EQUITABLE TOTAL COLORING OF SMALL CUBIC GRAPHS

Matheus Nunes Adauto

Dissertação de Mestrado apresentada ao Programa de Pós-graduação em Engenharia de Sistemas e Computação, COPPE, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Mestre em Engenharia de Sistemas e Computação.

Orientadores: Celina Miraglia Herrera de Figueiredo
Diana Sasaki Nobrega

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*"Words are, in my not-so-humble
opinion, our most inexhaustible
source of magic."
Albus Dumbledore.*

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COLORAÇÃO TOTAL EQUILIBRADA DE GRAFOS CÚBICOS PEQUENOS

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Fevereiro/2022

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Programa: Engenharia de Sistemas e Computação

Uma *k-coloração total* de um grafo simples $G = (V, E)$ atribui no máximo k cores aos vértices e arestas de G tal que cores distintas são atribuídas a cada par de vértices adjacentes em V, a cada par de arestas adjacentes em E, e cada vértice e suas arestas incidentes. Uma coloração total é *equilibrada* se as cardinalidades de quaisquer duas classes de cores diferem em no máximo 1.

O *número cromático total* $\chi''(G)$ é o mínimo k tal que G admite uma k-coloração total. A *Conjectura de Coloração Total* diz que para todo grafo simples, $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$, onde Δ é o grau máximo de G. Se $\chi''(G) = \Delta + 1$ o grafo é chamado *Tipo 1*, caso contrário, se $\chi''(G) = \Delta + 2$, é chamado *Tipo 2*.

Em 2020, Stemock considerou as colorações totais equilibradas de grafos cúbicos em seu artigo "On the equitable total $(k+1)$-coloring of k-regular graphs". O autor conjecturou que toda 4-coloração total de um grafo cúbico é equilibrada se $n < 20$. Este limite superior foi motivado por um grafo com $n = 20$ encontrado no artigo "On the equitable total chromatic number of cubic graphs" de Dantas et al. que é Tipo 1, mas não possui uma coloração total equilibrada com 4 cores. Esta conjectura torna-se relevante quando percebemos que se refere a mais de 40.000 grafos, fato que pode ser verificado no site "House of Graphs". Encontramos grafos cúbicos que servem como contraexemplos para a conjectura de Stemock e os exibimos no texto.

Em seguida, estudamos os grafos cúbicos para $n < 20$ e conseguimos determinar que uma 4-coloração total será necessariamente equilibrada em grafos cúbicos de 6, 8, 10, e 14 vértices. Além disso, provamos que um grafo cúbico de 12 vértices é o menor contraexemplo possível para a conjectura de Stemock.

Abstract of Dissertation presented to COPPE/UFRJ as a partial fulfillment of the requirements for the degree of Master of Science (M.Sc.)

EQUITABLE TOTAL COLORING OF SMALL CUBIC GRAPHS

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February/2022

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A *k-total coloring* of a simple graph $G = (V, E)$ assigns at most k colors to the vertices and edges of G such that distinct colors are assigned to every pair of adjacent vertices in V, to every pair of adjacent edges in E, and each vertex and its incident edges. A total coloring is *equitable* if the cardinalities of any two color classes differ by at most 1.

The *total chromatic number* $\chi''(G)$ is the minimum k such that G admits a k-total coloring. The *Total Coloring Conjecture* says that for every simple graph, $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$, where Δ is the maximum degree in G. If $\chi''(G) = \Delta + 1$ the graph is called Type 1, otherwise, if $\chi''(G) = \Delta + 2$, is called Type 2.

In 2020, Stemock considered equitable total colorings of cubic graphs in his article "On the equitable total $(k + 1)$-coloring of k-regular graphs". The author conjectured that every 4-total coloring of a cubic graph is equitable if $n < 20$. This upper bound was motivated by a graph with $n = 20$ in the article "On the equitable total chromatic number of cubic graphs" by Dantas et al. that is Type 1 but does not have an equitable total coloring with 4 colors. This conjecture becomes relevant when we realize that it refers to more than 40000 graphs, which can be verified in the website "House of Graphs". We find cubic graphs that serve as counterexamples to the Stemock's conjecture and display them in the text.

Then, we studied the cubic graphs for $n < 20$ and were able to determine that a 4-total coloring is necessarily equitable on cubic graphs of 6, 8, 10, and 14 vertices. Furthermore, we prove that a cubic graph of 12 vertices is the smallest possible counterexample to the Stemock's conjecture.

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Chapter 1

Graph Theory

We can model many real-life problems using a diagram made up of dots and lines connecting those dots. For example, we can say that points are cities and lines are roads that connect them[1]. That said, many questions can arise. What is the shortest way to travel from one city to another, if these are not neighbors? Once the shortest path is selected, is it the fastest and the most economical? This representation and these questions motivate one of the areas of combinatorics: graph theory.

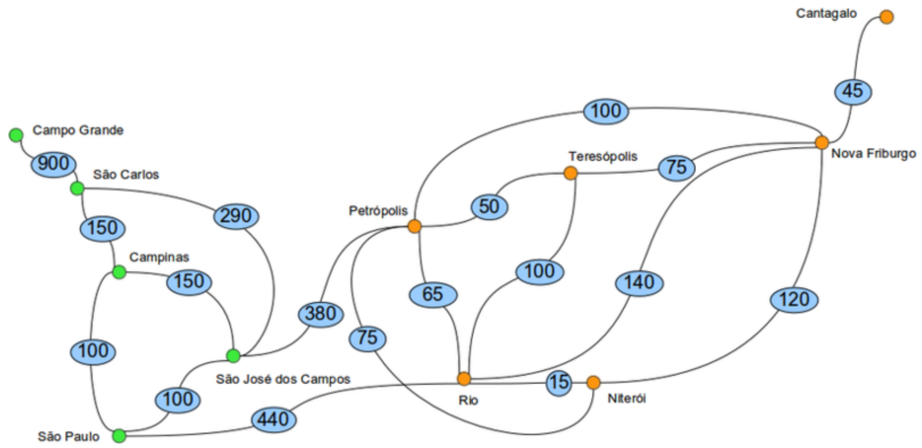

Figure 1.1: Some cities in Brazil represented in a graph.

