Duality for Generalized Equilibrium Problem

Flávia Morgana de O. Jacinto[*]
Departamento de Matemática, ICE, UFAM,
Manaus, AM, CEP 69077-000
Engenharia de Sistemas e Computação, COPPE, UFRJ,CP 68511,
Rio de Janeiro, RJ, CEP 21.445-970, Brazil.
morgana@cos.ufrj.br

Susana Scheimberg[†]
Departamento de Informática, IM,
Engenharia de Sistemas e Computação, COPPE, UFRJ, CP 68511,
Rio de Janeiro, RJ, CEP 21.445-970, Brazil.
susana@cos.ufrj.br

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Abstract

We introduce a Generalized Equilibrium Problem (GEP) that extends previous formulations given in the literature. We show that the (GEP) formulation contains problems not included in other equilibrium schemes, like the mixed variational inequality and the generalized quasi-variational inequality. We define a dual scheme for (GEP) based on the theory of conjugate functions that gives a unified dual analysis for interesting problems. Indeed, the lagrangian duality of a nonlinear program is a particular case of our dual scheme. We also establish necessary and sufficient optimality conditions for (GEP). These conditions become a well-known theorem given by Mosco and the dual

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results obtained by Morgan and Romaniello, which extend those introduced by Auslender and Teboulle for a variational inequality problem.

Key words: Equilibrium problems, Duality analysis, Conjugate functions.

1 Introduction

In this work, we introduce the following Generalized Equilibrium Problem:

$$(GEP)\begin{cases} \text{find } \bar{x} \in dom_X f \text{ such that} \\ f(\bar{x}, y) + \varphi(\bar{x}, y) + h(y) \geq \varphi(\bar{x}, \bar{x}) + h(\bar{x}) \text{ for all } y \in X, \end{cases} \tag{1.1}$$

where X is a real Hausdorff topological vector space, $f, \varphi : X \times X \to (-\infty, +\infty]$ and $h : X \to (-\infty, +\infty]$ are functions satisfying:

1. $dom_X f := \{x \in X : f(x, x) < +\infty\} \neq \phi$;

2. $f(x, x) \leq 0$ for all $x \in dom_X f$;

3. h is a convex function;

4. $dom f(x, .) \cap dom \varphi(x, .) \cap dom h \neq \phi$ for all $x \in dom_X f$.

Remark 1.1 *The function h is proper convex. In fact, it is a direct consequence of conditions (1), (3) and (4).*

Problem (GEP) extends the equilibrium problem given in [3] and generalizations appearing in the literature ([8], [10], [14]). The aim of this paper is twofold. Firstly, we present the advantage of (GEP) formulation that covers a wide range of important problems. Secondly, we introduce a dual scheme for the generalized equilibrium problem based on the theory of conjugate functions [20]. It gives a unified dual analysis for interesting problems that can be regarded as special cases of (GEP) like convex programming ([17], [18]), variational inequality problem ([1], [6], [9], [11], [13], [16]) and generalized quasi-variational inequality problem ([5], [15], [19]).
This paper is organized as follows. In section 2, we verify that the (GEP) problem contains previous equilibrium schemes. Furthermore, we show that there are problems belonging to our scheme but they are not included in

previous formulations. We start section 3 by considering basic definitions and results related to (GEP) problem. We also introduce a dual scheme for (GEP) and establish necessary and sufficient optimality conditions for primal-dual solutions. Finally, section 4 is devoted to ilustrate the powerful of our dual scheme. Indeed, we obtain the classical lagrangian dual problem of a nonlinear program as a particular case of it. Moreover, we get a well-known theorem given by Mosco in [16] and the dual results obtained by Morgan and Romaniello in [15] as special cases of the optimality conditions for (GEP) problem.
In this paper, any undefined terms or usage should be taken as in the Ekeland and Temam book [6] and the Van Tiel book [20].

2 The (GEP) problem and others schemes

In this section, we show that the (GEP) formulation provides a unified framework. Actually, the equilibrium problems given by Flores-Bazán in [8] and by Martínez-Legaz and Sosa in [14] can be considered particular schemes of (GEP). We also present two problems justifying our generalization.
Throughout this paper, we denote by X^* the topological dual space of X and by $\langle . , . \rangle$ the duality pairing between X^* and X.
Let us consider the *Flores-Bazán formulation* [8] defined by:

$$(FB) \text{ find } \bar{x} \in K \text{ such that } f_1(\bar{x}, y) + \varphi_1(\bar{x}, y) \geq \varphi_1(\bar{x}, \bar{x}) \text{ for all } y \in K,$$

where X is a reflexive Banach space, K is a nonempty closed convex set of X, $f_1 : K \times K \to \mathbb{R}$ and $\varphi_1 : K \times X \to (-\infty, +\infty]$ are functions satisfying $f_1(x, x) = 0$ and $K \cap dom\varphi_1(x, .) \neq \phi$ for all $x \in K$. Observe that the variables x and y are in the same set.
This formulation corresponds to a (GEP) problem by taking $f(x, y) := f_1(x, y) + \delta_{K\times K}(x, y)$, $\varphi(x, y) := \varphi_1(x, y) + \delta_K(x)$ and $h(y) := \delta_K(y)$, where δ_K denotes the indicator function of K. We note that the (FB) problem includes a class of quasi-variational inequality problems given by $f_1(x, y) = \langle F(x) - x^*, y - x\rangle$, where x^* is an element of the dual space X^*, F is an operator from K into X^* and $\varphi_1(x, y) = \delta_{Q(x)}(y)$ with Q a point-to-set operator from K into K. It is natural to assume that there exists $x \in K$ such that $x \in Q(x)$. In fact, this condition is satisfied when the problem has a finite

solution, which is the interesting case.
The *Martínez-Legaz and Sosa formulation* [14] is given by:

$$(MLS) \text{ find } \bar{x} \in K \text{ such that } f_1(\bar{x}, y) \geq 0 \text{ for all } y \in K,$$

where X is a real Hausdorff topological vector space, K is a nonempty convex set of X, $f_1 : X \times X \to [-\infty, +\infty]$ is a function such that $f_1(x, x) = 0$ and $f_1(x, .) : X \to (-\infty, +\infty]$ is a lower semicontinuous proper convex function for all $x \in K$. In this problem the variables x and y are also in the same set. There is no loss of generality if we assume $f_1 : X \times X \to (-\infty, +\infty]$. So, we can consider the (*MLS*) problem as a particular case of (*GEP*) with $f(x, y) := f_1(x, y) + \delta_K(x)$, $\varphi \equiv 0$ and $h \equiv \delta_K$. We note that the (*MLS*) formulation is an extension of the equilibrium problem given by Blum-Oettli in [3], which contains as special cases several problems, such as: convex optimization, complementarity, fixed point and variational inequality (see [3] for more details).

