Mixed Integer Non Linear Programming ($MINLP$) models for the Euclidean Steiner Tree Problem in R^n

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1 Introduction

An important history of the Euclidean Steiner tree problem is presented in [4], but nothing is said on the optimization models to solve it. An interesting application of Steiner tree problem using heuristics is in [18] and [17]. The first optimization model for the Euclidean Steiner Tree Problem (ESTP) was presented in [16]. From this formulation, another formulation of the ESTP is found in [12], [13], [10].

New formulations also derived from [16] will be presented in this report. These six models are Mixed Integer Non Linear Programming (MINLP).

An overview of exact algorithms for the Euclidean Steiner tree problem in n-space can be found in [5] and [11].

2 Definitions

Given p different points in $\mathbb{R}^n$, the ESTP seeks to find a minimum tree that spans these points using or not extra points, which are called Steiner points. The length of each edge is the Euclidean distance between its ends.

We consider a special graph $G = (V, E)$ as follows (see [16]):
Let $P = \{1, 2, ..., p-1, p\}$ be the set of indices associated with the given points

in $R^n: \ x^1, x^2, ..., x^{p-1}, x^p,$ and a set of indices $S = \{p+1, p+2, ..., 2p-3, 2p-2\}$ associated with the Steiner points also in $R^n: \ x^{p+1}, x^{p+2}, ..., x^{2p-3}, x^{2p-2}$. We take $V = P \cup S$. We denote $[i,j], \ i < j, \ i,j \in V$, an edge of G. Thus we define $E_1 = \{[i,j] \mid i \in P, \ j \in S\}, \ \ E_2 = \{[i,j] \mid i < j, \ i,j \in S\},$ and $E = E_1 \cup E_2$.

A tree which is an optimal solution for the ESTP is a sub-graph of $G = (V, E)$ (see [16]).

It is very easy to verify that all Steiner points belong to the convex hull of points $x^1, x^2, ..., x^{p-1}, x^p$.

Let $||x^i - x^j|| = \sqrt{\sum_{k=1}^{n}(x^i_k - x^j_k)^2}$ be the Euclidean distance between x^i and x^j. We compute $M = \max_{1 \leq i < j \leq p} ||x^i - x^j||$, which implies $||x^i - x^j|| \leq M, \ [i,j] \in E$.

For obtaining a better upper bound M, see Theorem 8.1 in [14].

$$\frac{1}{M}\sqrt{\sum_{k=1}^{n}(x^i_k - x^j_k)^2} \leq 1, \ [i,j] \in E,$$

$$\sqrt{\sum_{k=1}^{n}\frac{(x^i_k - x^j_k)^2}{M^2}} \leq 1, \ [i,j] \in E,$$

$$\sqrt{\sum_{k=1}^{n}(\frac{x^i_k}{M} - \frac{x^j_k}{M})^2} \leq 1, \ [i,j] \in E.$$

We put $x^i_k := \frac{x^i_k}{M}, \ i \in V, \ k = 1, 2, ..., n.$

Thus without loss of generality we can consider

$$\max_{1 \leq i < j \leq p} ||x^i - x^j|| = 1, \ x^i_k \geq 0, \ k = 1, 2, ..., n, \ i \in S.$$

In this case,

$$-1 \leq (x^i_k - x^j_k) \leq 1, \ k = 1, 2, ..., n, \ \ [i,j] \in E. \tag{1}$$

3 1^{st} Optimization Model, Maculan-Michelon-Xavier, 2000, [16]

$$(P1): \ \ Minimize \sum_{[i,j] \in E} ||x^i - x^j|| y_{ij}, \ subject \ to: \tag{2}$$

$$\sum_{j\in S} y_{ij} = 1, \quad i \in P = \{1,2,\dots,p\}, \tag{3}$$

$$\sum_{i\in P} y_{ij} + \sum_{k<j,k\in S} y_{kj} + \sum_{k>j,k\in S} y_{jk} = 3, \quad j \in S = \{p+1,\dots,2p-2\}, \tag{4}$$

$$\sum_{k<j,k\in S} y_{kj} = 1, \quad j \in S - \{p+1\}, \tag{5}$$

$$\sum_{i\in P} y_{ij} \leq 2, \quad j \in S, \tag{6}$$

$$x^i \in \mathbb{R}^n, \ i \in S, \tag{7}$$

$$y_{ij} \in \{0,1\}, \quad [i,j] \in E. \tag{8}$$

Constraints (6) are valid for $p > 3$.
Theoretically we do not need to consider constraints (4) and (6), but these constraints are valid for $(P1): \ y_{ij} \in \{0,1\}$.
The continuous relaxation of $(P1)$ is not convex.
About the use of $(P1)$, see [7].

