not contain a chordless cycle with an even number of vertices greater than 5. We show that 2-clique-colouring of (2, 1)-polar graphs is a $\mathcal{NP}$-complete problem. Finally, the class of (1, 1)-polar graphs is precisely the class of split graphs. It is interesting to recall that 2-clique-colouring of $(1, 1)$-polar graphs is in $\mathcal{P}$, since $(1, 1)$-polar are a subclass of chordal graphs, which are 2-clique-colourable.

Giving continuity to our results, we investigate an open problem left by Kratochvíl and Tuza [7] to determine the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3. Restricting the size of the cliques to be at least 3, we first show that 2-clique-colouring is still $\mathcal{NP}$-complete for (3, 1)-polar graphs, even if it is restricted to weakly chordal graphs with all cliques having size at least 3. Subsequently, we prove that the 2-clique-colouring of (2, 1)-polar graphs becomes polynomial when all cliques have size at least 3. Recall that the 2-clique-colouring of (2, 1)-polar graphs is $\mathcal{NP}$-complete when there are no restrictions on the size of the cliques.

We finish the paper answering the open problem of determining the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3 [7], by improving our proof that 2-clique-colouring is a Σ_2^P-complete problem for weakly chordal graphs. We replace each K_2 clique by a gadget with no clique of size 2, which forces distinct colours into two given vertices.

The paper is organized as follows. In Section 2, we show that 2-clique-colouring is still Σ_2^P-complete for weakly chordal graphs. We then determine a hierarchy of nested subclasses of weakly chordal graphs whereby each graph class is in a distinct complexity class, namely Σ_2^P-complete, $\mathcal{NP}$-complete, and $\mathcal{P}$. In Section 3, we determine the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3, answering a question of Kratochvíl and Tuza [7]. We then determine a hierarchy of nested subclasses of perfect graphs with all cliques having size at least 3 whereby each graph class is in a distinct complexity class. We refer the reader to Table 1 for our results and related work about 2-clique-colouring complexity of perfect graphs.

2. Hierarchical complexity of 2-clique-colouring of weakly chordal graphs

Défossez proved that 2-clique-colouring of perfect graphs is a Σ_2^P-complete problem [4]. In this section, we strengthen this result by showing that it is still Σ_2^P-complete for weakly chordal graphs. We show a subclass of perfect graphs (resp. of weakly chordal graphs) in which 2-clique-colouring is neither a Σ_2^P-complete problem nor in $\mathcal{P}$, namely $(3,1)$-polar graphs (resp. $(2,1)$-polar

Table 1: 2-clique-colouring complexity of perfect graphs and subclasses.

<table>
<tr><th colspan="3">Class</th><th>2-clique-colouring complexity</th></tr>
<tr><td rowspan="7">-</td><td rowspan="3">Perfect</td><td>-</td><td>Σ_2^P-complete [4]</td></tr>
<tr><td>K_4-free</td><td>$\mathcal{NP}$-complete [7]</td></tr>
<tr><td>K_3-free (Bipartite)</td><td>$\mathcal{P}$</td></tr>
<tr><td>Weakly chordal</td><td>-</td><td>Σ_2^P-complete</td></tr>
<tr><td>(3, 1)-polar</td><td>-</td><td rowspan="2">$\mathcal{NP}$-complete</td></tr>
<tr><td>(2, 1)-polar</td><td>-</td></tr>
<tr><td>Chordal (includes Split)</td><td>-</td><td>$\mathcal{P}$ [11]</td></tr>
<tr><td rowspan="4">All cliques having size at least 3</td><td>Perfect</td><td>-</td><td rowspan="2">Σ_2^P-complete</td></tr>
<tr><td rowspan="2">Weakly chordal</td><td>-</td></tr>
<tr><td>(3, 1)-polar</td><td>$\mathcal{NP}$-complete</td></tr>
<tr><td>(2, 1)-polar</td><td>-</td><td>$\mathcal{P}$</td></tr>
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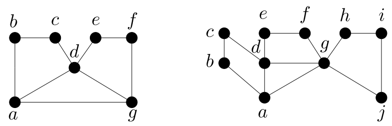


(a) $AK(a, g)$ (b) $NAS(a, j)$

Figure 2: Auxiliary graphs $AK(a, g)$ and $NAS(a, j)$

graphs). Recall that 2-clique-colouring of $(1, 1)$-polar graphs is in $\mathcal{P}$, since $(1, 1)$-polar are a subclass of chordal graphs, thereby 2-clique-colourable. Notice that weakly chordal, $(2, 1)$-polar, and $(1, 1)$-polar (resp. perfect, $(3, 1)$-polar, and $(1, 1)$-polar) are nested classes of graphs.

Given a graph $G = (V, E)$ and adjacent vertices $a, g \in V$, we say that we add to G a copy of an auxiliary graph $AK(a, g)$ of order 7 – depicted in Fig. 2a – if we change the definition of G by doing the following: we first change the definition of V by adding to it copies of the five vertices $b, \ldots, f$ of the auxiliary graph $AK(a, g)$; then we change the definition of E, adding to it copies of the eight edges (u, v) of $AK(a, g)$. Similarly, given a graph $G = (V, E)$ and non-adjacent vertices $a, j \in V$, we say that we add to G a copy of an auxiliary graph $NAS(a, j)$ of order 10 – depicted in Fig. 2b – if we change the definition of G by doing the following: we first change the definition of V by adding to it eight copies of the vertices $b, \ldots, i$ of the auxiliary graph $NAS(a, j)$; then we change the definition of E, adding to it copies of the thirteen edges (u, v) of $NAS(a, j)$.

The auxiliary graph $AK(a, g)$ is constructed to force the same colour (in a 2-clique-colouring) to adjacent vertices a and g, while the auxiliary graph $NAS(a, j)$ is constructed to force distinct colours (in a 2-clique-colouring) to non-adjacent vertices a and j (see Lemmas 1 and 2).

Lemma 1. *Let G be a graph and a, g be adjacent vertices in G. If we add to G a copy of an auxiliary graph $AK(a, g)$, then in any 2-clique-colouring of the resulting graph, adjacent vertices a and g have the same colour.*

PROOF. Follows from the fact that in $AK(a, g)$ there exists a path $abc \ldots g$ such that no edge lies in a triangle of G.

Lemma 2. *Let G be a graph and a, j be non-adjacent vertices in G. If we add to G a copy of an auxiliary graph $NAS(a, j)$, then in any 2-clique-colouring of the resulting graph, non-adjacent vertices a and j have distinct colours.*

PROOF. Follows from the fact that in $NAS(a, j)$ there exists a path $abc \ldots j$ such that no edge lies in a triangle of G.

We improve the proof of Défossez [4], in order to determine the complexity of 2-clique-colouring for weakly chordal graphs. Consider the QSAT2 problem, which is the Σ_2^P-complete canonical problem [9], as follows.

Problem 1. *Quantified 2-Satisfiability (QSAT2)*

Input: *A formula $\Psi = (X, Y, D)$ composed of a disjunction D of implicants (that are conjunctions of literals) over two sets X and Y of variables.*

Output: *Is there a truth assignment for X such that for every truth assignment for Y the formula is true?*

We prove that 2-clique-colouring weakly chordal graphs is Σ_2^P-complete by reducing the Σ_2^P-complete canonical problem QSAT2 to it. For a QSAT2 formula $\Psi = (X, Y, D)$, a weakly chordal graph G is constructed such that graph G is 2-clique-colourable if, and only if, there is a truth assignment of X, such that Ψ is true for every truth assignment of Y.

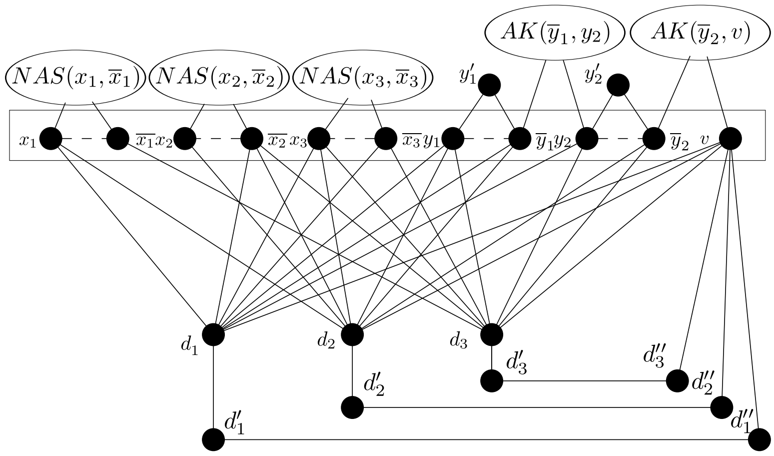
Theorem 3. *The problem of 2-clique-colouring is Σ_2^P-complete for weakly chordal graphs.*

Proof. A 2-partition of the graph is a certificate to decide whether a graph has a 2-clique-colouring. Moreover, a monochromatic clique is a certificate to check whether a 2-partition is not a 2-clique-colouring. Finally, it is easy to describe a polynomial-time algorithm to check whether a complete set is monochromatic and maximal. Hence, 2-clique-colouring is a Σ_2^P problem.

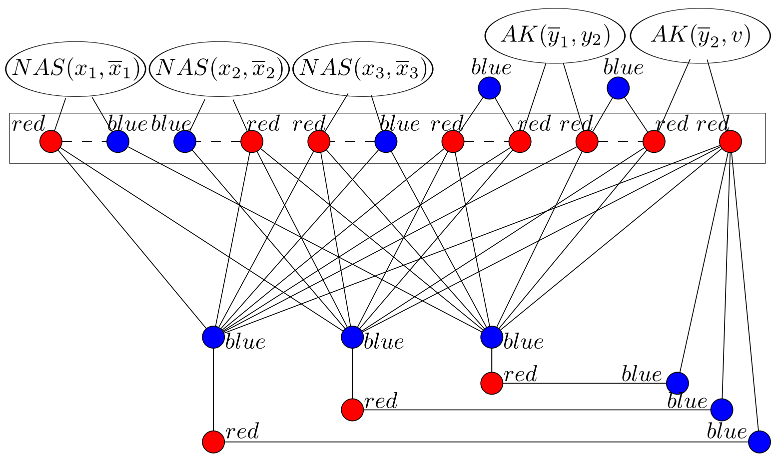
We prove that 2-clique-colouring weakly chordal graphs is Σ_2^P-hard by reducing QSAT2 to it. Let n, m, and p be the number of variables X, Y, and implicants, respectively, in formula Ψ. We define graph G, as follows.