Now, we show the advantage of the (*GEP*) problem over the schemes above. First, we consider the *Variational Inequality* problem focused by Mosco in [16]:

$$(VI) \text{ find } \bar{x} \in domA \text{ such that } \langle A(\bar{x}), y - \bar{x}\rangle + z(y) \geq z(\bar{x}) \text{ for all } y \in X,$$

where A is an operator from a locally convex Hausdorff topological vector space X into its dual X^* and $z : X \to (-\infty, +\infty]$ is a lower semicontinuous proper convex function. It is natural to assume that $domA \cap domz \neq \phi$ because this condition occurs when the problem has a nontrivial solution. Observe that domains of A and z may be different. So, neither (*FBF*) nor (*MLS*) formulations include this problem since variables x and y may not be in the same set.
The (*VI*) problem is a specific instance of the (*GEP*) problem for $f(x, y) := \langle A(x), y - x\rangle + \delta_{domA\cap domz}(x)$, $\varphi \equiv 0$ and $h \equiv z$. We recall that (*VI*) is also called mixed variational inequality problem [12].
Finally, let us consider the *Generalized Quasi-Variational Inequality* problem ([5], [15], [19]) given by:

$$(GQVI) \begin{cases} \text{find } \bar{x} \in G(\bar{x}) \text{ such that there exists } \bar{\xi} \in A(\bar{x}) \\ \text{satisfying } \langle \bar{\xi}, y - \bar{x}\rangle \geq 0 \text{ for all } y \in G(\bar{x}), \end{cases}$$

where X is a locally convex Hausdorff topological vector space, G and A are point-to-set operators from X into X and from X into its dual X^*, respectively, such that $G(x)$ is a convex set for all $x \in X$. It is also natural to assume $\{(x, \xi) : x \in G(x), \xi \in A(x)\} \neq \phi$. Again, neither (*FBF*) nor (*MLS*) formulations include this problem since variables x and y may not be in the same set.
This problem is a particular case of the (*GEP*) problem by considering the whole space by $E := X \times X^*$ and the functions $f, \varphi : E \times E \to (-\infty, +\infty]$ and $h : E \to (-\infty, +\infty]$ by $f(v, w) = f((x, \xi), (y, \rho)) := \langle \xi, y - x \rangle + \delta_{A(x)}(\xi) + \delta_{G(x)}(x)$, $\varphi(v, w) = \varphi((x, \xi), (y, \rho)) := \delta_{G(x)}(y)$ and $h \equiv 0$.

3 A dual scheme

We start this section by presenting basic results related to the solution set of (*GEP*), denoted by S. We introduce a dual scheme for the (*GEP*) problem that maintains classical duality properties. We conclude this part by establishing optimality conditions that extend a primal-dual result given in [14]. From now on, we consider the following condition.

Assumption 3.1 *Set S is nonempty.*

3.1 Preliminaries

Lemma 3.1 *For every $\bar{x} \in S$ it holds that $\bar{x} \in dom\varphi(\bar{x}, .) \cap domh$.*

Proof. Let $\bar{x} \in S$. By condition (4) of (*GEP*) there exists $y \in X$ such that $+\infty > f(\bar{x}, y) + \varphi(\bar{x}, y) + h(y) \geq \varphi(\bar{x}, \bar{x}) + h(\bar{x})$. So, we conclude that $\bar{x} \in dom\varphi(\bar{x}, .) \cap domh$. ∎

Lemma 3.2 *If $\bar{x} \in S$, then it verifies that $f(\bar{x}, \bar{x}) = 0$.*

Proof. Let $\bar{x} \in S$, that is, $\bar{x} \in dom_X f$ and $f(\bar{x}, y) + \varphi(\bar{x}, y) + h(y) \geq \varphi(\bar{x}, \bar{x}) + h(\bar{x})$ for all $y \in X$. Taking $y = \bar{x}$ in this inequality and using the Lemma above, we obtain that $f(\bar{x}, \bar{x}) \geq 0$. On the other hand, from condition (2) of (*GEP*) it holds that $f(\bar{x}, \bar{x}) \leq 0$. So, we obtain that $f(\bar{x}, \bar{x}) = 0$. ∎

Let us consider the bifunction F from X into X which assigns to each $x \in X$ the function $F_x : X \to (-\infty, +\infty]$ given by the sum of f and φ, that is,

$F_x(y) := F(x,y) = f(x,y) + \varphi(x,y)$. We also consider the function $v : X \to (-\infty, +\infty]$ defined by $v(x) := \varphi(x,x) + h(x)$.

Remark 3.1 *Condition (4) of (GEP) implies that F_x and v are proper functions for all $x \in dom_X f$.*

The result below extends the one given in [14]. It characterizes a solution of (GEP) as a solution of an optimization problem.

Lemma 3.3 *Let f, φ and h be functions verifying conditions (1)-(4) of (GEP). The point $\bar{x}$ is a solution of (GEP) if and only if it holds that*

$$\inf_{y \in X} \{F_{\bar{x}}(y) + h(y)\} = F_{\bar{x}}(\bar{x}) + h(\bar{x}) = v(\bar{x}) < +\infty. \tag{3.1}$$

Proof. Let $\bar{x} \in S$. Therefore, $\bar{x} \in dom_X f$ and $f(\bar{x}, y) + \varphi(\bar{x}, y) + h(y) \geq \varphi(\bar{x}, \bar{x}) + h(\bar{x}) = v(\bar{x})$ for all $y \in X$. Hence

$$\inf_{y \in X} \{F_{\bar{x}}(y) + h(y)\} \geq v(\bar{x}).$$

On the other hand, by Lemmas 3.1 and 3.2 we obtain that

$$+\infty > v(\bar{x}) = \varphi(\bar{x}, \bar{x}) + h(\bar{x}) = F_{\bar{x}}(\bar{x}) + h(\bar{x}) \geq \inf_{y \in X} \{F_{\bar{x}}(y) + h(y)\}.$$

Therefore, it results that $\inf_{y \in X} \{F_{\bar{x}}(y) + h(y)\} = F_{\bar{x}}(\bar{x}) + h(\bar{x}) = v(\bar{x}) < +\infty$. Conversely, assume that (3.1) holds. Hence,

$$F_{\bar{x}}(y) + h(y) \geq \inf_{y \in X} \{F_{\bar{x}}(y) + h(y)\} = v(\bar{x}) = \varphi(\bar{x}, \bar{x}) + h(\bar{x}) \text{ for all } y \in X.$$

The second equality in (3.1) implies that $f(\bar{x}, \bar{x}) = 0$, so, $\bar{x} \in dom_X f$. Therefore, we obtain that $\bar{x} \in S$. ■

3.2 A dual scheme for (GEP)

Let X^* be the dual space of X and let f^* be the conjugate of a function f defined on X (see for example [20]). Consider the (GEP) problem. According to Lemma 3.3, the function $v(.)$ can be called the *primal* function of (GEP). Next, we introduce dual concepts related to the (GEP) problem.

Definition 3.1 *The primal-dual function $L : X \times X^* \to [-\infty, +\infty]$ is given by*

$$L(x,\xi) = \begin{cases} -h^*(-\xi) - F_x^*(\xi), & if\ D_\xi \neq \phi \text{ and } x \in D_\xi \\ +\infty, & if\ D_\xi \neq \phi \text{ and } x \notin D_\xi \\ -\infty, & if\ D_\xi = \phi, \end{cases}$$

where $D_\xi := \{x \in dom_X f : F_x^*(\xi) < +\infty\}$.

Definition 3.2 *The dual problem of (GEP) is*

$$(DGEP) \begin{cases} \sup g(\xi) \\ \xi \in X^*, \end{cases}$$

where $g : X^* \to [-\infty, +\infty)$ is the dual function defined by

$$g(\xi) = \inf_{x \in X} L(x,\xi) = \begin{cases} -h^*(-\xi) - \sup_{x \in D_\xi} F_x^*(\xi), & if\ D_\xi \neq \phi \\ -\infty, & otherwise. \end{cases}$$

We observe that Remarks 1.1 and 3.1 imply that $h^*(\xi) > -\infty$ for all $\xi \in X^*$ and $F_x^*(\xi) \in \mathrm{I\!R}$ whenever $x \in D_\xi$. So, if $D_\xi \neq \phi$ then $\sup_{x \in D_\xi} F_x^*(\xi) > -\infty$. Therefore, the functions L and g are well defined.