4 2^{nd} Optimization Model, Fampa-Maculan, 2001, 2004, [12], [13]

$$(P2): \quad Minimize \sum_{[i,j]\in E} d_{ij}, \ subject\ to: \tag{9}$$

$$d_{ij} \geq ||x^i - x^j|| + y_{ij} - 1, \ [i,j] \in E. \tag{10}$$

$$\sum_{j\in S} y_{ij} = 1, \quad i \in P = \{1,2,\dots,p\}, \tag{11}$$

$$\sum_{i\in P} y_{ij} + \sum_{k<j,k\in S} y_{kj} + \sum_{k>j,k\in S} y_{jk} = 3, \quad j \in S = \{p+1,\dots,2p-2\}, \tag{12}$$

$$\sum_{k<j,k\in S} y_{kj} = 1, \quad j \in S - \{p+1\}, \tag{13}$$

$$\sum_{i\in P} y_{ij} \leq 2, \quad j \in S, \tag{14}$$

$$d_{ij} \geq 0, \quad [i,j] \in E, \tag{15}$$

$$x^i \in \mathbb{R}^n, \ i \in S, \tag{16}$$

$$y_{ij} \in \{0,1\}, \quad [i,j] \in E. \tag{17}$$

The continuous relaxation of $(P2)$ is convex.

5 3^{rd} Optimization Model, Ouzia-Maculan, 2018

We know that $y_{ij} = y_{ij}^2$. Then we can write $||x^i - x^j|| y_{ij} = y_{ij}^2 \sqrt{\sum_{k=1}^n (x_k^i - x_k^j)^2} =$

$\sqrt{y_{ij} \sum_{k=1}^n (x_k^i - x_k^j)^2} = \sqrt{\sum_{k=1}^n y_{ij}(x_k^i - x_k^j)^2} = \sqrt{\sum_{k=1}^n d_{ijk}^2}$,

where:
$-y_{ij} \leq d_{ijk} \leq y_{ij},\ [i,j] \in E,\ k = 1, 2, ..., n,$

$-(1 - y_{ij}) + (x_k^i - x_k^j) \leq d_{ijk} \leq (x_k^i - x_k^j) + (1 - y_{ij}),\ [i,j] \in E,\ k = 1, 2, ..., n.$

$$(P3): \quad Minimize \sum_{[i,j]\in E} \sqrt{\sum_{k=1}^{n} d_{ijk}^2},\ subject\ to: \tag{18}$$

$$-y_{ij} \leq d_{ijk} \leq y_{ij},\ [i,j] \in E,\ k = 1, 2, ..., n. \tag{19}$$

$$-(1 - y_{ij}) + (x_k^i - x_k^j) \leq d_{ijk} \leq (x_k^i - x_k^j) + (1 - y_{ij}),\ [i,j] \in E,\ k = 1, 2, ..., n. \tag{20}$$

$$\sum_{j\in S} y_{ij} = 1, \quad i \in P = \{1, 2, \ldots, p\}, \tag{21}$$

$$\sum_{i\in P} y_{ij} + \sum_{k<j, k\in S} y_{kj} + \sum_{k>j, k\in S} y_{jk} = 3, \quad j \in S = \{p+1, \ldots, 2p-2\}, \tag{22}$$

$$\sum_{k<j, k\in S} y_{kj} = 1, \quad j \in S - \{p+1\}, \tag{23}$$

$$\sum_{i\in P} y_{ij} \leq 2, \quad j \in S, \tag{24}$$

$$x^i \in \mathbb{R}^n,\ i \in S, \tag{25}$$

$$y_{ij} \in \{0, 1\}, \quad [i,j] \in E. \tag{26}$$

Constraints (24) are valid for $p > 3$.
$f_{ij} = \sqrt{\sum_{k=1}^n d_{ijk}^2}$ is a convex function but it is not differentiable. We can use $\hat{f}_{ij} = \sqrt{\sum_{k=1}^n d_{ijk}^2 + \lambda^2}$, where $\lambda = 10^{-12}$.