The text is organized into four chapters. In the first one, we will introduce some basic concepts concerning to graph theory, defining the main properties of this mathematical object and presenting the classes that will be relevant throughout the text. In the second, we will talk about the coloring problem, dividing it into three parts: vertex coloring, edge coloring, and total coloring. We will define the main results of each of the three parts since the coloring problem is central to our study. In the third, we will define cubic graphs and present the main subclasses of cubics that will be used throughout the text in addition to stating the results of the

[1]Figure from `http://wiki.nosdigitais.teia.org.br/Projeto_e_Analise_de_Algoritmos`. Accessed on: 09 Nov. 2021.

total coloring of these subclasses. In the last chapter, we will define equitable total coloring and present the results of this problem contextualizing it mainly to cubic graphs. In addition, we will enunciate the conjecture that motivated this work and show the results we obtained when studying it. Finally, we will summarize what we could extract from this study.

1.1 Königsberg Bridge Problem

To write any work on graphs without mentioning the following problem is a sacrilege. Thus, we will try to describe it in the most possible didactic way , using only intuitive notions since we do not yet present formal definitions.

In 1736, Leonhard Euler, a swiss mathematician, solved the problem entitled "Königsberg Bridge Problem" [5] which consisted of the following: There was a river that ran through Königsberg, the Prussian city, and divided it into four parts[2]. To connect them, there were seven bridges. Is it possible to go from one bank of the river to the other making a closed trail that passes only once on each of the seven bridges? Euler solved the problem by modeling the region to make viewing easier. He turned bridges into edges and their banks of river into vertices. That was the first diagram to represent a graph in history. Then he found that to go through any vertex, two edges are traveled, one to enter the vertex and the other to exit. Hence, each vertex must have an even number of edges incident to it. The graph of Königsberg bridges has vertices whose number of edges incident to it is odd. So the problem has no solution.

This problem marks the beginning of two branches of modern mathematics: Graph Theory and Topology.

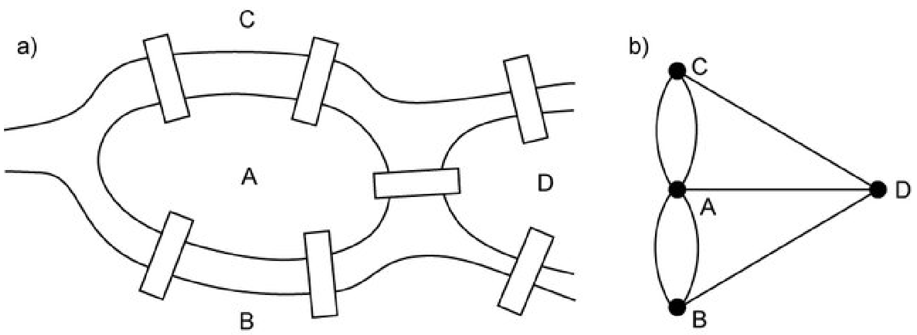

Figure 1.2: The Königsberg bridge problem: a) seven bridges of Königsberg; b) graph representation.

[2]Figure 1.2 from Pawel Boguslawski's Ph.D. thesis.

1.2 Four-Color Problem

The Four-Color Problem is the first known coloring problem and therefore, it is essential that we must mention it in this text.

This problem can be stated as follows: Given a flat map, divided into regions, how many colors are enough to color it so that neighboring regions do not have the same color? In 1852, the mathematician Francis Guthrie raised this question and conjectured that four colors were enough to color the map of England with four colors [5]. The next figure illustrates the map of Europe[3] colored as proposed by Guthrie. This conjecture remained unproved for many years, after attempts by Guthrie himself and other researchers. Only in 1976 did Kenneth Appel and Wolfgang Haken [1] prove that the conjecture was, in fact, true. To this end, they analyzed approximately 2000 maps using a computer. It was even the first time that a computer was used in a demonstration. So the conjecture came to be called the Four-Color Theorem. Note that we can model this problem as follows: each region of any map is a vertex and border regions receive an edge to connect the vertices that represent them. A color assignment takes place at the vertices of this graph. Thus, vertices connected to the same edge have different colors. This is a vertex coloring problem. This topic will be addressed later in the text.

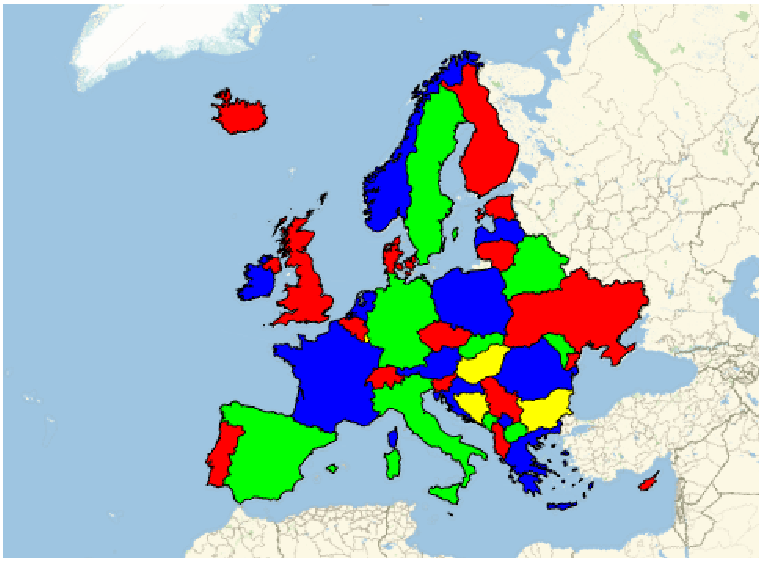

Figure 1.3: The map of Europe colored in four colors.

[3]Figure from `https://www.wolfram.com/mathematica/new-in-10/entity-based-geocomputation/find-a-four-coloring-of-a-map-of-europe.html`. Accessed on: 09 Nov. 2021.

1.3 Some Definitions

In this section, we will state the main definitions and results of graph theory that will be used throughout the text. The main results and definitions of this section were studied in [3], [13], and [15].

Definition 1. *A graph $G = (V(G), E(G))$ is an ordered pair, where $V(G)$ is a non-empty finite set of vertices and $E(G)$ is a set of edges, disjoint from $V(G)$, formed by unordered pairs of elements other than $V(G)$, that is, for every edge $e \in E(G)$ there are u and $v \in V(G)$ such that $e = \{u, v\}$, or simply $e = uv$. Also, $|V(G)| = n$ and $|E(G)| = m$.*

If $uv \in E$, we say that u and v are adjacent vertices or that u is a neighbor of v, and that the edge e is incident to both u and v, and u and v are said extremes of e. Two edges that have the same end are called adjacent edges.