Definition 3.3 *A point $(x,\xi) \in X \times X^*$ is called a primal-dual feasible point whenever $x \in D_\xi$.*

The next results show that our dual scheme preserves classical dual characteristics.

Proposition 3.1 *(weak duality) Let $(x,\xi) \in X \times X^*$ be a primal-dual feasible point. Then, for every $y \in X$ it holds that*

$$L(x,\xi) \leq F_x(y) + h(y) \ \text{ and } \ L(x,\xi) \leq v(x). \tag{3.2}$$

Moreover, $g(\xi) \leq F_x(y) + h(y)$ and $g(\xi) \leq v(x)$.

Proof. Let $x \in D_\xi$. So, it must x be in $dom_x f$ and $F_x^*(\xi) \in \mathrm{I\!R}$. By Fenchel's inequality [20] (or Generalized Young inequality [21]) we have

$$F_x(y) \geq \langle \xi, y \rangle - F_x^*(\xi) \ \text{ for all } y \in X.$$

So, it holds that

$$F_x(y) + h(y) \geq \langle \xi, y \rangle + h(y) - F_x^*(\xi) \text{ for all } y \in X.$$

Thus, we obtain that

$$\inf_{y \in X}\{F_x(y) + h(y)\} \geq \inf_{y \in X}\{\langle \xi, y \rangle + h(y)\} - F_x^*(\xi) =$$

$$-\sup_{y \in X}\{\langle -\xi, y \rangle - h(y)\} - F_x^*(\xi) = -h^*(-\xi) - F_x^*(\xi).$$

Therefore, by Definition 3.1, we get

$$L(x, \xi) \leq F_x(y) + h(y) \text{ for all } y \in X.$$

Now, taking $y := x$ in the inequality above and using condition (2) of (GEP) it follows that

$$L(x, \xi) \leq F_x(x) + h(x) \leq \varphi(x, x) + h(x) = v(x).$$

Then, we obtain (3.2). Moreover, for every $y \in X$, $x \in D_\xi$ we have

$$F_x(y) + h(y) \geq \inf_{x \in D_\xi}\{F_x(y) + h(y)\} \geq \inf_{x \in D_\xi}\{L(x, \xi)\} = \inf_{x \in X}\{L(x, \xi)\} = g(\xi),$$

where the first equality holds since $D_\xi \neq \phi$ and the last one corresponds to the definition of $g(.)$. Again, taking $y := x$ we get the inequality $g(\xi) \leq v(x)$. ■

Remark 3.2 *Observe that $L(x, \xi) < +\infty$ whenever $x \in D_\xi$. In fact, it is a consequence of the first inequality of (3.2) and condition (4) of (GEP).*

The next statement gives necessary optimality conditions to (GEP).

Theorem 3.1 (Necessary optimality conditions) *Assume that f, φ and h are functions satisfying conditions (1)-(4) of (GEP). If $\bar{x} \in X$ is a solution of (GEP) such that verifies the following conditions:*

(H1) *$F_{\bar{x}}$ is a convex function.*
(H2) *$\partial(F_{\bar{x}} + h) = \partial F_{\bar{x}} + \partial h$.*

Then there exists $\bar{\xi} \in X^*$ such that

$$(\bar{x}, \bar{\xi}) \text{ is a primal-dual feasible point and } L(\bar{x}, \bar{\xi}) \geq v(\bar{x}). \tag{3.3}$$

Proof. Let $\bar{x} \in S$ verifying conditions **(H1)** and **(H2)**. By conditions (1)-(4) of (*GEP*), Lemmas 3.1 and 3.3 and the characterization of a solution of a convex problem [20] we obtain that

$$0 \in \partial(F_{\bar{x}}(\bar{x}) + h(\bar{x})) = \partial F_{\bar{x}}(\bar{x}) + \partial h(\bar{x}).$$

So, there exists $\bar{\xi} \in X^*$ such that $\bar{\xi} \in \partial F_{\bar{x}}(\bar{x})$ and $-\bar{\xi} \in \partial h(\bar{x})$. Applying the characterization of a subgradient in terms of the conjugate function ([6], p. 21, Proposition 5.1) we have

$$F^*_{\bar{x}}(\bar{\xi}) + F_{\bar{x}}(\bar{x}) = \langle \bar{\xi}, \bar{x} \rangle \text{ and } h^*(-\bar{\xi}) + h(\bar{x}) = \langle -\bar{\xi}, \bar{x} \rangle. \tag{3.4}$$

Summing these two equalities and considering (3.1) we get

$$-F^*_{\bar{x}}(\bar{\xi}) - h^*(-\bar{\xi}) = F_{\bar{x}}(\bar{x}) + h(\bar{x}) = v(\bar{x}).$$

Moreover, since $\bar{x} \in dom_X f$ the first equality in (3.4) implies that $\bar{x} \in D_{\xi}$. The desired result follows from Definitions 3.1 and 3.3. ■

Remark 3.3 *Condition* **(H2)** *is a qualification constraint that is satisfied under different assumptions. For example, it holds in the following situations:*
(a) X is a real Hausdorff topological vector space and there exists $y \in domF_x \cap domh$ where F_x (or h) is continuous ([20], Theorem 5.38);
(b) X is a Banach space, F_x and h are lower semicontinuous convex functions with $domF_x \cap domh \neq \phi$ and $(epiF^*_x) + (epih^*)$ is a weak closed set in $X \times \mathbb{R}$, where $epi f$ means the epigraph of the function f [4];
(c) $X = \mathbb{R}^n$ and $ir(domF_x) \cap ir(domh) \neq \phi$, where $ir(C)$ denotes the relative interior of the set C [18].

Now, we establish sufficient optimality conditions to (*GEP*).

Theorem 3.2 (Sufficient optimality conditions) *Assume that f, φ and h are functions satisfying conditions (1)-(4) of (GEP). If a point $(\bar{x}, \bar{\xi}) \in X \times X^*$ satisfy (3.3) and the condition* **(H1)** *holds for $\bar{x}$, then $\bar{x}$ is a solution of (GEP).*

Proof. Using (3.3) and the first inequality in (3.2), we obtain that

$$F_{\bar{x}}(y) + h(y) \geq L(\bar{x}, \bar{\xi}) \geq v(\bar{x}) \text{ for all } y \in X. \tag{3.5}$$

Taking the infimum on y in (3.5), we get

$$+\infty > \inf_{y\in X}\{F_{\bar{x}}(y) + h(y)\} \geq v(\bar{x}), \tag{3.6}$$

where the first inequality above results from condition (4) of (GEP) since $\bar{x} \in D_{\bar{\xi}} = \{x \in dom_X f : F_x^*(\bar{\xi}) < +\infty\}$. Moreover, by condition (2) of (GEP) it follows that

$$v(\bar{x}) = \varphi(\bar{x}, \bar{x}) + h(\bar{x}) \geq F_{\bar{x}}(\bar{x}) + h(\bar{x}),$$

which together with (3.6) imply that

$$\inf_{y\in X}\{F_{\bar{x}}(y) + h(y)\} = F_{\bar{x}}(\bar{x}) + h(\bar{x}) = v(\bar{x}) < +\infty.$$

Therefore, by Lemma 3.3 we conclude that $\bar{x}$ is a solution of (GEP). ∎

Let us observe that Theorem 3.2 given in [14] is obtained from Theorems 3.1 and 3.2 when they are applied to the (MLS) problem.