- for each variable x_i, we create vertices x_i and $\overline{x}_i$;
- for each variable y_j, we create vertices y_j, y'_j, and $\overline{y}_j$ and edges $y_j y'_j$, and $y'_j \overline{y}_j$;
- we create a vertex v and edges so that the set $\{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n, y_1, \overline{y}_1, \ldots, y_m, \overline{y}_m, v\}$ induces a complete subgraph of G minus the matching $\{\{x_1, \overline{x}_1\}, \ldots, \{x_n, \overline{x}_n\}, \{y_1, \overline{y}_1\}, \ldots, \{y_m, \overline{y}_m\}\}$;
- add copies of the auxiliary graph $NAS(x_i, \overline{x}_i)$, for $i = 1, \ldots, n$;
- add copies of the auxiliary graph $AK(\overline{y}_j, y_{j+1})$, for $j = 1, \ldots, m-1$;
- add a copy of $AK(\overline{y}_m, v)$; and
- for each implicant d_k, we create vertices d_k, d'_k, d''_k, and we add the edges $d_k d'_k$, $d'_k d''_k$, $d''_k v$, and $d_k v$. Moreover, each vertex d_k is adjacent to a vertex l in $\{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n, y_1, \overline{y}_1, \ldots, y_m, \overline{y}_m, v\}$ if, and only if, the literal correspondent to $\overline{l}$ is not in the implicant correspondent to vertex d_k.

Refer to Fig. 3 for an example of such construction, given a formula $\Psi = (x_1 \wedge \overline{x}_2 \wedge y_2) \vee (x_1 \wedge x_3 \wedge \overline{y}_2) \vee (\overline{x}_1 \wedge \overline{x}_2 \wedge y_1)$.



(a) Graph constructed for a QSAT2 instance $\Psi = (x_1 \wedge \overline{x}_2 \wedge y_2) \vee (x_1 \wedge x_3 \wedge \overline{y}_2) \vee (\overline{x}_1 \wedge \overline{x}_2 \wedge y_1)$



(b) A satisfying truth assignment of $x_1 = \overline{x_2} = x_3 = T$

Figure 3: Example of a graph constructed for a QSAT2 instance, where NAS and AK denote the respectively auxiliary graphs

We claim that graph G is 2-clique-colourable if, and only if, Ψ has a solution. For every i, the vertices x_i and $\overline{x}_i$ have opposite colours in any 2-clique-colouring of G (see Lemma 2). The set $\{y_1, y_2, \ldots, y_m, \overline{y}_1, \overline{y}_2, \ldots, \overline{y}_m, v\}$ is monochromatic. Indeed, y_j, $\overline{y}_j$, y_{j+1} have the same colour, since $y_jy'_j$, $y'_j\overline{y}_j$ are cliques and, by Lemma 1, $\overline{y}_jy_{j+1}$ as well as $\overline{y}_mv$ have the same colour. Finally, d_1, d_2, $\ldots$, d_p all have the same colour, which is the opposite to the colour of v.

Assume there exists a valuation v_X of X such that Ψ is satisfied for any valuation of Y. We give a colouring to the graph G, as follows.

- assign colour 1 to y_j, $\overline{y}_j$, d'_k, and v,
- assign colour 2 to y'_j, d_k and d''_k,
- extend the unique 2-clique-colouring to the $m-1$ copies of the auxiliary graph $AK(\overline{y}_j, y_{j+1})$ and $AK(\overline{y}_m, v)$,
- assign colour 1 to x_i if the corresponding variable is $true$ in v_X, otherwise we assign colour 2 to it,
- assign colour 2 to $\overline{x}_i$ if the corresponding variable is $true$ in v_X, otherwise we assign colour 1 to it,
- extend the unique 2-clique-colouring to the n copies of the auxiliary graph $NAS(x_i, \overline{x}_i)$.

It still remains to be proved that this is indeed a 2-clique-colouring. Let us assume that it is not the case and that there exists a maximal clique K of G that is monochromatic. Clearly, K is not contained in a copy of any auxiliary graph, and that it does not contain any vertex of type y'_j, d'_k, or d''_k. As v is adjacent to all other vertices (which are the x_i, $\overline{x}_i$, y_j, $\overline{y}_j$, and d_k), we deduce that $v \in K$ and, subsequently, that all vertices of K have colour 1. Moreover, K contains exactly one vertex among x_i and $\overline{x}_i$, i.e. the one corresponding to the literal which is $true$ in v_X, and similarly exactly one vertex among y_j and $\overline{y}_j$. We remark that K does not contain any d_k since they have colour 2. Then we define a valuation v_Y in the following way. If $y_j \in K$, then v_Y assigns value $true$ to the corresponding variable, otherwise v_Y assigns the value $false$. Thus, the literals corresponding to the vertices of $K \setminus \{v\}$ are exactly those that are $true$ in the total valuation (v_X, v_Y). Let us consider now any d_k. Since K is maximal, each d_k is not adjacent to at least one vertex of K. By construction of G, this means that all implicants are $false$, which contradicts the definition of v_X. Hence, there is no monochromatic clique and we have a 2-clique-colouring.

For the converse, we now assume that G is 2-clique-colourable and we consider any 2-clique-colouring with colours 1 and 2. Without loss of generality, we can assume that v has colour 1. Then, y_j and $\overline{y}_j$ have colour 1 and d_k has colour 2. Vertices x_i and $\overline{x}_i$ have opposite colours and we define v_X in the following way. The literal x_i is assigned $true$ in v_X if the corresponding vertex has colour 1 in the clique-colouring, otherwise it is assigned $false$ in v_X. Let v_Y be any valuation of Y. Consider the clique K that contains v and the vertices corresponding to literals which are $true$ in the total valuation (v_X, v_Y). Since all those vertices have colour 1 and we have a 2-clique-colouring, it follows that K cannot be maximal. As a consequence, there exists some d_k which is adjacent to all vertices of K. Thus, the corresponding implicant is $true$ in that valuation and this proves that Ψ is satisfied for any valuation v_Y and that v_X has the right property.

It now remains to be proved that G is a weakly chordal graph. Fixing edge vd_i as an edge of a cycle, one can check that G has no chordless cycle of size greater than 5 as an induced subgraph. Now, we prove that G has no complement of a chordless cycle of size greater than 4 as an induced subgraph.

Let $\overline{H}$ be the complement of a chordless cycle of size greater than 5. Clearly, any vertex of H has degree at least 3. Hence, we analyse the vertices of G with degree at least 3. Let $S = \{x_i, \overline{x}_i, y_j, \overline{y}_j, v \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ and let $R = \{d_k | 1 \leq k \leq p\}$. All vertices of G with degree at least 3 are precisely the vertices of the auxiliary graphs, S, and R. We invite the reader to check that any vertex of an auxiliary graph that is not in S does not belong to H. Hence, every vertex of H belongs to S or R. First, we claim that R has at most 2 vertices of H. Indeed, 3 vertices of R induce a $\overline{K_3}$. Second, we claim that S has at most 2 vertices. Notice that $|R \cap H| > 0$, since a vertex in S has at most one non-neighbor in S and every neighbor of H has at least two non-neighbors in H. If $|R \cap H| = 1$, then $|S \cap H| \leq 2$, since a vertex in S has at most one non-neighbor in S and the unique vertex of $R \cap H$ has at most two non-neighbors in $S \cap H$. If $|R \cap H| = 2$, then $|S \cap H| \leq 2$, since a vertex in S has at most one non-neighbor in S and each vertex of $R \cap H$ has at most one non-neighbor in S. Hence, at most two vertices of H are in R and at most two vertices of H are in S, i.e. $|H| \leq 4$, which is a contradiction.

Now, our focus is on showing a subclass of weakly chordal graphs in which 2-clique-colouring is $\mathcal{NP}$-complete, namely (3, 1)-polar and (2, 1)-polar graphs.

Complements of bipartite graphs are a subclass of $(\infty, 1)$-polar graphs. Indeed, let $G = (V, E)$ be a complement of a bipartite graph, where (A, B) is a partition of V into two disjoint complete sets. Clearly, G is a $(\infty, 1)$-polar graph. Défossez [4] showed that it is co$\mathcal{NP}$-complete to check whether a 2-colouring of a complement of a bipartite graph is a 2-clique-colouring [4]. Hence, it is co$\mathcal{NP}$-hard to check if a colouring of the vertices of a $(\infty, 1)$-polar graph is a 2-clique-colouring. On the other hand, we show next that, if k is fixed, listing all cliques of a $(k, 1)$-polar graph and checking if each clique is polychromatic can be done in polynomial-time, although the constant behind the big O notation is impraticable. The outline of the algorithm follows. We create a subroutine in which, given a satellite K of G, we check whether every clique of G containing a subset of K is polychromatic. Lemma 4 determines the complexity of the subroutine and proves its correctness. The algorithm runs the subroutine for each satellite of G and, as a final step, check whether partition B is polychromatic if, and only if, partition B is a clique of G. Theorem 5 determines the complexity of the algorithm and prove its correctness.

Lemma 4. *There exists an $O(n)$-time algorithm to check whether every clique that contains a subset of a satellite S of a $(k, 1)$-polar graph, for a fixed $k \geq 1$,is polychromatic.*

PROOF. We prove the correctness of Algorithm 1 by induction. Let $A_1 = S$ and $B_1 = \bigcap_{v \in A_1} (N(v) \cap B)$. Notice that $A_1 \cup B_1$ is the unique clique of graph G that contains S. If $A_1 \cup B_1$ is monochromatic, then π is not a 2-clique-colouring of G.

Otherwise, i.e. $A_1 \cup B_1$ is polychromatic, we are done. Now, we need to check whether every clique of G containing a proper subset of S is polychromatic.

Let $A_{i+1} = A_1 \setminus \{x_1, \ldots, x_i\}$ and $B_{i+1} = \bigcap_{v \in A_{i+1}} (N(v) \cap B)$, for some $\{x_1, \ldots, x_i\} \subset A_1$. As an induction hyphotesis, suppose that every clique of G containing A_j, for every $1 \leq j \leq i$, is polychromatic.

For the induction step, consider a clique K of graph G containing A_{i+1}. By induction hyphotesis, if K contains A_j, for some $1 \leq j \leq i$, then K is polychromatic. Now, consider that K does not contain A_j, for any $1 \leq j \leq i$. Then, $K = A_{i+1} \cup B_{i+1}$. If K is monochromatic, then π is not a 2-clique-colouring of G. Otherwise, i.e. K is polychromatic, then every clique containing A_{i+1} is polychromatic and the proof of the correctness of Algorithm 1 is done.