Definition 2. *A graph is called simple if it has no loops or more than one edge connecting two vertices. A loop is an edge that connects a vertex to itself.*

The maximum number of edges of a simple graph G of order n is expressed by:

$$\binom{n}{2} = \frac{n!}{2!(n-2)!}$$

Definition 3. *The degree of a vertex v in G, represented by $d(v)$, is the number of edges incident to v. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees, respectively, of the vertices of the graph G. See Figure 1.4.*

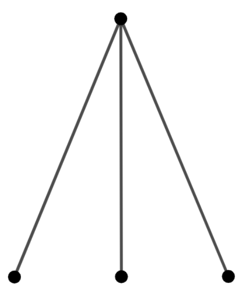

Figure 1.4: A graph with $\delta(G) = 1$ and $\Delta(G) = 3$.

Whenever there is no ambiguity, we will denote δ and Δ.

Theorem 1 ([3]). *For any graph G,*

$$\sum_{v \in V} d(v) = 2m$$

Proof. Since each edge has a vertex at each of its ends, it will contribute two units to the sum of the degrees. Hence, the result follows immediately. $\square$

Definition 4. *A graph H is called subgraph of G, denoted by $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. See Figure 1.5.*

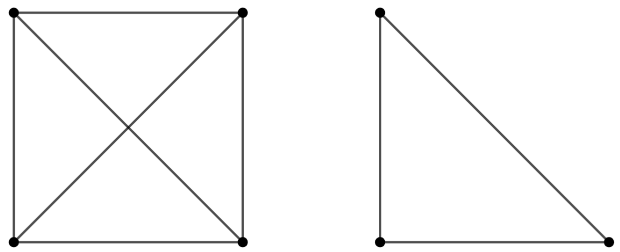

Figure 1.5: A graph and his subgraph, respectively.

Definition 5. *Let $G = (V, E)$ be a graph. A sequence $X = (v_1, v_2, ..., v_k)$ where $v_i \in V$ and $v_i v_{i+1} \in E$ is called a walk in G.*

1. *The k value is the length of X or the number of edges of X, counted with or without repetition.*
2. *If the walk X does not repeat edges, then X is called trail.*
3. *If the walk $X = (v_1, v_2, ..., v_k)$, $v_0 = v_k$, then X is closed.*
4. *If a trail does not repeat vertices then X is a path.*
5. *A cycle is a closed trail with a single repetition of vertices and with a length greater than two.*
6. *A graph $G = (V, E)$ is called acyclic is G does not have cycles.*

Definition 6. *A graph G is connected if there is a path between any pair of vertices. Otherwise it is disconnected, if there is at least one pair of vertices that it is not connected to any path. See Figure 1.6.*

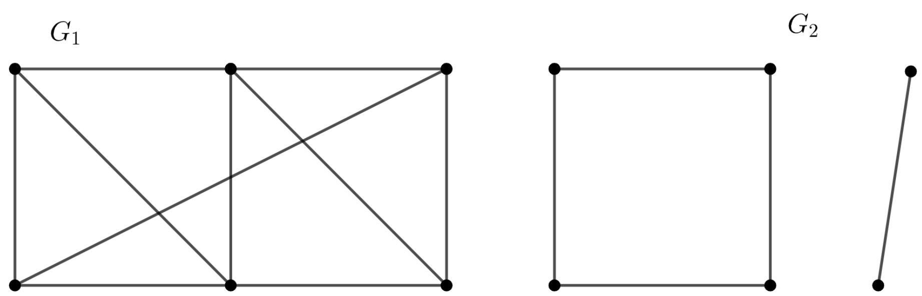

Figure 1.6: The connected graph G_1 and the disconnected graph G_2, respectively.

The next definition is fundamental in the text, since the graphs studied in this work are regular.

Definition 7. *A graph G is called regular of degree r, or r-regular when all its vertices have the same degree r. See Figure 1.7.*

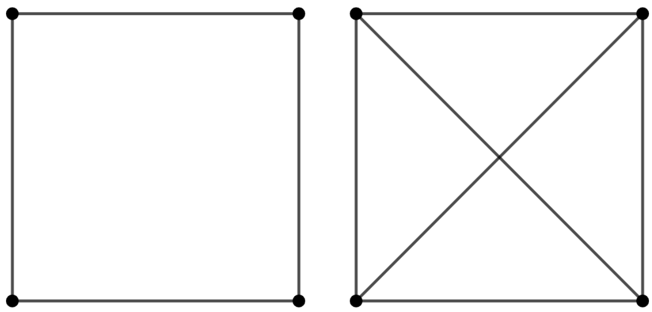

Figure 1.7: A 2-regular graph and a 3-regular graph, respectively.

Definition 8. *A set of vertices S is called independent set if no two vertices in set S are adjacent to each other. See Figure 1.8.*

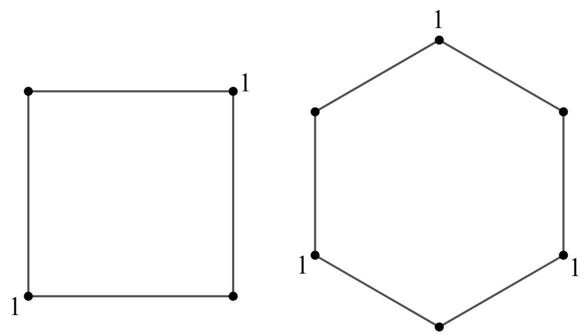

Figure 1.8: Vertices identified with 1 are those that are part of the independent set on each of the graphs.

Definition 9. *A set of edges M is called matching if no two edges in set M share a vertex. See Figure 1.9.*

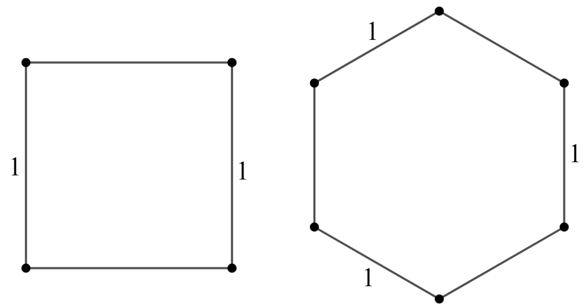

Figure 1.9: Edges identified with 1 are those that are part of the matching on each of the graphs.

Definition 10. *A set of vertices A is called total independent set if no two vertices in set A are adjacent to each other, no two edges in set A share a vertex, and no vertex is the extreme of an edge in set A. See Figure 1.10.*

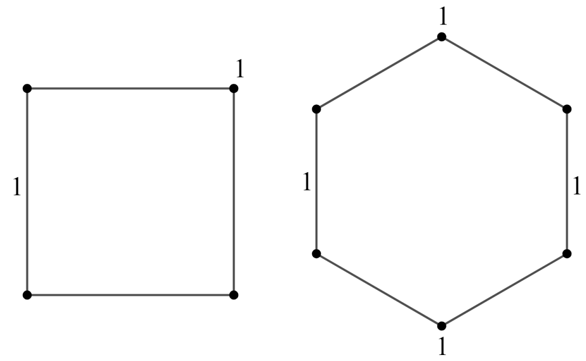

Figure 1.10: Elements identified with 1 are those that are part of the total independent set on each of the graphs.