4 Applications

In this section, we consider problems that belong to Optimization, Variational Inequalities and Generalized Quasi-Variational Inequalities. We give a suitable (GEP) formulation for each of them. We show that our dual scheme applied to nonlinear programming gives the classical lagrangian dual program (see for example [17]). When we use our necessary and sufficient conditions to the other problems we obtain the dual results introduced in [16] and [15]. Through this section, given two Hausdorff topological vector spaces V and W, the pairing between $V \times W$ and $V^* \times W^*$ is written, classically, as

$$\langle(\xi, \rho), (x, y)\rangle_{V\times W} = \langle\xi, x\rangle_V + \langle\rho, y\rangle_W.$$

Since in general there is no possibility of ambiguity, the pairing between any Hausdorff topological vector space and its dual space will be denoted by $\langle . \,, . \rangle$.

4.1 Convex Optimization

Let us consider the following *primal nonlinear programming problem* [17]:

$$(P) \quad \min_{y \in K \subset \mathbb{R}^n} \psi(y) \quad \text{such that } g_i(y) \leq 0, \;\; i = 1, ..., m,$$

where the functions $\psi, g_i : \mathbb{R}^n \to \mathbb{R}$ are convex and K is a nonempty closed convex set such that $K \cap \{y \in \mathbb{R}^n : g_i(y) \leq 0, \; i = 1, ..., m\} \neq \phi$. We replace (P) by the following problem:

$$(P_1) \begin{cases} \text{find } \bar{v} = (\bar{x}, \bar{z}) \in X \text{ such that} \\ \varphi(\bar{v}, w) + h(w) \geq \varphi(\bar{v}, \bar{v}) + h(\bar{v}) \text{ for all } w = (y, u) \in X, \end{cases} \tag{4.1}$$

where $X = \mathbb{R}^n \times \mathbb{R}^m$, $\varphi(v, w) = \psi(y) + \delta_{\mathbb{R}^m_-}(g(y) + u) + \delta_K(y)$ and $h(w) = \delta_{\mathbb{R}^m_+}(u)$. The i-component of $g(y) + u : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is $(g_i(y) + u_i)$.
Problems (P) and (P_1) are equivalent in the following way: $\bar{x} \in \mathbb{R}^n$ is a solution of (P) if and only if there exists $\bar{z} \in \mathbb{R}^m$ such that $(\bar{x}, \bar{z})$ solves (P_1). Observe that u is not a classical slack variable since we do not require $g(y) + u = 0$ but $g(y) + u \leq 0$. We say that u is a *quasi-slack variable.*
Problem (P_1) is a (GEP) problem where $f \equiv 0$. Let us observe that $F_v(w) = \varphi(v, w)$ does not depend on v. Therefore, we can drop v in F_v and in F_v^*. For each $\nu \in X^* = \mathbb{R}^n \times \mathbb{R}^m$ it holds that $D_\nu \neq \phi$ if and only if $F^*(\nu) < +\infty$. Thus, we have that $D_\nu = X$, or $D_\nu = \phi$.
Now, we determine the *primal-dual function associated with* (P_1). We denote $\nu = (\xi, \eta) \in X^* = \mathbb{R}^n \times \mathbb{R}^m$. If $D_\nu = \phi$ then $L(v, \nu) = -\infty$ for all $v \in X$, otherwise, it holds that $L(v, \nu) = -h^*(-\nu) - F^*(\nu)$ for all $v \in X$. In order to obtain $L(v, \nu)$, we calculate:

$$h^*(-\nu) = h^*(-\xi, -\eta) = \sup_{(y,u) \in \mathbb{R}^n \times \mathbb{R}^m} \{\langle -\xi, y\rangle + \langle -\eta, u\rangle - \delta_{\mathbb{R}^m_+}(u)\}.$$

Therefore, it results that

$$h^*(-\nu) = \begin{cases} 0, & \text{if } \xi = 0 \text{ and } \eta \geq 0 \\ +\infty, & \text{otherwise.} \end{cases}$$

So, $L(v, \nu) = -\infty$ if $\xi \neq 0$ or $\eta < 0$ and $D_\nu \neq \phi$. Thus, it is enough to calculate $F^*(\nu)$ for $\nu = (0, \eta)$ with $\eta \geq 0$:

$$F^*(0, \eta) = \sup_{y,u} \{\langle \eta, u\rangle - F(y, u)\} = \sup_{y,u} \{\langle \eta, u\rangle - \psi(y) - \delta_{\mathbb{R}^m_-}(g(y) + u) - \delta_K(y)\}.$$

Taking $s = g(y) + u$ it follows that $u = s - g(y)$ and

$$F^*(0,\eta) = \sup_{y,s}\{\langle -\eta, g(y)\rangle - \psi(y) - \delta_K(y) + \langle \eta, s\rangle - \delta_{\mathbb{R}^m_-}(s)\} = \sup_{y\in K}\{\langle -\eta, g(y)\rangle - \psi(y)\}.$$

So, we find that the primal-dual function is

$$L(v,\nu) = L(v,(\xi,\eta)) = \begin{cases} \inf_{y\in K}\{\psi(y) + \langle \eta, g(y)\rangle\}, & \text{if } \xi = 0,\ \eta \geq 0 \text{ and } D_\nu \neq \phi \\ -\infty, & \text{otherwise.} \end{cases}$$

Thus, the *dual function associated to* (P_1) is given by

$$g(\nu) = \inf_{v\in X} L(v,\nu) = \begin{cases} \inf_{y\in K}\{\psi(y) + \langle \eta, g(y)\rangle\}, & \text{if } \nu = (0,\eta),\ \eta \geq 0,\ D_\nu \neq \phi \\ -\infty, & \text{otherwise.} \end{cases}$$

Therefore, the following result holds:

Proposition 4.1 *The dual problem (DGEP) of* (P_1) *is the classical lagrangian dual problem of (P).*

Remark 4.1 *The linear programming problem defined by*

$$(LP)\ \min_{y\in\mathbb{R}^n} \langle c, y\rangle \text{ such that } Ay \geq b,\ y \geq 0,$$

is a particular case of (P), where the involved functions are linear and K *is the whole space. Hence, its classical dual program is obtained as above.*

4.2 Variational Inequality

In this subsection, we consider the variational inequality problem (VI) given in section 2. We show that Theorems 3.1 and 3.2 applied to the associated (GEP) problem give the sufficient and necessary conditions established by Mosco [16].

Here, we consider the (VI) problem in a more general framework where we drop the injectivity assumption on A, that is, A^{-1} defined by $A^{-1}(\xi) = \{y \in domA : A(y) = \xi\}$ can be a point-to-set operator from X^* into X. Under the blanket assumption $domA \cap domz \neq \phi$, we can replace (VI) by the following (GEP) problem:

$$(VI_1)\begin{cases} \text{find } \bar{x} \in dom_X f \text{ such that} \\ f(\bar{x},y) + h(y) \geq h(\bar{x}) \text{ for all } y \in X, \end{cases} \quad (4.2)$$

where $f(x,y) := \langle A(x), y - x\rangle + \delta_{domA}(x) + \delta_{domz}(x)$, $\varphi \equiv 0$, $h \equiv z$ and $dom_X f = domA \cap domz$.