Now, we give the time-complexity of Algorithm 1. First, there are at most $k!$ recursive calls. Second, the number of steps in an iteration of the algorithm is upper bounded by the complexity of calculating B_i. One can design an $O(|B| \log k)$-time algorithm to calculate B_i. Then, the algorithm is executed in $O(n)$ steps, since k is a constant and $|B|$ is upper bounded by n.

Algorithm 1: $O(n)$-time algorithm to output *yes*, if every clique of a $(k, 1)$-polar graph containing a subset of a satellite of G is polychromatic, for a fixed $k \geq 1$.

input : $G = (A, B)$, a $(k, 1)$-polar graph
π, a 2-colouring of G
A_i, a satellite of G

output: *yes*, if every clique of G containing a subset of a satellite A_i is polychromatic

begin
  if $|\pi(A_i)| \geq 2$ **then**
    for $i = 1$ **to** $|A_i|$ **do**
      $answer \longleftarrow recursive(A_i \setminus \{x_i\})$;
      if $answer = no$ **then**
        return *no;*
    return *yes;*
  else
    $B_i \longleftarrow \bigcap_{v \in A_i} (N(v) \cap B)$;
    if $|\pi(A_i \cup B_i)| \geq 2$ **then**
      return *yes;*
    else
      return *no;*
end

Theorem 5. *There exists an $O(n^2)$-time algorithm to check whether a colouring of the vertices of a $(k,1)$-polar graph, for a fixed $k \geq 1$, is a clique-colouring.*

Proof. The correctness of Algorithm 2 follows. A clique of G contains at least one vertex of a satellite of G or it is B. The first loop of Algorithm 2 checks whether all cliques in the former case are polychromatic. The second loop of Algorithm 2 checks whether B is a clique. If B is a clique, then we check whether B is polychromatic.

Now, we give the time-complexity of Algorithm 2. The first loop of Algorithm 2 runs at most n times the Algorithm 1, which runs in $O(n)$-time. The second loop of Algorithm 2 runs at most n times one comparison, which runs in $O(n)$-time. Then, Algorithm 2 is executed in at most $O(n^2)$ steps.

Algorithm 2: $O(n^2)$-time algorithm to output *yes*, if π is a clique-colouring of a $(k,1)$-polar graph, for a fixed $k \geq 1$.

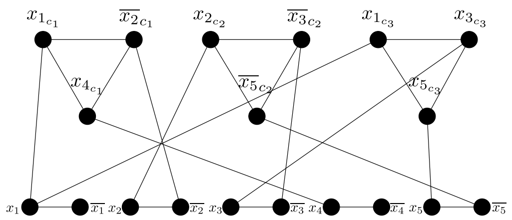
---

input : $G = (A, B)$, $(k,1)$-polar graph
π, a 2-colouring of G
output: *yes*, if π is a 2-clique-colouring o G.

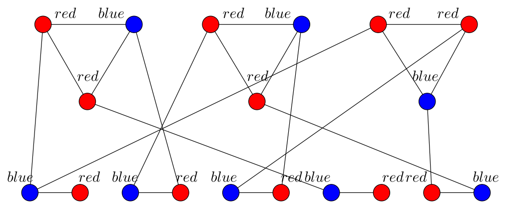
begin
  foreach *maximal complete set* $A' \in A$ **do**
    $answer \longleftarrow Algorithm\ 1(G, \pi, A')$;
    if $answer = no$ **then**
      return *no;*
  foreach *maximal complete set* $A' \in A$ **do**
    foreach $v \in A'$ **do**
      if $|N_B(v)| = |B|$ **then**
        return *yes;*
  if $|\pi(B)| \geq 2$ **then**
    return *yes;*
  else
    return *no;*
end

Consider the NAE-SAT problem, known to be $\mathcal{NP}$-complete [13].

Problem 2. *Not-all-equal satisfiability (NAE-SAT)*
Input: *A set X of boolean variables and a collection C of clauses (set of literals over U), each clause containing at most three different literals.*
Output: *Is there a truth assignment for X such that every clause contains at least one $true$ and at least one $false$ literal?*



(a) Graph constructed for a NAE-SAT instance $\phi = (x_1 \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge (x_1 \vee x_3 \vee x_5)$



(b) A satisfying truth assignment of $x_1 = x_2 = x_3 = x_4 = \overline{x_5} = T$ implies a 2-clique-colouring of G, where $x_1, x_2, x_3, x_4, \overline{x_5}$ receive blue colour

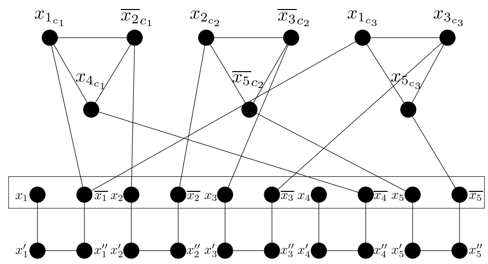
Figure 4: Example of a graph constructed following the framework given by Kratochvíl and Tuza [7] for a NAE-SAT instance ϕ

We first illustrate the framework of Kratochvíl and Tuza [7] to argue that 2-clique-colouring is $\mathcal{NP}$-hard with a reduction from NAE-SAT, as follows. Consider an instance ϕ of NAE-SAT. We construct a graph G, as follows. For every variable x, add an edge between vertices x and $\overline{x}$. For every clause c, add a triangle on vertices ℓ_c for all literals ℓ occurring in c. To finish the construction of G, for every literal ℓ and for every clause c containing ℓ, add an edge between ℓ and ℓ_c. The (maximal) cliques of G are the edges $x\overline{x}$, $\ell\ell_c$, and triangles $\{\ell_c \mid \ell \in c\}$. Hence, G is 2-clique-colourable if, and only if, ϕ is not-all-equal not-all-equal satisfiable. Refer to Fig. 4 for an example of such construction, given a formula $\phi = (x_1 \vee \overline{x}_2 \vee y_2) \wedge (x_1 \vee x_3 \vee \overline{y}_2) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee y_1)$.

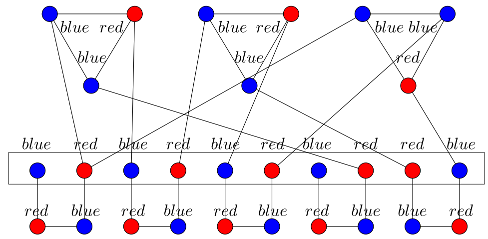
We apply the ideas of the framework of Kratochvíl and Tuza [7] to determine the complexity of 2-clique-colouring of (3, 1)-polar graphs. We prove that 2-clique-colouring (3, 1)-polar graphs is $\mathcal{NP}$-complete by reducing the NAE-SAT problem to it. For a NAE-SAT formula ϕ, a (3, 1)-polar graph G is constructed such that graph G is 2-clique-colourable if, and only if, ϕ is not-all-equal satisfiable. This is an intermediary step to achieve the complexity of 2-clique-colouring of (2, 1)-polar graphs, which are a subclass of weakly chordal graphs.

Theorem 6. *The problem of 2-clique-colouring is $\mathcal{NP}$-complete for (3, 1)-polar graphs.*

Proof. The problem of 2-clique-colouring a (3, 1)-polar graph is in $\mathcal{NP}$: Theorem 5 confirms that to check whether a colouring of a $(3, 1)$-polar graph is a



(a) Graph constructed for a NAE-SAT instance $\phi = (x_1 \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge (x_1 \vee x_3 \vee x_5)$



(b) A satisfying truth assignment of $x_1 = x_2 = x_3 = x_4 = \overline{x_5} = T$ implies a 2-clique-colouring of G, where $x_1, x_2, x_3, x_4, \overline{x_5}$ receive blue colour

Figure 5: Example of a (3, 1)-polar graph constructed for a NAE-SAT instance

2-clique-colouring is in $\mathcal{P}$.

We prove that 2-clique-colouring (3, 1)-polar graphs is $\mathcal{NP}$-hard by reducing NAE-SAT to it. The outline of the proof follows. For every formula ϕ, a graph G is constructed such that ϕ is not-all-equal satisfiable if, and only if, graph G is 2-clique-colourable. We finish the proof showing that G is (3, 1)-polar. Let n (resp. m) be the number of variables (resp. clauses) in formula ϕ. We define graph G, as follows.

- for each variable x_i, $1 \leq i \leq n$, we create four vertices x_i, x_i', x_i'', and $\overline{x}_i$ with edges $x_i x_i'$, $x_i' x_i''$, and $x_i'' \overline{x}_i$. Notice that vertices x_i and $\overline{x}_i$ correspond to the literals of variable x_i. Moreover, we create edges so that the set $\{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$ induces a complete subgraph of G;
- for each clause $c_j = (l_a, l_b, l_c)$, $1 \leq j \leq m$, we create a triangle c_j with three vertices $l_{a_{c_j}}$, $l_{b_{c_j}}$, and $l_{c_{c_j}}$. Notice that vertices $l_{a_{c_j}}$, $l_{b_{c_j}}$, and $l_{c_{c_j}}$ correspond to the literals of clause c_j. Moreover, each vertex $l \in \{l_{a_{c_j}}, l_{b_{c_j}}, l_{c_{c_j}}\}$ is adjacent to a vertex $\overline{l}$ in $\{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$ if, and only if, the literals correspondent to l and $\overline{l}$ are distinct literals of the same variable.

Refer to Fig. 5 for an example of such construction, given a formula $\phi = (x_1 \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge (x_1 \vee x_3 \vee x_5)$.

We claim that there exists a 2-clique-colouring in G if, and only if, formula ϕ is satisfiable. Assume that there exists a valuation v_ϕ such that ϕ is not-all-equal satisfied. We give a colouring to graph G, as follows.

- assign colour 1 to $l \in \{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$ if it corresponds to the literal which receives the *true* value in v_ϕ, otherwise we assign colour 2 to it,
- extend the unique 2-clique-colouring to vertices x'_i and x''_i, for each $1 \leq i \leq n$, i.e. assign colour 2 to x'_i and colour 1 to x''_i if the corresponding literal of x_i is *true* in v_ϕ. Otherwise, we assign colour 2 to x'_i and colour 1 to x''_i,
- extend the unique 2-clique-colouring to vertices $l_{a_{c_j}}$, $l_{b_{c_j}}$, and $l_{c_{c_j}}$, for each triangle $c_j = \{l_{a_{c_j}}, l_{b_{c_j}}, l_{c_{c_j}}\}$, $1 \leq j \leq m$, i.e. assign colour 1 to $l \in \{l_{a_{c_j}}, l_{b_{c_j}}, l_{c_{c_j}}\}$ if the corresponding opposite literal of the same variable is $false$ in v_ϕ, or else we assign colour 2 to l.