Next, we will define classes of graphs that will be relevant throughout the text.

Definition 11. *The cycle graph, denoted by C_n, is a graph on n vertices containing a single cycle through all vertices. See Figure 1.11.*

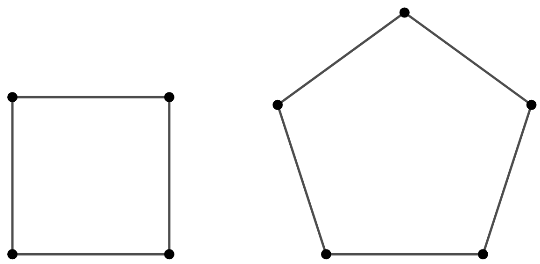

Figure 1.11: The cycle graphs C_4 and C_5, respectively.

Definition 12. *The complete graph, denoted by K_n, is a graph in which each pair of vertices is connected by an edge. See Figure 1.12.*

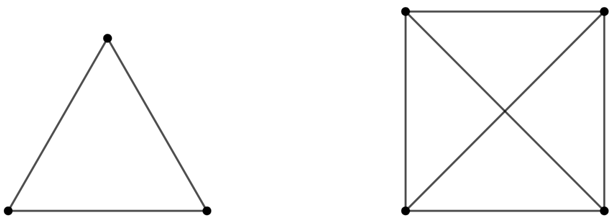

Figure 1.12: The complete graphs K_3 and K_4, respectively.

Definition 13. *The bipartite graph, denoted by $G = (V_1 \cup V_2, E)$, is a graph whose vertex set is partitioned into two disjoint sets (V_1 and V_2) such that each edge only connects vertices of different partitions. See Figure 1.13.*

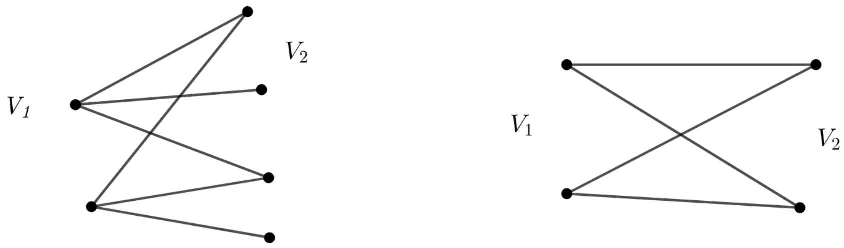

Figure 1.13: Two bipartite graphs with 6 and 4 vertices, respectively.

Having defined the main results and the main classes of graphs, we can proceed to study a very important problem in this work: graph coloring.

Chapter 2

Graph Coloring

The coloring problem is central to our work. It is important to note that one of the first problems in graph theory is a coloring problem: Four-Color Problem (more details in section 1.2). As an example of what was done in the previous chapter, we will bring to light basic concepts on the subject. To study graph coloring more completely, we will divide the subject into its three main topics: vertex coloring, edge coloring, and total coloring. The main results and definitions of this section were studied in [5], [11], and [16].

Observation 2. *The color classes of all the colorings present in the text will be represented by numbers. Different numbers represent different color classes.*

2.1 Vertex Coloring

Definition 14. *A vertex coloring is an assignment of colors to the vertices of a graph such that no two adjacent vertices have the same color.*

A k-coloring of a graph G is a coloring of the vertex set of G using a set of k colors, and a graph is k-colorable if there is a k-coloring of G.

Definition 15. *The smallest natural k for which G admits a k-coloring is called chromatic number of a graph G. This number is denoted by $\chi(G)$. See Figure 2.1.*

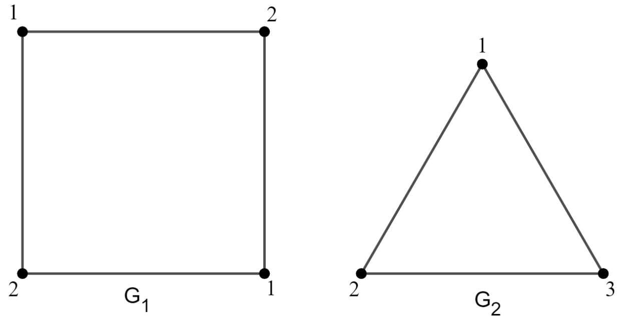

Figure 2.1: $\chi(G_1) = 2$, and $\chi(G_2) = 3$.

Observation 3. *Every graph of order n is n-colorable. Furthermore, $1 \leq \chi(G) \leq n$.*

We will now state results regarding the chromatic number of the graph classes defined in the first chapter.

Theorem 4 ([5])**.** *A graph G is 2-colorable if and only if G is bipartite.*

Proof. ($\Rightarrow$) Let G be a 2-colorable graph. Take a 2-coloring of G, with colors c_1 and c_2. Let V_1 and V_2 be the subsets of vertices that have the colors c_1 and c_2 respectively. Hence, V_1 and V_2 are bipartitions of G as there can be no edges between vertices of V_1 or between the vertices of V_2, since they have the same color.

($\Leftarrow$) Let G be a bipartite graph. Let $V_1, V_2 \subseteq V$ be the subsets of V that bipartition it. Just assign the color c_1 to the vertices of V_1 and the color c_2 to the vertices of V_2. Hence, we have that G is 2-colorable. □

Theorem 5 ([5])**.** *A graph C_n has $\chi(C_n) = 3$ if n is odd, and $\chi(C_n) = 2$ if n is even.*

Theorem 6 ([5])**.** *The chromatic number of a graph K_n is $\chi(K_n) = n$.*

Next, we will present an important definition that will be very useful when studying total coloring.

Definition 16. *The deficiency of G, denoted by $def(G)$, is defined as*

$$def(G) := \sum_{v \in V(G)} (\Delta(G) - d(v)).$$

Let C be a $(\Delta + 1)$-coloring of the vertices of G, and r the number of coloring classes of C where the size has the same parity as $|V(G)|$. If

$$def(G) \geq \Delta + 1 - r,$$

this coloring of vertices is called conformable. If G has some conformable vertex coloring, then we say that G is conformable.