Remark 4.2 *The function z^* is proper and convex. In fact, since z is a lower semicontinuous proper convex function, we have that $z^{**} = (z^*)^* = z$ ([6], p. 18, Proposition 4.1) and $z^*(\xi) > -\infty$ for all $\xi \in X^*$. In addition z^* is not identically $+\infty$ because if $z^* \equiv +\infty$ then $z = z^{**} \equiv -\infty$, what is a contradiction.*

In order to examine the necessary and sufficient optimality conditions for (VI_1), we calculate its *primal-dual function* L. We have that $F_x(y) = f(x, y)$. Let $x \in domA \cap domz$. Then, it results that

$$F_x^*(\xi) = \sup_{y\in X}\{\langle \xi - A(x), y\rangle\} + \langle A(x), x\rangle = \begin{cases} \langle \xi, x\rangle, & \text{if } x \in A^{-1}(\xi) \cap domz \\ +\infty, & \text{if } x \notin A^{-1}(\xi),\ x \in domA \cap domz. \end{cases}$$

Hence $D_\xi = \{x \in domA \cap domz : F_x^*(\xi) < +\infty\} = \{x \in X : x \in A^{-1}(\xi) \cap domz\}$. So, $D_\xi \neq \phi$ if and only if $A^{-1}(\xi) \cap domz \neq \phi$. Therefore, the *primal-dual function associated to* (VI_1) is given by

$$L(x,\xi) = \begin{cases} -z^*(-\xi) - \langle \xi, x\rangle, & \text{if } x \in A^{-1}(\xi) \cap domz \\ +\infty, & \text{if } x \notin A^{-1}(\xi) \cap domz \neq \phi \\ -\infty, & \text{if } A^{-1}(\xi) \cap domz = \phi. \end{cases} \tag{4.3}$$

Theorems 3.1 and 3.2 allow us to associate to (VI) a Dual Variational Inequality defined by

$$(DVI)\begin{cases} \text{find } \xi^* \in domA' \text{ such that there exists } u^* \in A'(\xi^*) \\ \text{satisfying } \langle \xi - \xi^*, u^*\rangle + z^*(\xi) \geq z^*(\xi^*) \text{ for all } \xi \in X^*, \end{cases} \tag{4.4}$$

where $A' : X^* \to \mathcal{P}(X)$ is given by $A'(\eta) := -A^{-1}(-\eta) = \{v \in X : \eta = -A(-v)\}$. Actually, we establish the following property:

Proposition 4.2 *Let $\bar{x} \in X$. The point $\bar{x}$ is a solution of (VI) if and only if $\bar{\xi} = -A(\bar{x})$ is a solution of (DVI) with $u^* = -\bar{x} \in A'(\bar{\xi})$ verifying the inequality of (4.4). Moreover, $\bar{x}$ and $\bar{\xi}$ are solutions of (VI) and (DVI), respectively, if and only if $\bar{x} \in -A'(\bar{\xi})$ or $\bar{\xi} = -A(\bar{x})$ and $z(\bar{x}) + z^*(\bar{\xi}) = \langle \bar{x}, \bar{\xi}\rangle$ holds.*

Proof. Let $\bar{x}$ be a solution of (VI), then $\bar{x}$ solves (VI_1). Hence, $\bar{x} \in domA \cap domz$. Since $A(\bar{x}) \in X^*$ we get $F_{\bar{x}}$ be a continuous linear functional on X. Therefore, $F_{\bar{x}}$ is convex and $\partial(F_{\bar{x}} + h) = \partial F_{\bar{x}} + \partial h$ ([20], Theorem

5.38). So, we can apply Theorem 3.1 to conclude that there exists $\tilde{\xi} \in X^*$ such that $(\bar{x}, \tilde{\xi})$ is a primal-dual feasible point and

$$L(\bar{x}, \tilde{\xi}) \geq v(\bar{x}). \tag{4.5}$$

From $\bar{x} \in D_{\tilde{\xi}}$, we have that $\bar{x} \in A^{-1}(\tilde{\xi}) \cap dom z$ and $L(\bar{x}, \tilde{\xi}) = -z^*(-\tilde{\xi}) - \langle \tilde{\xi}, \bar{x} \rangle$. Since $\varphi \equiv 0$, inequality (4.5) becomes

$$-z^*(-\tilde{\xi}) - \langle \tilde{\xi}, \bar{x} \rangle \geq z(\bar{x}). \tag{4.6}$$

Using the relation $z \equiv z^{**}$ and the definition of z^{**} we obtain that

$$z(\bar{x}) = z^{**}(\bar{x}) \geq \langle \xi, \bar{x} \rangle - z^*(\xi) \text{ for all } \xi \in X^*. \tag{4.7}$$

Thus, (4.6) and (4.7) imply

$$\langle -\tilde{\xi}, \bar{x} \rangle - z^*(-\tilde{\xi}) \geq \langle \xi, \bar{x} \rangle - z^*(\xi) \text{ for all } \xi \in X^*.$$

By Remark 3.2 we have that $-z^*(-\tilde{\xi})$ is finite. So, it holds that

$$\langle -\tilde{\xi} - \xi, \bar{x} \rangle + z^*(\xi) \geq z^*(-\tilde{\xi}) \text{ for all } \xi \in X^*. \tag{4.8}$$

Taking $\bar{\xi} := -\tilde{\xi}$ and $u^* := -\bar{x}$ we obtain that $u^* \in -A^{-1}(\tilde{\xi}) = -A^{-1}(-\bar{\xi}) = A'(\bar{\xi})$ and

$$\langle -\tilde{\xi} - \xi, \bar{x} \rangle = \langle \bar{\xi} - \xi, \bar{x} \rangle = \langle \xi - \bar{\xi}, u^* \rangle. \tag{4.9}$$

It follows from (4.8) and (4.9) that $\bar{\xi} = -A(\bar{x}) \in dom A'$ with $u^* = -\bar{x} \in A'(\bar{\xi})$ is a solution of (DVI).
On the other hand, assume that $\bar{\xi} = -A(\bar{x})$ is a solution of (DVI) with $u^* = -\bar{x} \in A'(\bar{\xi})$. Taking $\tilde{\xi} := -\bar{\xi}$, we obtain that $\bar{x} \in A^{-1}(\tilde{\xi})$ and the inequality of (DVI) is equivalent to

$$\langle -\xi - \tilde{\xi}, \bar{x} \rangle + z^*(\xi) \geq z^*(-\tilde{\xi}) \text{ for all } \xi \in X^*.$$

Using that z^* is a proper convex function (see Remark 4.2), the last inequality implies that $z^*(-\tilde{\xi})$ is finite. Hence, it holds that

$$+\infty > \langle -\tilde{\xi}, \bar{x} \rangle - z^*(-\tilde{\xi}) \geq \langle \xi, \bar{x} \rangle - z^*(\xi) \text{ for all } \xi \in X^*. \tag{4.10}$$

From the last inequality and the definition of z^{**} it results that

$$\langle -\tilde{\xi}, \bar{x} \rangle - z^*(-\tilde{\xi}) \geq z^{**}(\bar{x}). \tag{4.11}$$