It still remains to be proved that this is indeed a 2-clique-colouring.

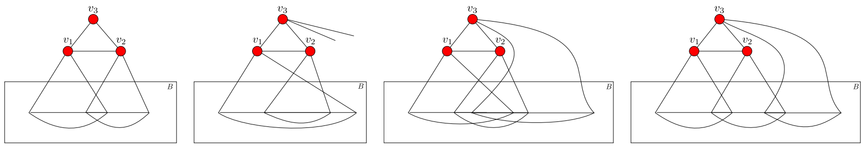
The cliques of size 2 are $x_i x'_i$, $x'_i x''_i$, and $x''_i \overline{x}_i$ and $l\overline{l}$, where $l \in c_j$ and $\overline{l}$ correspond to distinct literals of the same variable. The above colouring gives distinct colours to each vertex of a clique of size 2.

The cliques of size at least 3 are $\{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$ and triangle c_j. The former is polychromatic, since two vertices which represent distinct literals of the same variable receive distinct colours. The latter is also polychromatic, since (i) every vertex of triangle c_j represents a literal of some variable in ϕ, (ii) each clause c_j has at least one literal that receives the *true* value and at least one literal that receives the $false$ value (recall we are reducing from NAE-SAT problem), and (iii) a vertex in triangle c_j receives colour 1 if it corresponds to the literal in which the opposite literal of the same variable receives the $false$ value in v_ϕ. Otherwise, we assign colour 2 to it.

For the converse, we now assume that G is 2-clique-colourable and we consider any 2-clique-colouring. Recall the vertices x_i and $\overline{x}_i$ have distinct colours, since $x_i x'_i$, $x'_i x''_i$, and $x''_i \overline{x}_i$ are cliques. Hence, we define v_ϕ as follows. The literal x_i is assigned *true* in v_ϕ if the corresponding vertex has colour 1 in the clique-colouring, otherwise it is assigned $false$. Since we are considering a 2-clique-colouring, every triangle c_j is polychromatic. As a consequence, there exists at least one literal with *true* value in c_j and at least one literal with $false$ value in every clause c_j. This proves that ϕ is satisfied for valuation v_ϕ.

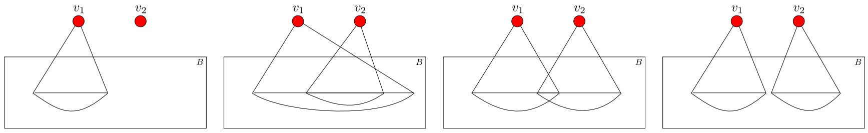
It now remains to be proved that G is a (3, 1)-polar graph. Let $A = \left(\bigcup_{i=1}^{n} \{x'_i, x''_i\}\right) \cup \left(\bigcup_{j=1}^{m} V(c_j)\right)$ and $B = \{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$ be the partition of $V(G)$ into two sets. Notice that each satellite is either a triangle or an edge. Hence, G is a (3, 1)-polar graph.

An additional requirement to the NAE-SAT problem is that all variables must be positive (no negated variables). This defines the known variant Pos-



(a) Case $\mathcal{K}_1$ (b) Case $\mathcal{K}_2$ (c) Case $\mathcal{K}_3$ (d) Case $\mathcal{K}_4$

Figure 6: A triangle satellite of an (α, β)-polar graph



(a) Case $\mathcal{K}_1$ (b) Case $\mathcal{K}_2$ (c) Case $\mathcal{K}_3$ (d) Case $\mathcal{K}_4$

Figure 7: An edge satellite of an (α, β)-polar graph

itive NAE-SAT [10, Chapter 7]. The variant Positive NAE-SAT is $\mathcal{NP}$-complete and the proof is by reduction from NAE-SAT. Replace every negated variable $\overline{l_i}$ by a fresh variable l_j, and add a new clause (l_i, l_j), to enforce the complement relationship. Notice that the new clause has only two literals. Hence, all that is needed here is (i) duplicate a clause with a negated variable, say $\overline{l_i}$, (ii) replace negated variable $\overline{l_i}$ by fresh variables l_j and l_k, respectively, and (iii) add a new clause (l_i, l_j, l_k), to enforce the complement relationship. An alternative proof that 2-clique-colouring is $\mathcal{NP}$-complete for (3, 1)-polar graphs can be obtained by a reduction from Positive NAE-SAT. The proof is analogous, but the constructed graph is as simple as the input (in constrast to NAE-SAT input).

We use a reduction from 2-clique-colouring of (3, 1)-polar graphs to determine the complexity of 2-clique-colouring of (2, 1)-polar graphs. In what follows, we provide some notation to classify the structure of 2-clique-colouring of (2, 1)-polar graphs and of (3, 1)-polar graphs. We capture their similarities and make it feasible a reduction from 2-clique-colouring $(3, 1)$-polar graphs to 2-clique-colouring (2, 1)-polar graphs.

Let $G = (V, E)$ be a (3, 1)-polar graph. Let K be a satellite of G. Consider the following four cases: ($\mathcal{K}_1$) there exists a vertex of K such that none of its neighbors is in partition B; ($\mathcal{K}_2$) the complementary case of $\mathcal{K}_1$, where there exists a pair of vertices of K, such that the closed neighborhood of one vertex of the pair is contained in the closed neighborhood of the other vertex of the pair; ($\mathcal{K}_3$) the complementary case of $\mathcal{K}_2$, where the intersection of the closed neighborhood of the vertices of K is precisely K; and ($\mathcal{K}_4$) the complementary case of $\mathcal{K}_3$. Clearly, any satellite K is either in case $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, or $\mathcal{K}_4$. Refer to Fig. 6 (resp. Fig. 7) for an example of each case of a triangle (resp. edge) satellite.

The following lemma is an important step to understand the role of triangles and edges that are either in case $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, or $\mathcal{K}_4$ in a 2-clique-colouring of (3, 1)-polar and of (2, 1)-polar graphs. The following lemma is also important towards the modification of a (3, 1)-polar graph to obtain a (2, 1)-polar graph, which is closely related to Theorem 8.

Lemma 7. *Let $G = (V, E)$ be a (3, 1)-polar graph, $\mathcal{K}$ be the set of satellites of G in case $\mathcal{K}_4$, and $K \in \mathcal{K}$.*

- *If G has a 2-clique-colouring, then $\bigcup_{v \in K} N_B(v)$ is polychromatic.*
- *If B has a 2-colouring that, for every $K' \in \mathcal{K}$, $\bigcup_{v \in K'} N_B(v)$ is polychromatic, then G is 2-clique-colourable.*

Proof. We begin by proving the former assertion. For the sake of contradiction, suppose $\bigcup_{v \in K} N_B(v)$ is monochromatic for some 2-clique-colouring of G. Without loss of generality, suppose that all vertices in $\bigcup_{v \in K} N_B(v)$ have colour 1. Hence, every vertex $v \in K$ has colour 2, otherwise $\{v\} \cup N_B(v)$ is a monochromatic clique. On the other hand, if every vertex $v \in K$ has the same colour, then K is a monochromatic clique, which is a contradiction.

Now, we prove the latter assertion. For each singleton satellite K, we extend the 2-colouring of B, as follows. Assign colour 1 to the vertex v of K if there exists a vertex of $N_B(v)$ with colour 2. Otherwise, assign colour 2 to v. For each edge satellite K, we extend the 2-colouring of G, as follows.

1. K is in case $\mathcal{K}_1$. Let u be a vertex of K, such that $N_B(u) = \emptyset$. Let v be a vertex of K, $u \neq v$. If $N_B(v) = \emptyset$, then assign colour 1 to v and colour 2 to u. Otherwise, i.e. $N_B(v) \neq \emptyset$, assign colour 1 to v if there exists a vertex in $N_B(v)$ with colour 2, otherwise assign colour 2 to v. Moreover, give colour 1 to u, if vertex v has colour 2, or else we assign colour 1 to u.
2. K is in case $\mathcal{K}_2$. Let v, w be distinct vertices of K, such that $N_B(v) \subseteq N_B(w)$. Assign colour 1 to v if there exists a vertex of $N_B(v)$ with colour 1, otherwise assign colour 2 to v. Assign colour 2 to w if vertex v received colour 1, otherwise assign colour 1 to w.
3. K is in case $\mathcal{K}_3$. Assign colour 1 to every vertex of K, if there exists a vertex in $\bigcap_{v \in K} N_B(v)$ with colour 2, otherwise assign colour 2 to every vertex of K.
4. K is in case $\mathcal{K}_4$. Let u, v be distinct vertices of K. Suppose $N_B(u)$ and $N_B(v)$ are monochromatic. Assign colour 1 to u if the vertices of $N_B(u)$ have colour 2. Otherwise, assign colour 2 to u. Assign colour 1 to v if the vertices of $N_B(v)$ have colour 2. Otherwise, assign colour 2 to v. Now, suppose, without loss of generality, that $N_B(u)$ is polychromatic.

Then, assign colour 1 to v if there exists a vertex of $N_B(v)$ with colour 2, otherwise assign colour 2 to v. Moreover, assign colour 2 to u if vertex v received colour 1, otherwise assign colour 2 to u.

For each triangle satellite K, we extend the 2-colouring of B, as follows.