From the definition of the conformable graph, for regular graphs where $def(G) = 0$, we obtain the following lemma:

Lemma 7 ([6]). *Let G be a regular graph. G is conformable if and only if it has a vertex coloring with $\Delta+1$ colors $c_1, c_2, ..., c_{\Delta+1}$, such that $i_1 \equiv i_2 \equiv ... \equiv i_{\Delta+1} \equiv |V(G)| \mod 2$, where i_j, $1 \leq j \leq \Delta+1$ is the number of vertices colored with the color c_j.*

See Figure 2.1. The graph G_1 has a conformable vertex coloring. Observe that the maximum degree of G_1 is 2, and for the conformable vertex coloring, the third color class has zero vertices, which is allowed since the number of vertices of G_1 is even. The coloring of G_2 is clearly conformable as well.

Theorem 8 (Brook's Theorem). *The chromatic number of a graph is $\chi(G) \leq \Delta$, unless the graph is complete or an odd cycle, in which case $\Delta+1$ colors are required.*

2.2 Edge Coloring

Definition 17. *An edge coloring is an assignment of colors to the edges of a graph such that no two adjacent edges have the same color.*

A k-edge coloring of a graph G is a coloring of the edge set of G using a set of k colors, and a graph is k-edge colorable if there is a k-edge coloring of G.

Definition 18. *The smallest natural k for which G admits a k-edge coloring is called chromatic index of a graph G. This number is denoted by $\chi'(G)$. See Figure 2.2.*

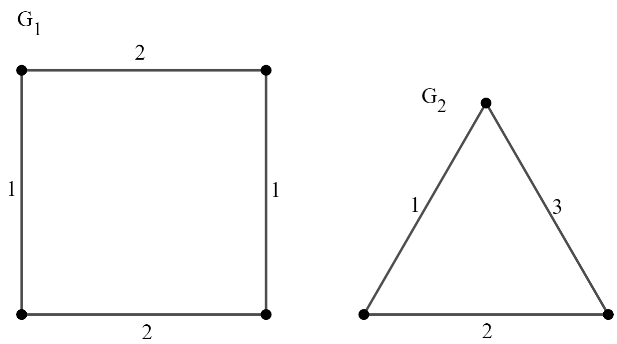

Figure 2.2: $\chi'(G_1) = 2$, and $\chi'(G_2) = 3$.

We will now state results regarding the chromatic index of the graph classes defined in the first chapter.

Theorem 9 ([5]). *A graph C_n has $\chi'(C_n) = 3$ if n is odd, and $\chi'(C_n) = 2$ if n is even.*

Theorem 10 ([5]). *A graph K_n has $\chi'(K_n) = n$ if n is odd, and $\chi'(K_n) = n - 1$ if n is even.*

Theorem 11 (König's Theorem). *If G is a non-empty bipartite graph, then $\chi'(G) = \Delta$.*

The following is an important theorem that classifies a graph according to its chromatic index.

Theorem 12 (Vizing's Theorem). *For every non-empty graph G,*

$$\Delta \leq \chi'(G) \leq \Delta + 1.$$

This theorem originated the classification of graphs into two classes: G is *Class 1* if $\chi'(G) = \Delta$; G is *Class 2* if $\chi'(G) = \Delta + 1$. See Figure 2.3.

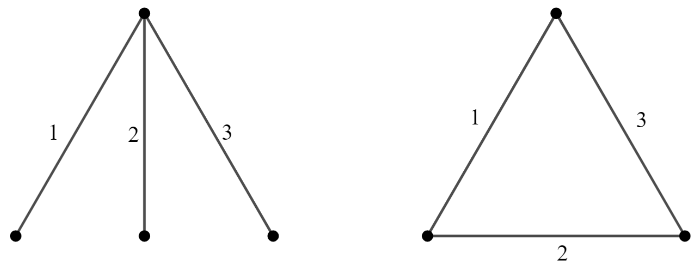

Figure 2.3: A class 1 graph and a class 2 graph, respectively.

In order to state that a graph G has $\chi'(G) = \Delta$, we just need to display an edge coloring with Δ colors. In order to state $\chi'(G) = \Delta + 1$ we must first prove that there is no edge coloring with Δ colors and later, show an edge coloring with $\Delta + 1$ colors.

2.3 Total Coloring

Definition 19. *A total coloring C^T of a graph G is a color assignment to the set of all elements $E \cup V$ in a set of colors $C = \{c_1, c_2, ..., c_k\}$, $k \in \mathbb{N}$, so that different colors are assigned to:*

- *Every pair of vertices that are adjacent;*
- *All edges that are adjacent;*
- *Each vertex and its incident edges.*

A k-total coloring of a G graph is a total coloring of G using a set of k colors, and a graph is k-total colorable if there is a k-total coloring of G.

Definition 20. *The smallest natural k for which G admits a k-total coloring is called total chromatic number of a graph G. This number is denoted by $\chi''(G)$. See Figure 2.4.*

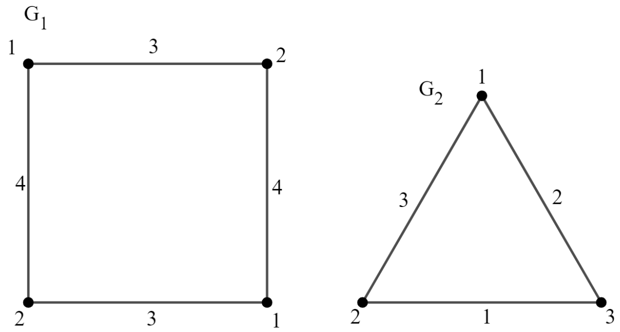

Figure 2.4: $\chi''(G_1) = 4$, and $\chi''(G_2) = 3$.

Behzad and Vizing independently conjectured the same upper bound for the total chromatic number.

Conjecture 1 (Total Coloring Conjecture)**.** *For every simple graph G,*

$$\Delta + 1 \leq \chi''(G) \leq \Delta + 2.$$

If $\chi''(G) = \Delta + 1$ the graph is called Type 1 and if $\chi''(G) = \Delta + 2$ the graph is called Type 2. We refer to a total coloring with $\Delta + 1$ colors as a Type 1 total coloring. We refer to the Total Coloring Conjecture as TCC.

In order to state that a graph G has $\chi''(G) = \Delta + 1$, we just need to display a total coloring with $\Delta+1$ colors. In order to state $\chi''(G) = \Delta+2$ we must first prove that there is no total coloring with $\Delta+1$ colors and later, show a total coloring with $\Delta + 2$ colors. However, if TCC holds for the graph class in question, this second step is not necessary.