When we consider all given points $x^i \in R^n, i \in P$, such that $0 \leq x_k^i \leq 1,\ k = 1, 2, .., n,\ i \in P$, (1) is also valid.

The continuous relaxation of $(P3)$ is also convex.

6 4^{th} Optimization Model, Ouzia-Maculan, 2018

$$(P4): \quad Minimize \sum_{[i,j]\in E} \sqrt{d_{ij}},\ subject\ to: \tag{27}$$

$$d_{ij} \geq \sum_{k=1}^{n}(x_k^i - x_k^j)^2 + (y_{ij}-1)n, \ [i,j] \in E. \tag{28}$$

$$\sum_{j\in S} y_{ij} = 1, \ \ i \in P = \{1,2,\ldots,p\}, \tag{29}$$

$$\sum_{i\in P} y_{ij} + \sum_{k<j,k\in S} y_{kj} + \sum_{k>j,k\in S} y_{jk} = 3, \ \ j \in S = \{p+1,\ldots,2p-2\}, \tag{30}$$

$$\sum_{k<j,k\in S} y_{kj} = 1, \ \ j \in S - \{p+1\}, \tag{31}$$

$$\sum_{i\in P} y_{ij} \leq 2, \ \ j \in S, \tag{32}$$

$$d_{ij} \geq 0, \ \ [i,j] \in E, \tag{33}$$

$$x^i \in \mathbb{R}^n, \ i \in S, \tag{34}$$

$$y_{ij} \in \{0,1\}, \ \ [i,j] \in E. \tag{35}$$

The objective function (27) is a concave function. Thus the continuous relaxation of $(P4)$ is not convex.

7 5^{th} Optimization Model, Maculan-Ouzia, 2019

We consider the constraints (10), (15), (16) and (17) in the $(P2)$ model:

$$d_{ij} \geq ||x^i - x^j|| + y_{ij} - 1, \ [i,j] \in E;$$

$$d_{ij} \geq 0, \ \ [i,j] \in E; \ x^i \in \mathbb{R}^n, \ i \in S; \ y_{ij} \in \{0,1\}, \ \ [i,j] \in E;$$

which can be written:

$$d_{ij} + (1 - y_{ij}) \geq ||x^i - x^j||, \ [i,j] \in E.$$

We can write: $z_{ij} = d_{ij} + (1 - y_{ij}), \ [i,j] \in E$. It is easy to observe that $z_{ij} \geq 0, \ [i,j] \ \in E$.
Let us define $t_{ijk} = x_k^i - x_k^j, \ [i,j] \in E, \ k = 1,2,...,n$.

$$||x^i - x^j|| = \sqrt{\sum_{k=1}^{n}(x_k^i - x_k^j)^2} = \sqrt{\sum_{k=1}^{n} t_{ijk}^2}, \ [i,j] \in E.$$

We replace constraints (10) by:

$$z_{ij} \geq \sqrt{\sum_{k=1}^{n} t_{ijk}^2} \quad \text{and} \quad z_{ij} = d_{ij} + (1 - y_{ij}), \ \ [i,j] \in E.$$

As $z_{ij} \geq 0, \quad [i,j] \in E:$

$$z_{ij} \geq \sqrt{\sum_{k=1}^{n} t_{ijk}^2} \quad \text{can be replaced by} \quad z_{ij}^2 \geq \sum_{k=1}^{n} t_{ijk}^2 \ [i,j] \in E.$$

Remark: $z_{ij}^2 \geq \sum_{k=1}^{n} t_{ijk}^2, \ z_{ij} \geq 0, \ [i,j] \in E$, are second order cone constraints, see [22].

$$(P5): \quad Minimize \sum_{[i,j]\in E} d_{ij}, \ subject \ to: \tag{36}$$

$$z_{ij}^2 \geq \sum_{k=1}^{n} t_{ijk}^2 \ [i,j] \in E. \tag{37}$$

$$z_{ij} = d_{ij} + (1 - y_{ij}), \ [i,j] \in E. \tag{38}$$

$$-y_{ij} \leq t_{ijk} \leq y_{ij}, \ [i,j] \in E, \ k = 1, 2, ..., n. \tag{39}$$

$$-(1 - y_{ij}) + (x_k^i - x_k^j) \leq t_{ijk} \leq (x_k^i - x_k^j) + (1 - y_{ij}), \ [i,j] \in E, \ k = 1, 2, ..., n. \tag{40}$$