1. K is in case $\mathcal{K}_1$. Let u, v, w be the distinct vertices of K, such that $N_B(u) = \emptyset$. If $N_B(v) \cup N_B(w) = \emptyset$, then assign colours 1, 1, and 2 to u, v, and w respectively. Now, if $N_B(v) \cup N_B(w)$ is monochomatic, then assign colour 1 to v and w if the colour assigned to every vertex of $N_B(v) \cup N_B(w)$ is 2, otherwise we assign colour 1 to v and w. Moreover, assign colour 1 to u if v and w have colour 2, otherwise assign colour 2 to u. Now, if $N_B(v) \cup N_B(w)$ is polychromatic, we have two cases. If $N_B(v) \cap N_B(w) \neq \emptyset$, then assign colour 1 to v and w if there exists a vertex in $N_B(v) \cap N_B(w)$ with colour 2, otherwise assign colour 1 to v and w. Moreover, assign colour 1 to u if v and w have colour 2, otherwise assign colour 2 to u. If $N_B(v) \cap N_B(w) = \emptyset$, assign colour 1 to v (resp. to w) if there exists a vertex of $N_B(v)$ (resp. $N_B(w)$) with colour 2, otherwise assign colour 2 to v (resp. w). Moreover, assign colour 1 to u if v or w have colour 2, otherwise assign colour 2 to u.
2. K is in case $\mathcal{K}_2$. Let v, w be distinct vertices of K, such that $N_B(v) \subseteq N_B(w)$. Assign colour 1 to v if there exists a vertex in $N_B(v)$ with colour 1, otherwise assign colour 2 to v. Assign colour 2 to w if vertex v received colour 1, otherwise assign colour 1 to w. Let $u \in K \setminus \{v, w\}$. By hyphotesis, $N_B(u) \neq \emptyset$. Assign colour 1 to u if there exists a vertex of $N_B(u)$ with colour 2, otherwise assign colour 2 to u.
3. K is in case $\mathcal{K}_3$. Assign colour 1 to every vertex of K, if there exists a vertex of $\bigcap_{v \in K} N_B(v)$ with colour 2, otherwise assign colour 2 to every vertex of K.
4. K is in case $\mathcal{K}_4$. Let u, v, w be distinct vertices of K. Suppose $N_B(u)$, $N_B(v)$, and $N_B(w)$ are monochromatic. Assign colour 1 to u if the vertices of $N_B(u)$ have colour 2. Otherwise, assign colour 2 to u. Assign colour 1 to v if the vertices of $N_B(v)$ have colour 2. Otherwise, assign colour 2 to v. Assign colour 1 to w if the vertices of $N_B(w)$ have colour 2. Otherwise, assign colour 2 to w. Now, suppose, without loss of generality, that $N_B(u)$ is polychromatic. Assign colour 1 to v if there exists a vertex of $N_B(v)$ with colour 2, otherwise assign colour 2 to v. Assign colour 1 to w if there exists a vertex of $N_B(w)$ with colour 2, otherwise assign colour 2 to w. Finally, assign colour 2 to u if vertex v and w received colour 1, otherwise assign colour 2 to u.

We invite the reader to check that the given colouring is a 2-clique-colouring to graph G.

For a given a (3, 1)-polar graph G, we proceed to obtain a (2, 1)-polar graph G' that is 2-clique-colourable if, and only if, G is 2-clique-colourable, as

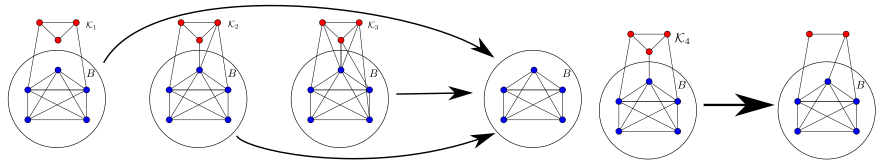


Figure 8: An iteration to obtain a (2, 1)-polar graph G', given a (3, 1)-polar graph G, such that G is 2-clique-colourable if and only if G' is 2-clique-colourable

follows. For each triangle satellite, if it is in case $\mathcal{K}_4$, we replace it by an edge in which (i) both complete sets have the same neighboorhod contained in B and (ii) the edge is also in case $\mathcal{K}_4$, otherwise we just delete triangle K. See Fig. 8 for examples. Such construction is done in polynomial-time and we depict it as Algorithm 3. See Fig. 9 for an application of Algorithm 3 given as input a (3,1)-polar graph, which is not (2,1)-polar, with clique-chromatic number 3. Algorithm 3 and Theorem 5 imply the following theorem.

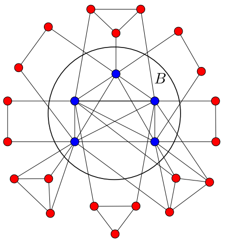
Algorithm 3: An $O(n^2)$-time algorithm to output a (2, 1)-polar graph G', such that graph G is 2-clique-colourable if, and only if, graph G' is 2-clique-colourable.

input : $G = (A, B)$, a (3, 1)-polar graph.
output: G', a (2, 1)-polar graph that is 2-clique-colourable if, and only if, G is 2-clique-colourable.

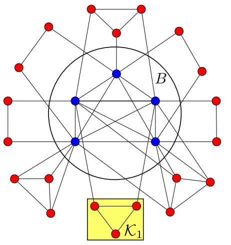
```
begin
    foreach satellite K = {v1, v2, v3} do
        V' ⟵ ∅;
        E' ⟵ ∅;
        if K is in case K4 then
            V' ⟵ {u1, u2};
            E' ⟵ {(u1, u2)};
            E' ⟵ E' ∪ {(u1, x) | x ∈ N_B(v1)};
            E' ⟵ E' ∪ {(u2, x) | x ∈ ((N_B(v2) ∪ N_B(v3)) \ N_B(v1))};
        G ⟵ G[V(G) \ K];
        V(G) ⟵ V(G) ∪ V';
        E(G) ⟵ E(G) ∪ E';
    return G;
end
```

Theorem 8. *The problem of 2-clique-colouring is $\mathcal{NP}$-complete for (2, 1)-polar graphs.*

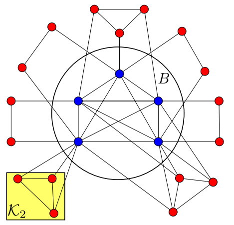
Proof. The problem of 2-clique-colouring a (2, 1)-polar graph is in $\mathcal{NP}$: Theorem 5 confirms that it is in $\mathcal{P}$ to check whether a colouring of a (2, 1)-polar graph is a 2-clique-colouring.



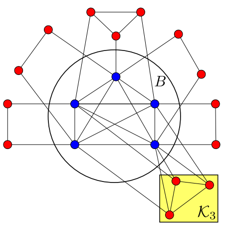
(a) A (3,1)-polar graph, which is not (2,1)-polar, with clique-chromatic number 3.



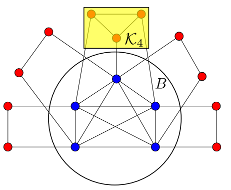
(b) A satellite K_1 that belongs to $\mathcal{K}_1$.



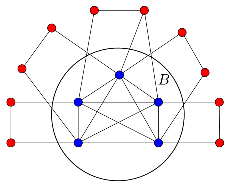
(c) A satellite K_2 that belongs to $\mathcal{K}_2$.



(d) A satellite K_3 that belongs to $\mathcal{K}_3$.



(e) A satellite K_4 that belongs to $\mathcal{K}_4$.



(f) A (2,1)-polar graph with clique-chromatic number 3.

Figure 9: Application of Algorithm 3 given as input a (3,1)-polar graph, which is not (2,1)-polar, with clique-chromatic number 3.

We claim that Algorithm 3 is an $O(n^2)$-time algorithm to output a (2, 1)-polar graph G', given a (3, 1)-polar graph G, such that G' is 2-clique-colourable if, and only if, graph G is 2-clique-colourable.

Without loss of generality, suppose that B is a (maximal) clique of the graph. We use induction on the number of triangle satellites of G in order to prove that G is 2-clique-colourable if, and only if, G' is 2-clique-colourable. For the sake of conciseness, we prove only one way of the basis induction. The converse and the step induction follows analogously.

Suppose that there exists only one triangle satellite $K = \{v_1, v_2, v_3\}$ of G. Let $K' = \{u_1, u_2\}$ be the edge that replaced satellite K to obtain graph G'. Suppose that there exists a 2-clique-colouring π of G. Every clique of G containing a subset of a satellite $S \neq K$ of G is polychromatic. Let π' be a 2-colouring of G', such that $\pi'(v) = \pi(v)$, for each $v \in V(G) \cap V(G')$. Hence, every clique of G' containing a subset of a satellite $S' \neq K'$ of G is polychromatic. If $B(G)$ is a clique of G, then B is polychromatic. Clearly, $B(G')$ is polychromatic. Hence, we are left to prove that every clique of G' containing a subset of K' is polychromatic. If K is either in case $\mathcal{K}_1$, $\mathcal{K}_2$ or $\mathcal{K}_3$, then we are done. If K is in case $\mathcal{K}_4$, then $\bigcup_{v \in K} N_B(v)$ is polychromatic. Since $\bigcup_{v \in K} N_B(v) = \bigcup_{v \in K'} N_B(v)$ and $\pi'(v) = \pi(v)$, for each $v \in B$, $\bigcup_{v \in K'} N_B(v)$ is polychromatic. By Lemma 7, G' is 2-clique-colourable.

We noticed strong connections between hypergraph 2-colorability and 2-clique-colouring (2, 1)-polar graphs. Indeed, we have a simpler alternative proof showing that 2-clique-colouring (2, 1)-polar graphs is $\mathcal{NP}$-complete by a reduction from hypergraph 2-colouring. In constrast to graphs, deciding if a given hypergraph is 2-colourable is an $\mathcal{NP}$-complete problem, even if all edges have cardinality at most 3 [8]. The reader may ask why we did not exploit only the alternative proof that is quite shorter than the original proof. The reason is related to be consistent with the next section, where we show that even restricting the size of the cliques to be at least 3, the 2-clique-colouring of (3, 1)-polar graphs is still $\mathcal{NP}$-complete, while 2-clique-colouring of (2, 1)-polar graphs becomes a problem in $\mathcal{P}$.

Proof. The problem of 2-clique-colouring a (2, 1)-polar graph is in $\mathcal{NP}$: Theorem 5 confirms that it is in $\mathcal{P}$ to check whether a colouring of a (2, 1)-polar graph is a 2-clique-colouring.

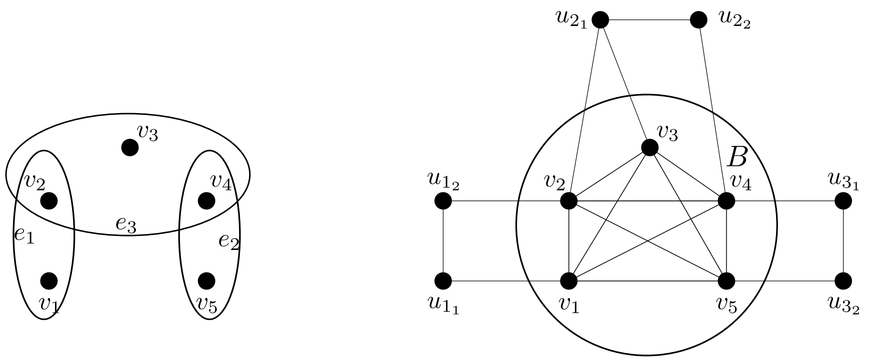
We prove that 2-clique-colouring (2, 1)-polar graphs is $\mathcal{NP}$-hard by reducing hypergraph 2-colouring to it. The outline of the proof follows. For every hypergraph $\mathcal{H}$, a (2, 1)-polar graph G is constructed such that $\mathcal{H}$ is 2-colourable if, and only if, graph G is 2-clique-colourable. Let n (resp. m) be the number of hypervertices (resp. hyperedges) in hypergraph $\mathcal{H}$. We define graph G, as follows.