Observation 13. *In every total coloring of G with $\Delta + 1$ colors, at every vertex v of maximum degree, we must have all colors represented, i.e., either by coloring v or by coloring one of its Δ incident edges.*

The following lemma is very important to the results that we will obtain in Chapter 4.

Lemma 14 ([6])**.** *If G is a Type 1 graph, then G is conformable.*

Compare Figure 2.1 with two conformable colorings, and Figure 2.4 with two optimal total colorings for the same graphs.

Theorem 15 ([16])**.** *A graph C_n is Type 1 if n is a multiple of 3 and, it is Type 2 otherwise.*

Theorem 16 ([16])**.** *A graph K_n is Type 1 if n is odd and, it is Type 2 if n is even.*

Chapter 3

Cubic Graphs

Cubic graphs will be the main object of study of this work and, therefore, it is important that after we define them, we present classes of cubic graphs that will be discussed throughout the text.

Definition 21. *The* 3*-regular graphs are called cubic graphs.*

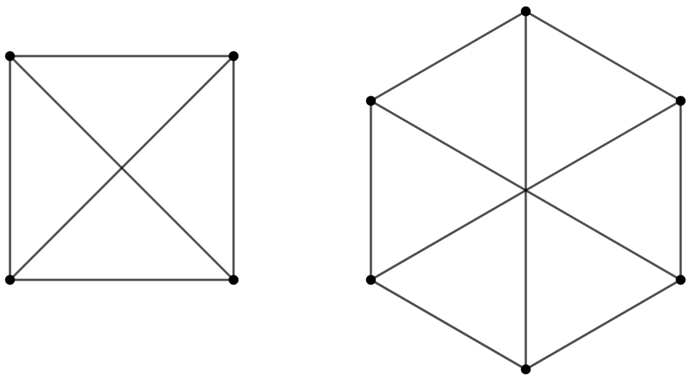

Figure 3.1: Two cubic graphs.

Definition 22. *A graph G is called subcubic if* $\Delta \leq 3$.

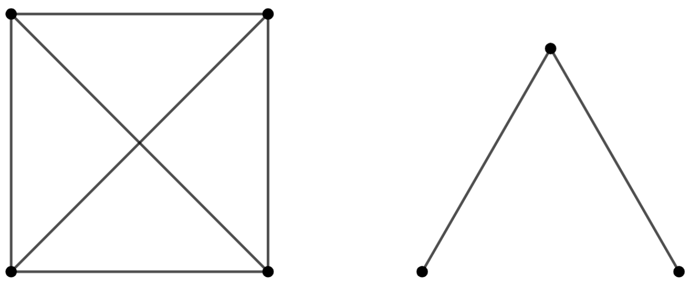

Figure 3.2: Two subcubic graphs.

3.1 Known Classes of Cubic Graphs

In this section, we will present classes of cubic graphs that will be important throughout the work.

Definition 23. *The Petersen graph is the cubic graph with* 10 *vertices and* 15 *edges depicted on the right in Figure 3.3.*

Definition 24. *The generalized Petersen graph, denoted by $G(n, k)$ for $n \geq 3$ and $1 \leq k \leq \frac{n-1}{2}$, is a cubic graph consisting of an inner star polygon and an outer cycle graph with corresponding vertices in the inner and outer polygons connected with edges. Note that a generalized Petersen graph has $2n$ vertices and $3n$ edges.*

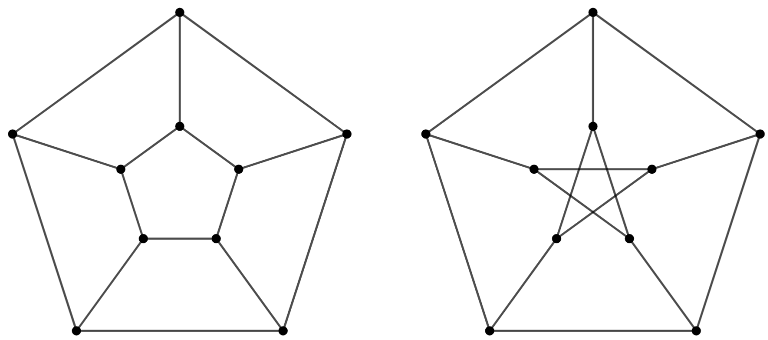

Figure 3.3: The graphs $G(5, 1)$ and $G(5, 2)$, respectively.

Note that the graph $G(5, 2)$ is the Petersen graph.

Definition 25. *The circular ladder graph, denoted by L_n, is a ladder with n vertices and two extra edges that connect each top vertex with its respective bottom vertex. This creates two cycles, one inside and the other outside, that are connected by edges.*

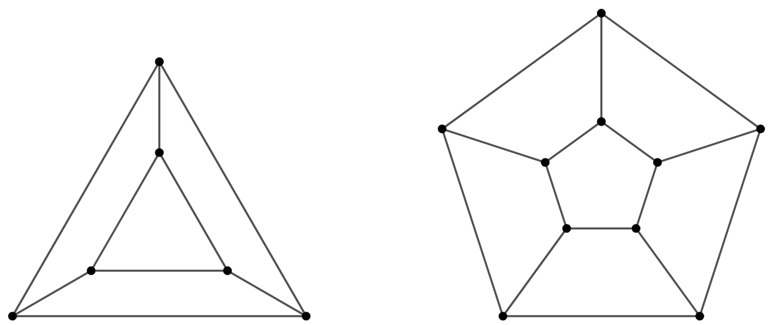

Figure 3.4: The graphs L_6 and L_{10}, respectively.

Note that circular ladder graphs are a subclass of the generalized Petersen graphs.

Definition 26. *The Möbius ladder graph, denoted by M_n, is a cubic graph with n vertices, formed from a n-cycle by adding edges, called "rungs", connecting opposite pairs of vertices in the cycle.*

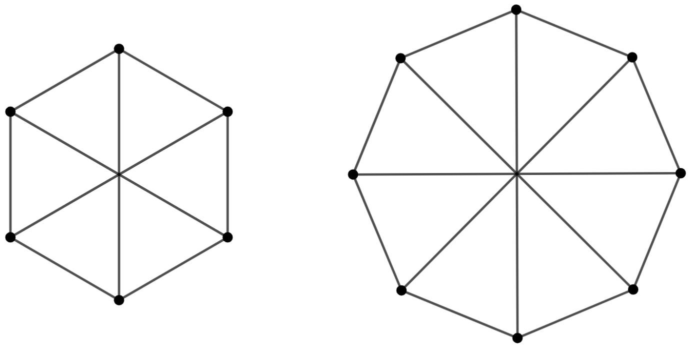

Figure 3.5: The graphs M_6 and M_8, respectively.