$$\sum_{j\in S} y_{ij} = 1, \quad i \in P = \{1, 2, \ldots, p\}, \tag{41}$$

$$\sum_{i\in P} y_{ij} + \sum_{k<j, k\in S} y_{kj} + \sum_{k>j, k\in S} y_{jk} = 3, \quad j \in S = \{p+1, .., 2p-2\}, \tag{42}$$

$$\sum_{k<j, k\in S} y_{kj} = 1, \quad j \in S - \{p+1\}, \tag{43}$$

$$\sum_{i\in P} y_{ij} \leq 2, \quad j \in S, \tag{44}$$

$$d_{ij} \geq 0, \quad [i,j] \in E, \tag{45}$$

$$x^i \in \mathbb{R}^n, \ i \in S, \tag{46}$$

$$y_{ij} \in \{0, 1\}, \quad [i,j] \in E. \tag{47}$$

8 6^{th} Optimization Model, Maculan-Ouzia-Pinto, 2020

From (39), (40), (47) we can write $d_{ij} = \sqrt{\sum_{k=1}^{n} t_{ijk}^2}$, and we will define a new model $(P6)$ from $(P5)$:
$Minimize \ \sum_{[i,j]\in E} d_{ij}, \ suject \ to:$

$d_{ij} \geq \sqrt{\sum_{k=1}^{n} t_{ijk}^2}$, $[i,j] \in E$ *and* (39-47).

Remark: $d_{ij} \geq \sqrt{\sum_{k=1}^{n} t_{ijk}^2}$ will be replaced by

$d_{ij}^2 \geq \sum_{k=1}^{n} t_{ijk}^2$, $d_{ij} \geq 0$, $[i,j] \in E$.

Thus we will have:

$$(P6): \quad Minimize \sum_{[i,j]\in E} d_{ij}, \; subject\ to: \tag{48}$$

$$d_{ij}^2 \geq \sum_{k=1}^{n} t_{ijk}^2 \; [i,j] \in E. \tag{49}$$

$$-y_{ij} \leq t_{ijk} \leq y_{ij}, \; [i,j] \in E, \; k = 1,2,...,n. \tag{50}$$

$$-(1-y_{ij}) + (x_k^i - x_k^j) \leq t_{ijk} \leq (x_k^i - x_k^j) + (1-y_{ij}), \; [i,j] \in E, \; k = 1,2,...,n. \tag{51}$$

$$\sum_{j\in S} y_{ij} = 1, \quad i \in P = \{1,2,\ldots,p\}, \tag{52}$$

$$\sum_{i\in P} y_{ij} + \sum_{k<j,k\in S} y_{kj} + \sum_{k>j,k\in S} y_{jk} = 3, \quad j \in S = \{p+1,..,2p-2\}, \tag{53}$$

$$\sum_{k<j,k\in S} y_{kj} = 1, \quad j \in S - \{p+1\}, \tag{54}$$

$$\sum_{i\in P} y_{ij} \leq 2, \quad j \in S, \tag{55}$$

$$d_{ij} \geq 0, \quad [i,j] \in E, \tag{56}$$

$$x^i \in \mathbb{R}^n, \; i \in S, \tag{57}$$

$$y_{ij} \in \{0,1\}, \quad [i,j] \in E. \tag{58}$$

The continuous relaxations of $(P5)$ and $(P6)$ are smooth and also convex. For these continuous relaxations the XPRESS [2] software can recognize the constraints (37) with $d_{ij} \geq 0$ in (45) and the constraints (49) with $d_{ij} \geq 0$ in (56) as a second order cones, and uses an interior point algorithm to solve them, see [22].

8.1 Just an example using models $(P3)$ and $(P5)$

Let the coordinates of the 8 vertices of a unit cube be given in R^3 :
$(0\ 0\ 0), (1\ 0\ 0), (0\ 1\ 0), (0\ 0\ 1), (0\ 1\ 1), (1\ 0\ 1), (1\ 1\ 0), (1\ 1\ 1)$. We would like

to find the Euclidean Steiner Tree which spans these vertices. For solving $(P3)$ we use BONMIN [1] software which does not take into account the existence of second order conical restrictions, as XPRESS does. However, XPRESS cannot be used with algebraic expressions in which there are square roots. So we will solve $(P5)$ with XPRESS, and for $(P3)$ we used BONMIN. These two models used NEOS Server [3, 6, 8, 15].
Model $(P3)$: 6.9896414 best solution, best possible 4.1227728 (28,083 seconds)
Model $(P5)$: 6.1961524 optimal solution (2,668 seconds). From this optimal topology we will be able to compute exactly the size of this optimal tree [19]:
$1 + 3\sqrt{3} = 6.19615242271...$

9 Geometric cuts for all formulations

Let $x^i,\ i \in P$ given points (terminals), and $d_i = \min_{j \in P,\, j \neq i} \|x^i - x^j\|$, for all $i \in P$.