- for each hypervertex v_i, $1 \leq i \leq n$, we create a vertex v_i in G, so that the set $\{v_1, \ldots, x_n\}$ induces a complete subgraph of G, which is the partition B of graph G;
- for each hyperedge $e_j = \{v_1, \ldots, v_l\}$, $1 \leq j \leq m$, we create two vertices u_{j_1} and u_{j_2}. Moreover, we create edges $u_{j_1}v_1, \ldots, u_{j_1}v_{l-1}$, and $u_{j_2}v_l$ so that $\{u_{j_1}u_{j_2}\}$ is a satellite in case $\mathcal{K}_4$.

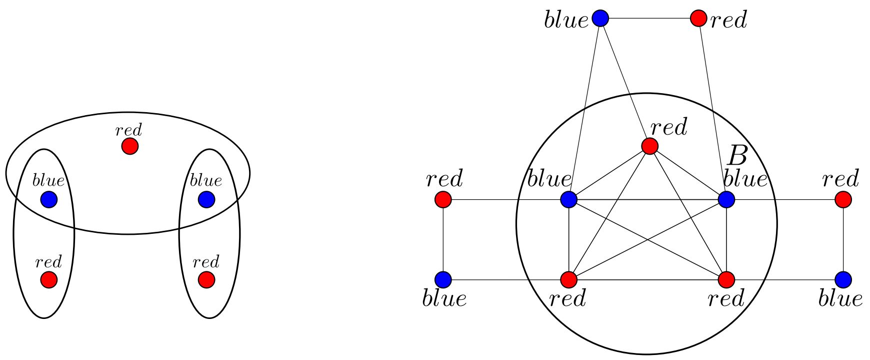
Clearly, graph G is a (2, 1)-polar graph and such construction is done in polynomial-time. Refer to Fig. 10 for an example of such construction.

We claim that hypergraph $\mathcal{H}$ is 2-colourable if, and only if, graph G is 2-clique-colourable. Assume that there exists a proper 2-colouring π of $\mathcal{H}$. We give a colouring to the graph G, as follows.

- assign colour $\pi(v)$ for each v of partition B,
- extend the 2-clique-colouring for each clique $(\{u_{j_1}, u_{j_2}\})$ that is a satellite of G.



(a) Hypergraph instance and its corresponding constructed (2, 1)-polar graph

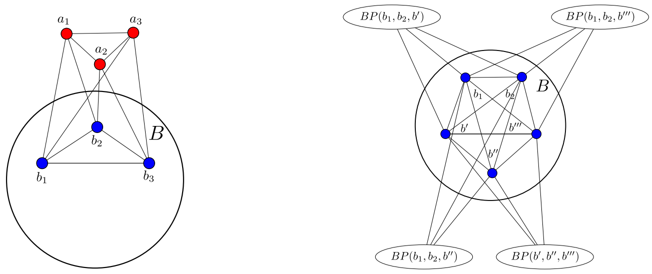


(b) A proper 2-colouring assignment of the hypergraph and its corresponding 2-colouring assignment of the constructed graph, as well as a 2-clique-colouring assignment of the constructed graph and its corresponding 2-colouring assignment of the hypergraph

Figure 10: Example of a (2, 1)-polar graph constructed for a given hypergraph instance

It still remains to be proved that this is indeed a 2-clique-colouring. Consider the partition $B = \{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$. Clearly, the above colouring assigns 2 colours to this set. Each satellite K of G is in case $\mathcal{K}_4$ and $\bigcup_{v \in K} N_B(v)$ is polychromatic, since $\bigcup_{v \in K} N_B(v) = e_j$. By Lemma 7, graph G is 2-clique-colourable.

For the converse, we now assume that G is 2-clique-colourable and we consider any 2-clique-colouring π' of G. We give a colouring to hypergraph $\mathcal{H}$, as follows. Assign colour $\pi'(v)$ for each hypervertex v. By Lemma 7 $\bigcup_{v \in K} N_B(v)$ is polychromatic for each satellite K of G. Then, hypergraph $\mathcal{H}$ is 2-colourable, since $\bigcup_{v \in K} N_B(v) = e_j$ for every hyperedge e_j.



(a) Auxiliary graph $BP(b_1, b_2, b_3)$ (b) Auxiliary graph $BS(b_1, b_2)$

Figure 11: Auxiliary graphs $BP(b_1, b_2, b_3)$ and $BS(b_1, b_2)$

3. Restricting the size of the cliques

Kratochvíl and Tuza [7] are interested in determining the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3. We determine what happens with the complexity of 2-clique-colouring of (2, 1)-polar graphs, of (3, 1)-polar graphs, and of weakly chordal graphs, respectively, when all cliques are restricted to have size at least 3. The latter result address Kratochvíl and Tuza's question.

Given graph G and $b_1, b_2, b_3 \in V(G)$, we say that we add to G a copy of an auxiliary graph $BP(b_1, b_2, b_3)$ of order 6 – depicted in Fig. 11a – if we change the definition of G by doing the following: we first change the definition of V by adding to it copies of the vertices a_1, a_2, a_3 of the auxiliary graph $BP(b_1, b_2, b_3)$; second, we change the definition of E by adding to it copies of the edges (u, v) of $BP(b_1, b_2, b_3)$.

Similarly, given a graph G and $b_1, b_2 \in V(G)$, we say that we add to G a copy of an auxiliary graph $BS(b_1, b_2)$ of order 17 – depicted in Fig. 11b – if we change the definition of G by doing the following: we first change the definition of V by adding to it copies of the vertices b', b'', b''' of the auxiliary graph $BS(b_1, b_2)$; second, we change the definition of E by adding to it edges so that $B(G) \cup \{b_1, b_2, b', b'', b'''\}$ is a complete set; finally, we add copies of the auxiliary graphs $BP(b_1, b_2, b')$, $BP(b_1, b_2, b'')$, $BP(b_1, b_2, b''')$, $BP(b', b'', b''')$.

Lemma 9. *Let G be a weakly chordal graph (resp. (3, 1)-polar graph) and $b_1, b_2, b_3 \in V(G)$ (resp. $b_1, b_2, b_3 \in B(G)$). If we add to G a copy of an auxiliary graph $BP(b_1, b_2, b_3)$, then the following assertions are true.*

- *The resulting graph G' is weakly chordal (resp. (3, 1)-polar).*
- *If all cliques of G have size at least 3, then all cliques of G' have size at least 3.*
- *Any 2-clique-colouring of G' assigns at least 2 colours to b_1, b_2, b_3.*
- *G is 2-clique-colourable if G' is 2-clique-colourable.*

- *G' is 2-clique-colourable if there exists a 2-clique-colouring of G that assigns at least 2 colours to b_1, b_2, b_3.*

Proof. Let G be a weakly chordal graph and $b_1, b_2, b_3 \in V(G)$. Add to G a copy of an auxiliary graph $BP(b_1, b_2, b_3)$ in order to obtain graph G'.

Suppose, by contradiction, that G' has a chordless cycle H with an odd number of vertices greater than 4 or the complement $\overline{H}$ of a chordless cycle with an odd number of vertices greater than 5. Clearly, $BP(b_1, b_2, b_3)$ is a weakly chordal graph. Since G and $BP(b_1, b_2, b_3)$ are weakly chordal graphs, H and $\overline{H}$ contains a vertex of $BP(b_1, b_2, b_3) \setminus G$ and a vertex of $G \setminus BP(b_1, b_2, b_3)$. Since $\{b_1, b_2, b_3\}$ is a complete set that is a cutset of G' that disconnects $BP(b_1, b_2, b_3) \setminus G$ from $G \setminus BP(b_1, b_2, b_3)$. Then, every cycle with vertices of $BP(b_1, b_2, b_3) \setminus G$ and of $G \setminus BP(b_1, b_2, b_3)$ contains a chord, i.e. there is no such H. Since a_1, a_2, and a_3 have at most 3 neighbors, $|\overline{H}| \leq 6$. If $\overline{H}$ has only vertex a_i of $BP(b_1, b_2, b_3) \setminus G$, then a_i has at most 2 neighbors in $\overline{H}$, which is a contradiction. If $\overline{H}$ has only vertices a_i, a_j, $i \neq j$, of $BP(b_1, b_2, b_3) \setminus G$, then $\overline{H}$ contains $\{b_1, b_2, b_3\}$, otherwise a_i or a_j have at most 2 neighbors in $\overline{H}$. Let u be a vertex of $G \setminus BP(b_1, b_2, b_3)$ in $\overline{H}$. Since a_i and a_j are not neighbors of u and a vertex in $\overline{H}$ has at most two non-neighbors in $\overline{H}$, $\{u, b_1, b_2, b_3\}$ is a complete set, which is a contradiction. If $\overline{H}$ has all three vertices of $BP(b_1, b_2, b_3) \setminus G$, then a vertex of $G \setminus BP(b_1, b_2, b_3)$ in H has 3 non-neighbors in $\overline{H}$, which is a contradiction. Hence, there is no such $\overline{H}$. Finally, G' is weakly chordal. If G is a (3, 1)-polar graph and $b_1, b_2, b_3 \in B(G)$, then G' is a (3, 1)-polar graph with $A(G') = A(G) \cup \{a_1, a_2, a_3\}$ and $B(G') = B(G)$ as the partition of $V(G')$ into two sets. Notice that the added satellite is a triangle. Hence, G' is a (3, 1)-polar graph.

Let $\mathcal{C}(G)$ be the set of cliques of graph G. We have $\mathcal{C}(G) \cap \mathcal{C}(G') = \mathcal{C}(G)$ and $\mathcal{C}(G') \setminus \mathcal{C}(G) = \{\{a_1, b_1, b_2\}, \{a_2, b_2, b_3\}, \{a_3, b_1, b_3\}, \{a_1, a_2, b_2\}, \{a_2, a_3, b_3\}, \{a_1, a_3, b_1\}, \{a_1, a_2, a_3\}\}$. Clearly, if all cliques of G have size at least 3, then all cliques of G' have size at least 3.

Since $\{a_1, a_2, a_3\}$ is a clique of G', any 2-clique-colouring π' of G' assigns at least 2 colours to a_1, a_2, a_3. Let $i, j, k, \ell \in \{1, 2, 3\}$ and $\pi'(a_i) \neq \pi'(a_j)$. Since $\{a_i, b_i, b_k\}$ (resp. $\{a_j, b_j, b_\ell\}$) is a clique of G', π' assigns a colour which is not $\pi'(a_i)$ to b_i or b_k (resp. π' assigns a colour which is not $\pi'(a_j)$ to b_j or b_ℓ). Hence, π' assigns 2 distinct colours to b_1, b_2, b_3.