3.2 Total Coloring of Cubic Graphs

In this section, we will show results regarding the total coloring of families of cubic graphs that were presented in the previous section. It is important to emphasize that Theorem 18 will provide us with important arguments for the analysis of a conjecture that we will make in the next chapter.

The following theorem guarantees that Conjecture 1 holds for cubic graphs.

Theorem 17 ([8]). *For any subcubic graph G, $\chi''(G) \leq 5$.*

For the benefit of the reader for the well known proof by Chetwynd and Hilton published in 1988, we use the same notation $2n$ for the number of vertices. So in the proof below, n denotes half of the number of vertices, and we prove that besides L_{10}, all circular ladders admit 4-total coloring.

Theorem 18 ([6]). *Every circular ladder graph L_{2n}, with $n \geq 3$ and $n \neq 5$, is Type 1.*

Proof. First, to facilitate the understanding of the proof, we will represent the circular ladder graphs by their notation as generalized Petersen graphs. In addition, we will present another representation of circular ladder graphs. Let the circular ladder graph be $G(3, 1)$. See in Figure 3.6 two ways of representing $G(3, 1)$.

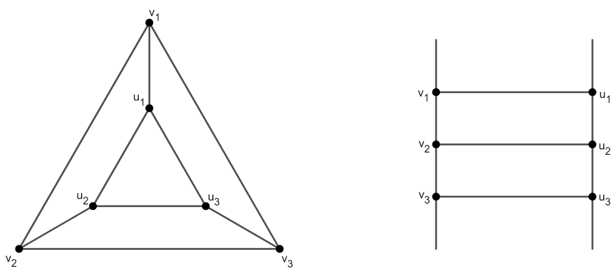

Figure 3.6: Two ways of representing $G(3, 1)$.

Note that in the representation on the right, the line segments connected to vertex v_1 and vertex v_3 represent an edge between them. The same goes for u_1 and u_3. In addition, the edges connecting v_i to u_i are called rung. To prove that a graph is Type 1, we just need to show a total coloring with $\Delta + 1$ colors. Let a graph $G(n, 1)$, with n multiple of 3, that is, $n \equiv 0 \mod 3$. To the graph $G(3, 1)$, we have the following total coloring in Figure 3.7.

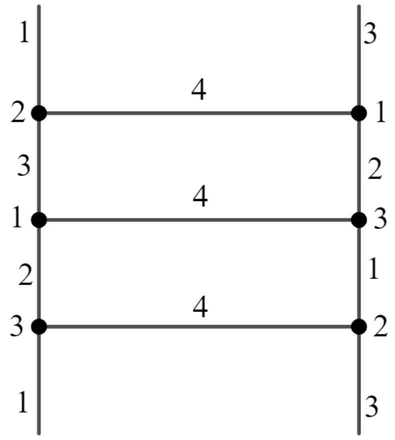

Figure 3.7: A 4-total coloring of $G(3, 1)$.

To color the other $G(n, 1)$ with $n \equiv 0 \mod 3$, just paste copies of $G(3, 1)$ with their 4-total coloring.

Now consider the graphs $G(n, 1)$ with $n \equiv 1 \mod 3$. The first graph that meets this configuration is $G(4, 1)$. We have the following total coloring in Figure 3.8 for it.

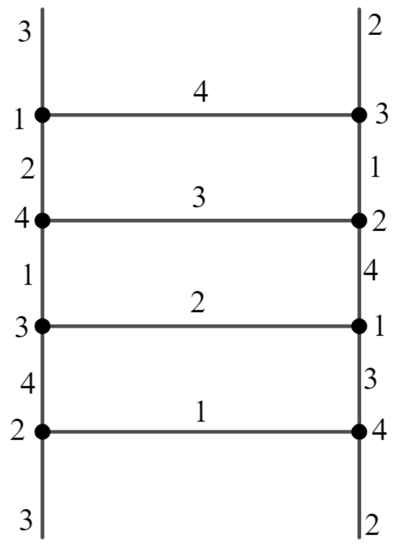

Figure 3.8: A 4-total coloring of $G(4, 1)$.

To obtain the total coloring of the other graphs with $n \equiv 1 \mod 3$, we simply paste the following structure (see Figure 3.9) to the colored graph $G(4, 1)$, which standardizes the coloring whenever we add six more vertices, repeating it according to the number of vertices of the graph in question.

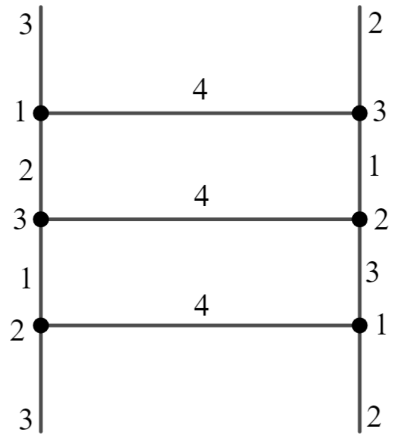

Figure 3.9: Another 4-total coloring for $G(3,1)$ which provides the colored structure that must be pasted repeatedly to the graph $G(4,1)$ to generate a 4-total coloring of the other graphs with $n \equiv 1 \mod 3$. Note that these two 4-total colorings are compatible, whereas the 4-total coloring for $G(3,1)$ presented in Figure 3.7 cannot be used.

Finally, consider the graphs $G(n,1)$ with $n \equiv 2 \mod 3$. The first graph that meets this configuration is $G(8,1)$, since we excluded the graph $G(5,1)$. We have the following total coloring in Figure 3.10 for it.

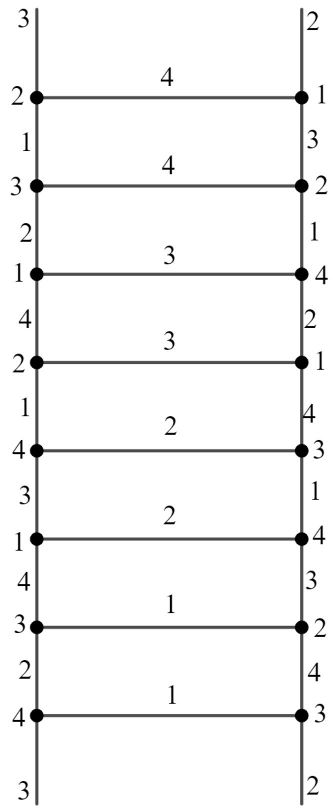

Figure 3.10: A 4-total coloring of $G(8,1)$.