We consider two terminals x^i and x^j, they may be connected to the same Steiner point only if

$$\|x^i - x^j\| \leq d_i + d_j. \tag{59}$$

The proof is in [21]. It is also shown that this property can be strengthened to

$$\|x^i - x^j\| \leq \sqrt{d_i^2 + d_j^2 + d_i\, d_j}. \tag{60}$$

Let $S = \{p+1, \dots, 2p-2\}$. Using (59), we can add to all proposed models the following constraints:

$$y_{is} + y_{js} \leq 1, \quad \forall i < j \in P, \forall s \in S \text{ such that } \|x^i - x^j\| > d_i + d_j. \tag{61}$$

Or, using (60),

$$y_{is} + y_{js} \leq 1, \quad \forall i < j \in P, \forall s \in S \text{ such that } \|x^i - x^j\| > \sqrt{d_i^2 + d_j^2 + d_i\, d_j}. \tag{62}$$

Preliminary results

We generated two instances, each one with 8 random terminals in the unit cube. These instances were solved using AMPL Version 20191015 (Linux x86_64 (gcc 4.2.1)) and solver XPRESS 8.6.0(34.01.03).

		(P5)	(P5)+(61)	(P5)+(62)
Instance1	Time(s)	1817	784	307
	Nodes	829363	274163	107221
Instance2	Time(s)	1019	178	103
	Nodes	471119	73895	38173

Using geometric cuts for the vertices of unit cubes in R^n

Let $x^1,\ x^2,\ x^3,\ ...\ x^{2^n},\ x^i \in R^n, i = 1, 2, ..., 2^n = p$, be the vertices of an unit cube in R^n. It is ease to observe that $d_i = 1, i = 1, 2,\ ...,\ p$.
For $n \geq 4$ two vertices x^i, x^j, such that $||x^i - x^j|| > \sqrt{3}$, we introduce the valid inequalities (62): $y_{is} + y_{js} \leq 1, s \in S$.

10 Lower bound for $(PI),\ I = 1, 2, 3, 4, 5, 6$

We remember that $0 \leq x_k^i \leq 1,\ i \in V,\ k = 1, 2, ..., n$.

10.1 Theoretical results

We consider the complete graph K_p formed from the p given points for which each edge is associated with the Euclidean distance between the two points connected by this edge.

A minimum spanning tree of this K_p has length $val(MST)$. For $0 \leq \rho \leq 1$, we can write

$$\rho \times val(MST) \leq val(PI) \leq val(MST).$$

Let ρ_n be the infimum of ρ, in $\mathbb{R}^n$, defined as the Steiner ratio. In [9] we have $\rho_n \geq 0.615$, where 0.615 is the best known lower bound. Du, in this same paper, presented a conjecture for $\rho_2 = \frac{\sqrt{3}}{2}$. Smith and MacGregor Smith [20] also presented another conjecture for $\rho_3 \approx 0.7841937$.

10.2 1^{st} Lower bound

We consider

$$z_{ij} = \max_{k=1,2,\ldots,n} |x_k^i - x_k^j|,\ [i, j] \in E.$$

It is easy to observe in $(P3)$ that

$$\sqrt{\sum_{k=1}^{n} d_{ijk}^2} \geq z_{ij},\ [i, j] \in E.$$

Thus we define a mixed integer linear programming problem as follows:

$$(LBP1): \quad Minimize \sum_{[i,j] \in E} z_{ij},\ subject\ to: \tag{63}$$

$$z_{ij} \geq d_{ijk},\ k = 1, 2, ..., n,\ [i, j] \in E, \tag{64}$$

$$z_{ij} \geq -d_{ijk},\ k = 1, 2, ..., n,\ [i, j] \in E, \tag{65}$$

$$-y_{ij} \leq d_{ijk} \leq y_{ij},\ [i, j] \in E,\ k = 1, 2, ..., n. \tag{66}$$

$$-(1 - y_{ij}) + (x_k^i - x_k^j) \leq d_{ijk} \leq (x_k^i - x_k^j) + (1 - y_{ij}),\ [i, j] \in E,\ k = 1, 2, ..., n. \tag{67}$$