Finally, $\pi'(G)$ is a 2-clique-colouring of G, since $\mathcal{C}(G) \subset \mathcal{C}(G')$. Now, consider a 2-clique-colouring of G that assigns 2 colours to b_1, b_2, b_3. It is easy to extend π in order to assign colours to the vertices of $BP(b_1, b_2, b_3) \setminus G$, such that all cliques of $\mathcal{C}(G') \setminus \mathcal{C}(G)$ are polychromatic.

Lemma 10. *Let G be a weakly chordal graph (resp. (3, 1)-polar graph) and $b_1, b_2 \in V(G)$ (resp. $b_1, b_2 \in B(G)$). If we add to G a copy of an auxiliary graph $BS(b_1, b_2)$, then the following assertions are true.*

- *The resulting graph G' is weakly chordal (resp. (3, 1)-polar).*
- *If all cliques of G have size at least 3, then all cliques of G' have size at least 3.*

- *Any 2-clique-colouring of G' assigns 2 colours to b_1 and b_2.*
- *G is 2-clique-colourable if G' is 2-clique-colourable.*
- *G' is 2-clique-colourable if there exists a 2-clique-colouring of G that assigns 2 colours to b_1 and b_2.*

PROOF. Let G be a weakly chordal graph and $b_1, b_2 \in V(G)$. Add to G a copy of an auxiliary graph $BS(b_1, b_2)$ in order to obtain graph G'.

Suppose, by contradiction, that G' has a chordless cycle H with an odd number of vertices greater than 4 or the complement $\overline{H}$ of a chordless cycle with an odd number of vertices greater than 5. First, we prove that $BS(b_1, b_2)$ is a weakly chordal (3, 1)-polar graph. A complete graph K_5 with vertices b_1, b_2, b', b'', b''' is a weakly chordal (3, 1)-polar graph with $A(K_5) = \emptyset$ and $B(K_5) = \{b_1, b_2, b', b'', b'''\}$. By Lemma 9, if we add copies of the auxiliary graphs $BP(b_1, b_2, b')$, $BP(b_1, b_2, b'')$, $BP(b_1, b_2, b''')$, and $BP(b', b'', b''')$, then we have a (weakly chordal) (3, 1)-polar graph that corresponds to $BS(b_1, b_2)$. Hence, $BS(b_1, b_2)$ is a (weakly chordal) (3, 1)-polar graph. Since G and $BS(b_1, b_2)$ are weakly chordal graphs, H and $\overline{H}$ contains a vertex of $BS(b_1, b_2) \setminus G$ and a vertex of $G \setminus BS(b_1, b_2)$. Since $\{b_1, b_2, b', b'', b'''\}$ is a complete set that is a cutset of G' that disconnects $BS(b_1, b_2) \setminus G$ from $G \setminus BS(b_1, b_2)$. Then, every cycle with vertices of $BS(b_1, b_2) \setminus G$ and of $G \setminus BS(b_1, b_2)$ contains a chord, i.e there is no such H. Subgraph $\overline{H}$ have vertices b_1 or b_2, otherwise $\overline{H}$ is disconnected. Hence, $\overline{H}$ has a 1-cutset or a 2-cutset, which is a contradiction since the complement of a chordless cycle is triconnected. Hence, G' is weakly chordal. If G is a (3, 1)-polar graph and $b_1, b_2 \in B(G)$, then G' is a (3, 1)-polar graph with $A(G') = A(G) \cup (V(BS(b_1, b_2)) \setminus \{b_1, b_2, b', b'', b'''\})$ and $B(G') = B(G) \cup \{b', b'', b'''\}$ as the partition of $V(G')$ into two sets. Notice that all added satellites are triangles. Hence, G' is a (3, 1)-polar graph.

Let $\mathcal{C}(G)$ be the set of cliques of graph G. If $B(G) = \emptyset$, then we have $\mathcal{C}(G) \cap \mathcal{C}(G') = \mathcal{C}(G) \setminus \{b_1, b_2\}$ and $\mathcal{C}(G') \setminus \mathcal{C}(G)$ is precisely $\{b_1, b_2, b', b'', b'''\}$, and all cliques added by the inclusion of copies of the auxiliary graphs $BP(b_1, b_2, b')$, $BP(b_1, b_2, b'')$, $BP(b_1, b_2, b''')$, and $BP(b', b'', b''')$. Otherwise, i.e. $B(G) \neq \emptyset$, then we have $\mathcal{C}(G) \cap \mathcal{C}(G') = \mathcal{C}(G) \setminus \{B(G)\}$ and $\mathcal{C}(G') \setminus \mathcal{C}(G)$ is precisely $B(G')$, and all cliques added by the inclusion of copies of the auxiliary graphs $BP(b_1, b_2, b')$, $BP(b_1, b_2, b'')$, $BP(b_1, b_2, b''')$, and $BP(b', b'', b''')$. By Lemma 9, all cliques added by the auxiliary graphs have size at least 3. Then, all cliques of G' have size at least 3.

Consider any 2-clique-colouring π' of G'. Since we added a copy of the auxiliary graph $BP(b', b'', b''')$, Lemma 9 states that π' assigns at least 2 colours to b', b'', b'''. Without loss of generality, suppose that π' assigns distinct colours to b' and b''. Since we added copies of the auxiliary graphs $BP(b', b_1, b_2)$ and $BP(b'', b_1, b_2)$, Lemma 9 states that π' assigns at least 2 colours to $\{b', b_1, b_2\}$ and at least 2 colours to $\{b'', b_1, b_2\}$, i.e. π' assigns a colour which is not $\pi'(b')$ to b_1 or b_2 and a colour which is not $\pi'(b'')$ to b_1 or b_2. Hence, π' assigns 2 distinct colours to b_1, b_2.

If $B(G) = \emptyset$, then $\pi'(G)$ is a 2-clique-colouring of G, since $\mathcal{C}(G) \setminus \mathcal{C}(G') = \mathcal{C}(G) \setminus \{b_1, b_2\}$ and π' assigns distinct colours to b_1, b_2. Otherwise, i.e. $B(G) \neq \emptyset$, $\pi'(G)$ is a 2-clique-colouring of G since π' assigns at least 2 colours to $\{b_1, b_2\} \subset B(G)$ and to every clique of $\mathcal{C}(G) \cap \mathcal{C}(G') = \mathcal{C}(G) \setminus B(G)$.

Now, consider a 2-clique-colouring of G that assigns 2 colours to b_1, b_2. Assign the same colour of b_1 to b' and b''. Assign the same colour of b_2 to b'''. The sets $\{b_1, b_2, b'\}$, $\{b_1, b_2, b''\}$, $\{b_1, b_2, b'''\}$, and $\{b', b'', b'''\}$ have 2 colours each. By Lemma 9, all cliques added by the inclusion of copies of the auxiliary graphs $BP(b_1, b_2, b')$, $BP(b_1, b_2, b'')$, $BP(b_1, b_2, b''')$, and $BP(b', b'', b''')$ are polychromatic. Hence, we have a 2-clique-colouring of G'.

We strengthen the result that 2-clique-colouring of (3, 1)-polar graphs is $\mathcal{NP}$-complete, now even restricting all cliques to have size at least 3, which gives a subclass of weakly chordal graphs.

Theorem 11. *The problem of 2-clique-colouring is $\mathcal{NP}$-complete for (weakly chordal) (3, 1)-polar graphs with all cliques having size at least 3.*

Proof. The problem of 2-clique-colouring a (3, 1)-polar graph with all cliques having size at least 3 is in $\mathcal{NP}$: Theorem 5 confirms that to check whether a colouring of a (3, 1)-polar graph is a 2-clique-colouring is in $\mathcal{P}$.

We prove that 2-clique-colouring (3, 1)-polar graphs with all cliques having size at least 3 is $\mathcal{NP}$-hard by reducing NAE-SAT to it. The outline of the proof follows. For every formula ϕ, a (3, 1)-polar graph G with all cliques having size at least 3 is constructed such that ϕ is satisfiable if, and only if, graph G is 2-clique-colourable. Let n (resp. m) be the number of variables (resp. clauses) in formula ϕ. We define graph G as follows.

- for each variable x_i, $1 \leq i \leq n$, we create two vertices x_i and $\overline{x}_i$. Moreover, we create edges so that the set $\{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$ induces a complete subgraph of G.

- for each variable x_i, $1 \leq i \leq n$, add a copy of the auxiliary graph $BS(x_i, \overline{x}_i)$. Vertices x_i and $\overline{x}_i$ correspond to the literals of variable x_i.

- for each clause $c_j = (l_a, l_b, l_c)$, $1 \leq j \leq m$, we add a copy of the auxiliary graph $BP(l_a, l_b, l_c)$.

Refer to Fig. 12 for an example of such construction, given a formula $\phi = (x_1 \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge (x_1 \vee x_3 \vee x_5)$.

First, we prove that the graph G is a (3, 1)-polar graph with all cliques having size at least 3.

Consider the set $\{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$. Clearly, this set is a clique with size at least 3 and also a (3, 1)-polar graph. Lemma 10 states that, for each added auxiliary graph $BS(x_i, \overline{x}_i)$ to a (3, 1)-polar graph with all cliques having size at least 3, every obtained graph remains in the class. Lemma 9 states that, for each added auxiliary graph $BP(l_{a_{c_j}}, l_{b_{c_j}}, l_{c_{c_j}})$ to a (3, 1)-polar graph with all

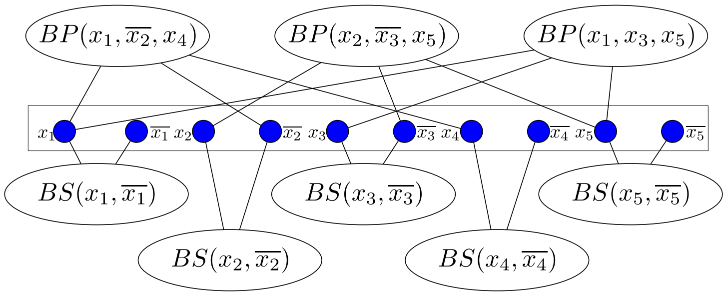


Figure 12: Example of a (3, 1)-polar graph with all cliques having size at least 3 constructed for a NAE-SAT instance $\phi = (x_1 \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge (x_1 \vee x_3 \vee x_5)$

cliques having size at least 3, every obtained graph remains in the class. Hence, G is a (3, 1)-polar graph with all cliques having size at least 3.