The relation $z^{**} \equiv z$ together with (4.10) and (4.11) imply that $\bar{x} \in domz$. So, we obtain that $\bar{x} \in D_{\tilde{\xi}}$, that is, $(\bar{x}, \tilde{\xi})$ is a primal-dual feasible point. Furthermore, it holds that

$$L(\bar{x}, \tilde{\xi}) = -z^*(-\tilde{\xi}) - \langle \tilde{\xi}, \bar{x} \rangle \geq z(\bar{x}) = v(\bar{x}).$$

Then, by Theorem 3.2 we have that $\bar{x}$ solves (VI_1). So, $\bar{x}$ is a solution of (VI).
Now, we prove the last part. If $\bar{x}$ is a solution of (VI), that is, it solves (VI_1), then $\bar{\xi} = -A(\bar{x})$ with $\bar{u} = -\bar{x}$ solves (DVI). On the other hand, if $\bar{\xi}$ is a solution of (DVI) with $\bar{u} \in A'(\bar{\xi})$, then $\bar{x} = -\bar{u}$ solves (VI_1). Moreover, $\bar{x}$ solves (VI) and $\bar{\xi}$ solves (DVI) if and only if inequalities (4.2) and (4.4) are hold which are equivalent to $\bar{\xi} \in \partial z(\bar{x})$, $\bar{x} \in \partial z^*(\bar{\xi})$ with $\bar{x} \in A'(\bar{\xi})$ or $\bar{\xi} = -A(\bar{x})$. Thus, using the characterization of a subgradient in terms of the conjugate function ([6], p. 21, Proposition 5.1) we have

$$z(\bar{x}) + z^*(\bar{\xi}) = \langle \bar{\xi}, \bar{x} \rangle \text{ and } z^*(\bar{\xi}) + z^{**}(\bar{x}) = \langle \bar{\xi}, \bar{x} \rangle.$$

From $z^{**} = z$ we obtain the desired result. ■

The property above becomes the sufficient and necessary condition considered by Mosco ([16], Theorem 1) when the operator A is injective.
If A is a point-to-set operator from X into X^* an analog of Proposition 4.2 holds.

4.3 Generalized Quasi-Variational Inequality

In this subsection, we get the dual scheme presented by Morgan and Romaniello which extends the one obtained by Auslender and Teboulle in [1] for a variational inequality problem. Indeed, they are obtained as corollaries of Theorems 3.1 and 3.2 applied to a (GEP) problem associated to the *Generalized Quasi-Variational Inequality* considered in [15].
Through this part, we consider the $(GQVI)$ problem given in section 2 under the conditions established in [15], that is, X is a Banach space, $G(x) = \{t \in X : q_i(x,t) \leq 0,\ i = 1, ..., m\}$ and $q_i(x,.) : X \to (-\infty, +\infty]$ is a lower semicontinuous proper convex function for all i.
In [15] is introduced the following dual problem for this (GQVI):

$$(DGQVI) \begin{cases} \text{find } \eta^* \in \mathbb{R}^m_+ \text{ such that there exists } d^* \in K(\eta^*) \\ \text{satisfying } \langle d^*, \eta - \eta^* \rangle \geq 0 \text{ for all } \eta \in \mathbb{R}^m_+, \end{cases} \tag{4.12}$$

where $K(\eta) := \{ -Q(x,x) : 0 \in A(x) + \sum_{j=1}^{m} \eta_i \partial_2 q_i(x,x)\}$ for all $\eta \in \mathrm{I\!R}^m_+$ and $Q(x,y) = (q_1(x,y), ..., q_m(x,y))$. Here, $\partial_2 q_i(x,t)$ is the subdifferential of the function $q_i(x,.)$ at the point t, that is,

$$\partial_2 q_i(x,t) = \{g^* \in X^* : q_i(x,y) \geq q_i(x,t) + \langle g^*, y-t\rangle \ \forall\, y \in X\}.$$

In order to study optimality conditions for primal solution, it is natural to ask for $\{(x,\xi) \in X \times X^* : \xi \in A(x), x \in G(x)\} \neq \phi$.
We follow the ideas considered in subsection 4.1 . We take advantage of the particular structure of the point-to-set operator G by introducing *quasi-slack variables*. Instead of $(GQVI)$ formulation, we consider the following problem:

$$(GQVI_1) \left\{ \begin{array}{c} \text{find } \bar{s} \in dom_{\mathcal{X}} f \text{ such that} \\ f(\bar{s},t) + \varphi(\bar{s},t) + h(t) \geq \varphi(\bar{s},\bar{s}) + h(\bar{s}) \text{ for all } t \in \mathcal{X}, \end{array} \right. \tag{4.13}$$

where $\mathcal{X} := X \times X^* \times \mathrm{I\!R}^m$, $\bar{s} := (\bar{v}, \bar{z}) = (\bar{x}, \bar{\xi}, \bar{z})$, $t := (w,u) = (y,\rho,u)$, $f, \varphi : \mathcal{X} \times \mathcal{X} \to (-\infty, +\infty]$ and $h : \mathcal{X} \to (-\infty, +\infty]$ are functions defined by $f(s,t) := \langle \xi, y - x\rangle + \delta_{A(x)}(\xi) + \delta_{G(x)}(x)$, $\varphi(s,t) := \delta_{\mathrm{I\!R}^m_-}(Q(x,y) + u)$, $h(t) := \delta_{\mathrm{I\!R}^m_+}(u)$ and $dom_{\mathcal{X}} f = \{(x,\xi,z) \in \mathcal{X} : \xi \in A(x),\ x \in G(x)\}$. Note that $dom_X f \neq \phi$.
Problems $(GQVI)$ and $(GQVI_1)$ are equivalent in the following sense: $\bar{x}$ is a solution of $(GQVI)$ if and only if there exist $\bar{\xi} \in X^*$ and $\bar{z} \in \mathrm{I\!R}^m$ such that $(\bar{x}, \bar{\xi}, \bar{z})$ is a solution of $(GQVI_1)$.
Let us recall that $\mathcal{X}^* = \mathcal{X}^* \times \mathcal{X}^{**} \times \mathrm{I\!R}^m$ ([7], p. 68, Proposition 2). In order to obtain dual results related to $(GQVI_1)$, we calculate the conjugate function of h and F_s for $s \in dom_X f$. Let $\nu = (\nu_1, \nu_2, \nu_3) \in \mathcal{X}^*$. By the definition of h^* it can be shown that

$$h^*(-\nu) = \left\{ \begin{array}{cc} 0, & \text{if } \nu_1 = 0,\ \nu_2 = 0 \text{ and } \nu_3 \geq 0 \\ +\infty, & \text{otherwise.} \end{array} \right. \tag{4.14}$$

Hence, it is enough to calculate $F_s^*(\nu)$ for $s \in dom_{\mathcal{X}} f$ and $\nu = (0,0,\nu_3)$ with $\nu_3 \geq 0$. Therefore, we have

$$F_s^*(\nu) = \sup_{(y,\rho,u)\in\mathcal{X}} \{\langle \nu_3, u\rangle - \langle \xi, y-x\rangle - \delta_{\mathrm{I\!R}^m_-}(Q(x,y)+u)\}.$$

Following the same argument used in subsection 4.1, we take $n = Q(x,y)+u$. Thus,

$$\begin{array}{rcl} F_s^*(0,0,\nu_3) & = & \sup_{(y,\rho,n)\in\mathcal{X}} \{\langle \nu_3, n - Q(x,y)\rangle - \langle \xi, y-x\rangle - \delta_{\mathrm{I\!R}^m_-}(n)\} \\ & = & \sup_{y\in X} \{\langle -\nu_3, Q(x,y)\rangle - \langle \xi, y-x\rangle\}. \end{array} \tag{4.15}$$