$$\sum_{j \in S} y_{ij} = 1, \quad i \in P = \{1, 2, \ldots, p\}, \tag{68}$$

$$\sum_{i \in P} y_{ij} + \sum_{k<j, k \in S} y_{kj} + \sum_{k>j, k \in S} y_{jk} = 3, \quad j \in S = \{p+1, \ldots, 2p-2\}, \tag{69}$$

$$\sum_{k<j, k \in S} y_{kj} = 1, \quad j \in S - \{p+1\}, \tag{70}$$

$$\sum_{i \in P} y_{ij} \leq 2, \quad j \in S, \tag{71}$$

$$x^i \in \mathbb{R}^n, \; i \in S, \tag{72}$$

$$y_{ij} \in \{0, 1\}, \quad [i, j] \in E. \tag{73}$$

We define $val(\cdot)$ as the optimum value of the objective function of problem $(\cdot)$. Thus $val(LBP1) \leq val(P3)$.

10.3 2^{nd} Lower bound

If we replace the objective function (63) by

$$\sum_{[i,j] \in E} \left(\sum_{k=1}^{n} (x_k^i - x_k^j)^2) \right),$$

we have a new optimization model as follows:

$$(LBP2): \quad Minimize \sum_{[i,j] \in E} \left(\sum_{k=1}^{n} (x_k^i - x_k^j)^2 \right), \; subject\ to: \tag{74}$$

$$(64 - 73).$$

We observe that

$$\sum_{k=1}^{n} (x_k^i - x_k^j)^2 \leq \sqrt{\sum_{k=1}^{n} (x_k^i - x_k^j)^2}, \; [i, j] \in E.$$

Thus $val(LBP2) \leq val(P3)$.

10.3.1 Comparison between the lower bounds

Property: If $0 \leq x_k^i \leq \frac{1}{n}, \; i \in V, k = 1, 2, ..., n$ then $val(LBP1) \geq val(LBP2)$.

Proof:
We chose an edge $[i, j] \in E, \; x_k^i - x_k^j = a_k$,

$$-\frac{1}{n} \leq a_k \leq \frac{1}{n} \; \rightarrow \; |a_k| \leq \frac{1}{n},$$

$$\sum_{k=1}^{n}(x_k^i - x_k^j)^2 = \sum_{k=1}^{n} a_k^2 \neq 0.$$

We define

$$a = \max_{k=1,\ldots,n}\{|a_k|\}.$$

$$\sum_{k=1}^{n} a_k^2 \leq na^2,\ na^2 \leq a \ \rightarrow a \leq \frac{1}{n} \ .$$

We compare now $val(LBP1),\ val(LBP2)$ and $0.615 \times val(MST)$ for some examples.

First example: four points in $R^2:\ (0\ 0), (1\ 0), (0\ 1), (1, 1)$. These points are the vertices of a unit square. We know that $val(P5) = 1+\sqrt{3} \approx 2.732,\ val(LBP1) = 2,\ val(LBP2) = 1.5,\ val(MST) = 3,\ 0.615 \times 3 = 1,845$. Let the Du conjecture $(\rho_2 = \frac{\sqrt{3}}{2})$ be true, then a lower bound for the length of the Steiner tree will be $\rho_2 = \frac{\sqrt{3}}{2} \times val(MST) = 2.598$.

Second example: 8 points (vertices of the unit cube in R^3. We know that $val(P5) = 1 + 3\sqrt{3} \approx 6.196,\ val(LBP1) = 4,\ val(LBP2) =?,\ val(MST) = 7,\ 0.615 \times 7 = 4.305$.

Third example: 8 points (vertices of a cube in R^3, for which each edge has length $= \frac{1}{3}$. We know that $val(P5) = \frac{1+3\sqrt{3}}{3} \approx 2.098,\ val(LBP1) = \frac{8}{6} \approx 1.333 \geq val(LBP2),\ val(MST) = \frac{7}{3} \approx 2.333,\ 0.615 \times \frac{7}{3} \approx 1.435$.

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