Such construction is done in polynomial-time. One can check with Lemmas 9 and 10 that G has $3m + 17n$ vertices.

We claim that formula ϕ is satisfiable if, and only if, there exists a 2-clique-colouring of G. Assume there exists a valuation v_ϕ such that ϕ is satisfied. We give a colouring to graph G, as follows.

- assign colour 1 to $l \in \{x_1, \overline{x}_1, \dots, x_n, \overline{x}_n\}$ if it corresponds to the literal which receives the $true$ value in v_ϕ, otherwise we assign colour 2 to it.
- extend the 2-clique-colouring to the copy of the auxiliary graph $BS(x_i, \overline{x}_i)$, for each variable x_i, $1 \leq i \leq n$, according to Lemma 10. Notice that the necessary condition to extend the 2-clique-colouring is satisfied.
- extend the 2-clique-colouring to the copy of the auxiliary graph $BP(l_a, l_b, l_c)$, for each triangle $c = \{l_a, l_b, l_c\}$, $1 \leq j \leq m$, according to Lemma 9. Notice that the necessary condition to extend the 2-clique-colouring is satisfied.

It still remains to be proved that this is indeed a 2-clique-colouring.

Consider the set $\{x_1, \overline{x}_1, \dots, x_n, \overline{x}_n\}$. Clearly, the above colouring assigns 2 colours to this set. Lemma 10 states that, for each added auxiliary graph $BS(x_i, \overline{x}_i)$ to a 2-clique-colourable weakly chordal (3, 1)-polar graph, we obtain a 2-clique-colourable graph. Lemma 9 states that, for each added auxiliary graph $BP(l_a, l_b, l_c)$ to a 2-clique-colourable weakly chordal (3, 1)-polar graph, we obtain a 2-clique-colourable graph. Hence, graph G is 2-clique-colourable.

For the converse, we now assume that G is 2-clique-colourable and we consider any 2-clique-colouring. Recall that the vertices x_i and $\overline{x}_i$ have distinct colours, since we added the auxiliary graph $BS(x_i, \overline{x}_i)$, for each variable x_i. Hence, we define v_ϕ as follows. The literal x_i is assigned $true$ in v_ϕ if the corresponding vertex has colour 1 in the clique-colouring, otherwise it is assigned $false$. Since we are considering a 2-clique-colouring, every triangle (clique) c_j,

$1 \leq j \leq m$, is polychromatic. As a consequence, there exists at least one literal with $true$ value in c_j and at least one literal with $false$ value in every clause c_j. This proves that ϕ is satisfied for valuation v_ϕ.

As an remark, a shorter alternative proof that 2-clique-colouring is $\mathcal{NP}$-complete for weakly chordal (3, 1)-polar graphs with all cliques having size at least 3 can be obtained by a reduction from POSITIVE NAE-SAT. The alternative proof follows analogously to alternative proof of Theorem 6.

On the other hand, we prove that 2-clique-colouring (2, 1)-polar graphs becomes polynomial when all cliques have size at least 3.

Theorem 12. *The problem of 2-clique-colouring is polynomial for (2, 1)-polar graphs with all cliques having size at least 3.*

PROOF. Let S be a satellite of G. First, S is an edge, since G is a (2, 1)-polar graph. Second, every clique of G has size at least 3. Then, there is a vertex of partition B that is a neighbor of both vertices of S. This implies that every satellite of G is in case $\mathcal{K}_3$. Notice that Algorithm 3 outputs a complete graph, when G is given as input. Hence, G is 2-clique-colourable (see Theorem 8).

The polynomial-time algorithm to give a 2-clique-colouring for (2, 1)-polar graphs with all cliques having size at least 3 follows. Give any 2-colouring to the vertices of B. For each satellite S, assign colour 1 to a vertex of S, if it has a neighbor in B with colour 2, otherwise assign colour 2. It is easy to check that it is a 2-clique-colouring of G, since every satellite of G is in case $\mathcal{K}_3$.

In the proof that 2-clique-colouring weakly chordal graphs is a Σ_2^P-complete problem (Theorem 3), we constructed a weakly chordal graph with K_2 cliques to force distinct colours in their extremities (in a 2-clique-colouring). We can obtain a weakly chordal graph with no cliques of size 2 by adding copies of the auxiliary graph $BS(u, v)$, for every K_2 clique $\{u, v\}$. Auxiliary graphs AK and NAS become AK' and NAS', both depicted in Fig. 13.

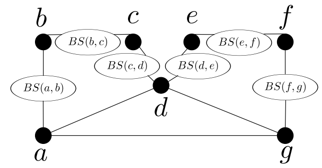
Finally, the weakly chordal graph constructed in Theorem 3 becomes a weakly chordal graph with no K_2 clique, depicted in Fig. 14.

Such construction is done in polynomial-time. Notice that, in the constructed graph of Theorem 3, every K_2 clique $\{u, v\}$ has 2 distinct colours in a clique-colouring. Hence, one can check with Lemmas 9 and 10 that the obtained graph is weakly chordal and it is 2-clique-colourable if, and only if, the constructed graph of Theorem 3 is 2-clique-colourable. This implies the following theorem.

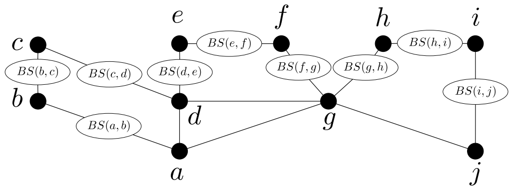
Theorem 13. *The problem of 2-clique-colouring is Σ_2^P-complete for weakly chordal graphs with all cliques having size at least 3.*

As a direct consequence of Theorem 13, we have that 2-clique-colouring is Σ_2^P-complete for perfect graphs with all cliques having size at least 3.

Corollary 14. *The problem of 2-clique-colouring is Σ_2^P-complete for perfect graphs with all cliques having size at least 3.*



(a) $AK'(a,g)$



(b) $NAS'(a,j)$

Figure 13: Auxiliary graphs $AK'(a,g)$ and $NAS'(a,j)$

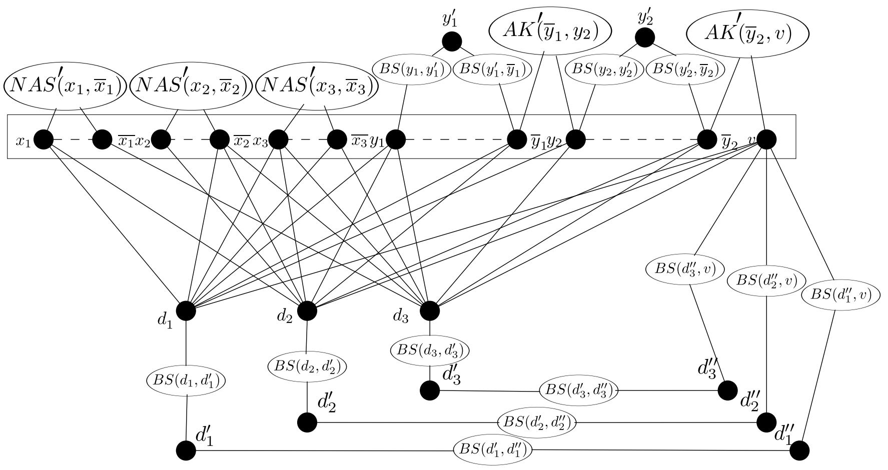


Figure 14: Graph constructed for a QSAT2 instance $\Psi = (x_1 \wedge \overline{x}_2 \wedge y_2) \vee (x_1 \wedge x_3 \wedge \overline{y}_2) \vee (\overline{x}_1 \wedge \overline{x}_2 \wedge y_1)$

4. Final considerations

Marx [9] proved complexity results for k-clique-colouring, for fixed $k \geq 2$, and related problems that lie in between two distinct complexity classes, namely Σ_2^P-complete and Π_3^P-complete. Marx approaches the complexity of clique-colouring by fixing the graph class and diversifying the problem. In the present work, our point of view is the opposite: we rather fix the (2-clique-colouring) problem and we classify the problem complexity according to the inputted graph class, which belongs to nested subclasses of weakly chordal graphs. We achieved complexities lying in between three distinct complexity classes, namely Σ_2^P-

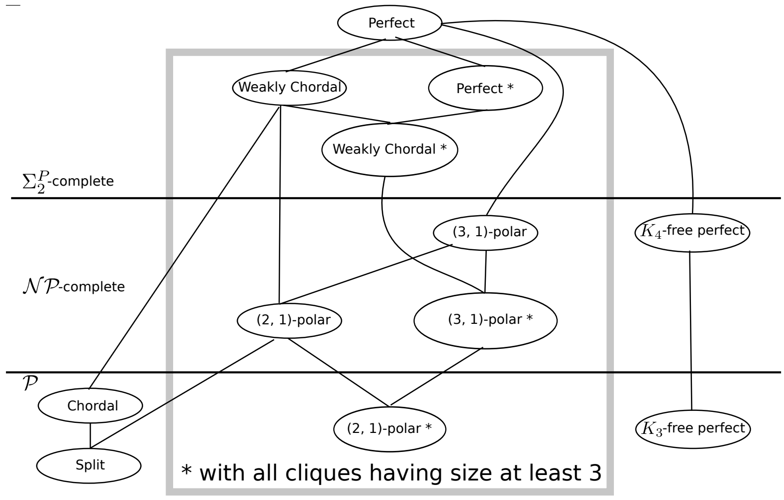


Figure 15: 2-clique-colouring complexity of perfect graphs and subclasses.

complete, $\mathcal{NP}$-complete and $\mathcal{P}$. Fig. 15 shows the relation of inclusion among the classes of graphs of Table 1. The 2-clique-colouring complexity for each class is highlighted.

Notice that the perfect graph subclasses for which the 2-clique-colouring problem is in $\mathcal{NP}$ mentioned so far in the present work satisfy that the number of cliques is polynomial. We remark that the complement of a matching has an exponential number of cliques and yet the 2-clique-colouring problem is in $\mathcal{NP}$, since no such graph is 2-clique-colourable. Now, notice that the perfect graph subclasses for which the 2-clique-colouring problem is in $\mathcal{P}$ mentioned so far in the present work satisfy that all graphs in the class are 2-clique-colourable. Macêdo Filho et al. [6] have proved that unichord-free graphs are 3-clique-colourable, but a unichord-free graph is 2-clique-colourable if and only if it is perfect. As a future work, we aim to find subclasses of perfect graphs where not all graphs are 2-clique-colourable and yet the 2-clique-colouring problem is in $\mathcal{P}$ when restricted to the class.

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