We observe that condition **(H1)** is verified for all $s = (x, \xi, z) \in dom_{\mathcal{X}} f$. Indeed, it holds that $x \in G(x)$ and $\xi \in A(x)$. Thus, for $t = (w, u) = (y, \rho, u)$ we have

$$F_s(t) = f(s,t) + \varphi(s,t) = \langle \xi, y - x \rangle + \delta_{\mathbb{R}^m_-}(Q(x,y) + u).$$

Considering application $\gamma(.) := \delta_{\mathbb{R}^m_-}(Q(x,.) + u)$, it is easy to verify that its epigraph $epi\gamma = \{(t,\alpha) \in \mathcal{X} \times \mathbb{R} : \gamma(t) \leq \alpha\}$ is a convex set, since $q_i(x,.)$ is convex for all i. So, $F_s(.)$ is a convex function too.
Now, we present three results under additional conditions on G assumed by Morgan and Romaniello in [15].

Proposition 4.3 *Consider problem* $(GQVI_1)$. *Let* $(\bar{s}, \bar{\nu}) = ((\bar{x}, \bar{\xi}, -Q(\bar{x}, \bar{x})), \bar{\nu})$ *be a primal-dual feasible point such that*

$$L(\bar{s}, \bar{\nu}) \geq v(\bar{s}) \tag{4.16}$$

and let $\cap_{i=1}^m dom(q_i(\bar{x}, .))$ *be a nonempty open subset of* X. *Then, there exists* $\bar{\eta} \in \mathbb{R}^m_+$ *such that* $(\bar{x}, \bar{\eta})$ *satisfies the following "Generalized Karush-Kuhn-Tucker conditions" :*

$(KKT)_1 :\ \bar{x} \in G(\bar{x})$;
$(KKT)_2 :\ 0 \in A(\bar{x}) + \sum_{i=1}^m \bar{\eta}_i \partial_2 q_i(\bar{x}, \bar{x})$;
$(KKT)_3 :\ Q(\bar{x}, \bar{x}) \in N_{\mathbb{R}^m_+}(\bar{\eta})$.

Proof. Since $(\bar{s}, \bar{\nu})$ is a primal-dual feasible point, we have that $\bar{s} = (\bar{x}, \bar{\xi}, -Q(\bar{x}, \bar{x})) \in D_{\bar{\nu}} = \{s \in dom_{\mathcal{X}} f : F_s(\bar{\nu}) < +\infty\}$. From $dom_{\mathcal{X}} f = \{s = (x, \xi, z) \in \mathcal{X} : \xi \in A(x),\ x \in G(x)\}$ yields $(KKT)_1$. Furthermore, the definition of $G(.)$ implies that $-Q(\bar{x}, \bar{x}) \geq 0$. Moreover, we have

$$\varphi(\bar{s}, \bar{s}) = \delta_{\mathbb{R}^m_-}(Q(\bar{x}, \bar{x}) + (-Q(\bar{x}, \bar{x}))) = 0 \text{ and } h(\bar{s}) = \delta_{\mathbb{R}^m_+}(-Q(\bar{x}, \bar{x})) = 0.$$

So, it results that $v(\bar{s}) = 0$. Hence, by Definition 3.1, inequality (4.16) leads up to

$$-h^*(-\bar{\nu}) - F^*_{\bar{s}}(\bar{\nu}) \geq 0. \tag{4.17}$$

Combining this inequality and (4.14), we obtain that

$$\bar{\nu} = (0, 0, \bar{\eta}) \text{ with } \bar{\eta} \geq 0.$$

Hence, we get $h^*(-\bar{\nu}) = 0$. Therefore, by (4.15) and (4.17) we have

$$\inf_{y \in X} \{\langle \bar{\eta}, Q(\bar{x}, y)\rangle + \langle \bar{\xi}, y - \bar{x}\rangle\} \geq 0.$$

It follows from $Q(\bar{x}, \bar{x}) \leq 0$ and $\bar{\eta} \geq 0$ that $\langle \bar{\eta}, Q(\bar{x}, \bar{x})\rangle \leq 0$. Thus, taking $y = \bar{x}$, we obtain that

$$0 \geq \langle \bar{\eta}, Q(\bar{x}, \bar{x})\rangle \geq \inf_{y \in X} \{\langle \bar{\eta}, Q(\bar{x}, y)\rangle + \langle \bar{\xi}, y - \bar{x}\rangle\} \geq 0 \tag{4.18}$$

Hence, the equality is satisfied and $\bar{x}$ is a solution of

$$\inf_{y \in X} \{\langle \bar{\eta}, Q(\bar{x}, y)\rangle + \langle \bar{\xi}, y - \bar{x}\rangle\},$$

which implies

$$0 \in \partial\{\sum_{i=1}^{m} \bar{\eta}_i q_i(\bar{x}, .) + \langle \bar{\xi},\ . - \bar{x}\rangle\}(\bar{x}). \tag{4.19}$$

Using that a lower semicontinuous convex function over a Banach space is continuous on the interior of its effective domain ([6], Corollary 2.5) and that $\cap_{i=1}^{m} dom(q_i(\bar{x}, .))$ is a nonempty open set, we can apply the Moreau-Rockafellar Theorem ([21], 47.B, Vol. III) to conclude that (4.19) is equivalent to

$$0 \in \sum_{i=1}^{m} \partial\{\bar{\eta}_i q_i(\bar{x}, .)\}(\bar{x}) + \partial\{\langle \bar{\xi},\ . - \bar{x}\rangle\}(\bar{x}) = \sum_{i=1}^{m} \bar{\eta}_i \partial_2 q_i(\bar{x}, \bar{x}) + \bar{\xi}.$$

Hence, condition $(KKT)_2$ holds since $\bar{\xi} \in A(\bar{x})$. Moreover, from the first inequality of (4.18), it follows that

$$0 = \langle \bar{\eta}, Q(\bar{x}, \bar{x})\rangle = \inf_{\eta \in \mathrm{I\!R}^m_+} \{\langle \eta, -Q(\bar{x}, \bar{x})\rangle\}.$$

Thus, it must be $0 \in \nabla(\langle .\ , -Q(\bar{x}, \bar{x})\rangle)(\bar{\eta}) + N_{\mathrm{I\!R}^m_+}(\bar{\eta})$. So, we obtain condition $(KKT)_3$, since $-Q(\bar{x}, \bar{x}) = \nabla(\langle .\ , -Q(\bar{x}, \bar{x})\rangle)(\bar{\eta})$. ∎

Proposition 4.4 *Consider problem* $(GQVI_1)$*. If a point* $(\bar{x}, \bar{\eta}) \in X \times \mathrm{I\!R}^m_+$ *satisfies conditions* $(KKT)_1$ *-* $(KKT)_3$*, then there exists* $\bar{\xi} \in X^*$ *such that* $((\bar{x}, \bar{\xi}, -Q(\bar{x}, \bar{x})), (0, 0, \bar{\eta}))$ *is a primal-dual feasible point verifying (4.16).*

Proof. From condition $(KKT)_2$ we obtain that there exists $\bar{\xi} \in A(\bar{x})$ such that $\bar{\xi} = -\sum_{i=1}^{m} \bar{\eta}_i g_i^*$ with $g_i^* \in \partial_2 q_i(\bar{x}, \bar{x})$ for $i = 1, ..., m$. Hence, by defining $\bar{s} = (\bar{x}, \bar{\xi}, -Q(\bar{x}, \bar{x}))$ and from $(KKT)_1$ we conclude that

$$\bar{s} \in dom_{\mathcal{X}} f. \tag{4.20}$$

Using the same argument in Proposition 4.3, condition $(KKT)_2$ can be rewritten in the following way

$$0 \in \partial\{\langle \bar{\eta}, Q(\bar{x}, .) \rangle\}(\bar{x}) + \partial\{\langle \bar{\xi}, . - \bar{x} \rangle\}(\bar{x}).$$

It follows that

$$0 \in \partial\{\langle \bar{\eta}, Q(\bar{x}, .) \rangle + \langle \bar{\xi}, . - \bar{x} \rangle\}(\bar{x}).$$

In other words, we have

$$\min_{y \in X}\{\langle \bar{\eta}, Q(\bar{x}, y) \rangle + \langle \bar{\xi}, y - \bar{x} \rangle\} = \langle \bar{\eta}, Q(\bar{x}, \bar{x}) \rangle. \tag{4.21}$$

Now, from condition $(KKT)_1$ we also have that $Q(\bar{x}, \bar{x}) \leq 0$. Thus, using the hypothesis $\bar{\eta} \geq 0$, we obtain that

$$\langle \bar{\eta}, Q(\bar{x}, \bar{x}) \rangle \leq 0. \tag{4.22}$$

On the other hand, condition $(KKT)_3$ implies

$$\langle Q(\bar{x}, \bar{x}), \eta - \bar{\eta} \rangle \leq 0 \; \forall \, \eta \in \mathbb{R}^m_+.$$

Taking $\eta = 0$ in the last inequality, we have that $\langle Q(\bar{x}, \bar{x}), \bar{\eta} \rangle \geq 0$ which together with (4.22) imply that $\langle \bar{\eta}, Q(\bar{x}, \bar{x}) \rangle = 0$. Thus, by defining $\bar{\nu} = (0, 0, \bar{\eta})$ with $\bar{\eta} \geq 0$ and by using (4.15) and (4.21) we obtain that

$$F^*_{\bar{s}}(\bar{\nu}) = \inf_{y \in X}\{\langle \bar{\eta}, Q(\bar{x}, y) \rangle + \langle \bar{\xi}, y - \bar{x} \rangle\} = 0.$$

Therefore, it follows from these equalities and (4.20) that $(\bar{s}, \bar{\nu})$ is a primal-dual feasible point, that is, $\bar{s} \in D_{\bar{\nu}}$. Moreover, we have that $h(\bar{s}) = \varphi(\bar{s}, \bar{s}) = 0$. Thus, $v(\bar{s}) = 0$ and by Definition 3.1 we get

$$L(\bar{s}, \bar{\nu}) = -h^*(-\nu) - F^*_{\bar{s}}(\nu) = 0 = \varphi(\bar{s}, \bar{s}) + h(\bar{s}),$$

which is the desired result. ■

Lemma 4.1 *Let* $\bar{s} = (\bar{x}, \bar{\xi}, \bar{z})$ *be a point of* $dom_{\mathcal{X}} f$ *such that there exists* $y_0 \in X$ *verifying* $q_i(\bar{x}, y_0) < 0$ *for all* $i = 1, ..., m$*. Then,* $\bar{s}$ *verifies condition* **(H2)**.

Proof. By hypothesis, it holds that $-Q(\bar{x}, y_0) > 0$. We define $t_0 := (y_0, \xi_0, u_0)$ where $u_0 \in \{u \in \mathrm{I\!R}^m : 0 < u < -Q(\bar{x}, y_0)\}$ and $\xi_0 = \bar{\xi}$. Thus, we obtain that $u_0 \in \mathrm{I\!R}^m_{++} := \{u \in \mathrm{I\!R}^m : u > 0\}$, $h(t_0) = 0$ and $F_{\bar{s}}(t_0) = \langle \bar{\xi}, y_0 - \bar{x} \rangle \in \mathrm{I\!R}$. So, $t_0 \in domF_{\bar{s}}$ and $t_0 \in int(domh) = X \times X^* \times \mathrm{I\!R}^m_{++}$. Recall that h is continuous on $int(domh)$. Therefore, by Remark 3.3(a), we have that $\partial(F_{\bar{s}} + h) = \partial F_{\bar{s}} + \partial h$ obtaining the desired result. ■

Using Theorems 3.1 and 3.2, Propositions 4.3 and 4.4 and Lemma 4.1 we obtain the following results.

Corollary 4.1 *Let* $\bar{x} \in X$*. If there exists* $\bar{\eta} \in \mathrm{I\!R}^m_+$ *such that* $(\bar{x}, \bar{\eta})$ *satisfies* $(KKT)_1$*,* $(KKT)_2$ *and* $(KKT)_3$*. Then,* $\bar{x}$ *is a solution of (GQVI) and* $\bar{\eta}$ *is a solution of (DGQVI).*

Proof. Let $\bar{\eta} \in \mathrm{I\!R}^m_+$ such that $(\bar{x}, \bar{\eta})$ satisfies $(KKT)_1$ - $(KKT)_3$. By Proposition 4.4 there exists $\bar{\xi} \in X^*$ such that $\bar{s} = (\bar{x}, \bar{\xi}, -Q(\bar{x}, \bar{x}))$ and $\bar{\nu} = (0, 0, \bar{\eta})$ verify (4.16). By Theorem 3.2 we obtain that $\bar{s}$ is a solution of $(GQVI_1)$, in other words, $\bar{x}$ is a solution of $(GQVI)$. Clearly, $\bar{\eta}$ is a solution of $(DGQVI)$. ■

Corollary 4.2 *Let* $\bar{x}$ *be a solution of (GQVI) such that it verifies:*

(i) $\cap_{i=1}^m dom(q_i(\bar{x}, .))$ *is a nonempty open subset of* X

(ii) $\exists\, y_0 \in X$ *verifying* $q_i(\bar{x}, y_0) < 0$ *for all* $i = 1, ..., m$*.*

Then there exists $\bar{\eta} \in \mathrm{I\!R}^m_+$ *such that* $(\bar{x}, \bar{\eta})$ *verifies conditions* $(KKT)_1$*,* $(KKT)_2$ *and* $(KKT)_3$ *(moreover,* $\bar{\eta}$ *solves (DGQVI)).*

Proof. If $\bar{x}$ is a solution of $(GQVI)$, then there exists $\bar{\xi} \in A(\bar{x})$ such that $\bar{s} = (\bar{x}, \bar{\xi}, -Q(\bar{x}, \bar{x}))$ is a solution of $(GQVI_1)$. Since $\bar{x}$ verifies (ii) and $\bar{s} \in dom_{\mathcal{X}} f$, by Lemma 4.1 we have that $\bar{s}$ satisfies condition **(H2)**. Thus, Theorem 3.1 says that there exists $\bar{\nu} \in \mathcal{X}^*$ such that $(\bar{s}, \bar{\nu})$ is a primal-dual feasible point satisfying (4.16). So, by Proposition 4.3, we obtain the desired result. ■

We observe that these two last results are those obtained by Morgan and Romaniello in [15] with a slight difference: we do not use hypothesis (i) to obtain sufficient optimality conditions.

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