To obtain the total coloring of the other graphs with $n \equiv 2 \mod 3$, we simply paste the following structure (see Figure 3.11) to the colored graph $G(8,1)$, which standardizes the coloring whenever we add six more vertices, repeating it according to the number of vertices of the graph in question.

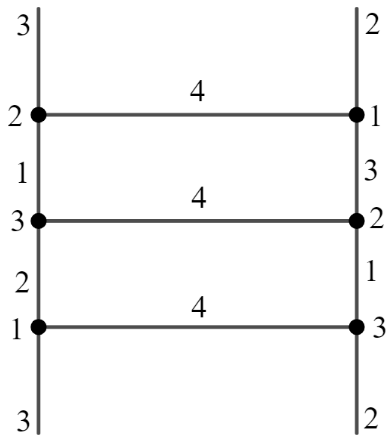

Figure 3.11: Yet another 4-total coloring for $G(3,1)$ which provides the colored structure that must be pasted repeatedly to the graph $G(8,1)$ to generate a 4-total coloring of the other graphs with $n \equiv 2 \mod 3$. Note that these two 4-total colorings are compatible, whereas the 4-total colorings for $G(3,1)$ previously cannot be used.

We conclude that every circular ladder graph L_{2n}, with $n \geq 3$ and $n \neq 5$, admits a 4-total coloring. □

Please refer to Figure 4.7 for the 4-total colorings of L_{12} and L_{18} provided by the above proof. We will consider these colorings again in the next chapter about equitable colorings.

Theorem 19 ([6]). *Every Möbius Ladder graph is Type 2.*

Theorem 20 ([10]). *All generalized Petersen graphs $G(n,k)$, with $k \leq 6$, are Type 1, but $G(5,1)$ and $G(9,3)$.*

It is important to note that the total coloring problem for generalized Petersen graphs is not fully resolved.

3.3 Small Cubic Graphs

In the next chapter, we will present the conjecture that motivated this work. This conjecture has as object cubic graphs with up to 20 vertices. So, we consulted the website "House of graphs", please refer to link `http://hog.grinvin.org/Cubic` or to the link `http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html`. See also the article [4] which talks about this interesting database, so that we could get an idea of how many graphs were being covered by the conjecture and how many graphs there are for each number of vertices. For the convenience of the reader, in Table 3.1 we have the number of connected cubic graphs which are small in the sense that $n < 20$.

Connected Cubic Graphs	
Number of vertices	Number of graphs
4	1
6	2
8	5
10	19
12	85
14	509
16	4060
18	41301

Table 3.1: The exponential growth of connected cubic graphs according to number of vertices.

It is interesting to see that as the number of vertices increases, the number of cubic graphs that have this number of vertices increases abruptly. This makes the analysis of the conjecture that we will present in the next chapter a very challenging problem. See in Figure[1] 3.12 some examples of small cubic graphs.

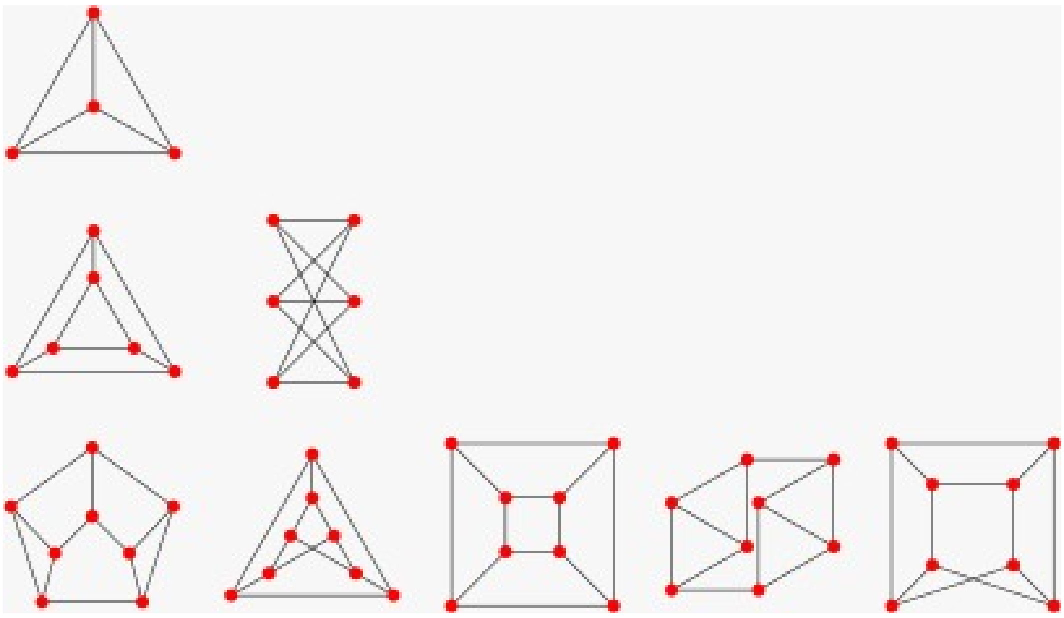

Figure 3.12: The small cubic graphs for $n \leq 8$.

[1]Figure from `https://mathworld.wolfram.com/CubicGraph.html`. Accessed on: 04 Jan. 2022.

Chapter 4

Equitable Total coloring

After presenting definitions and results of graph theory, graph coloring and cubic graphs, we can finally talk about equitable total coloring, the main object of study of this dissertation.

4.1 Basic Definitions and Results

In this section, we will define equitable total coloring and present results in relation to the equitable total coloring of cubic graphs that are our research object.

Definition 27. *A total coloring is called equitable if the cardinalities of any two classes of color differ from at most 1. We denote the equitable total chromatic number by $\chi''_e(G)$.*

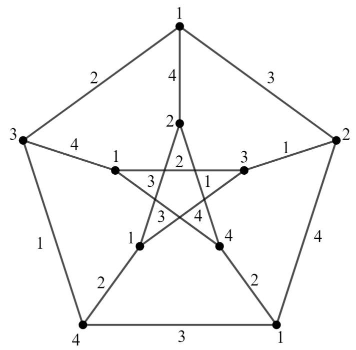

Figure 4.1: An equitable 4-total coloring of the Petersen graph.

See Figure 4.1, where we present an optimal equitable total coloring for the Petersen graph, proving that its equitable total chromatic number is 4. Note that the color 1 has 7 elements, whereas the other color classes have 6 elements each.

For equitable total coloring, we have an important conjecture, corresponding to the Total Coloring Conjecture TCC:

Conjecture 2 (Equitable Total Coloring Conjecture)**.** *For every simple graph G,*

$$\Delta + 1 \leq \chi''_e(G) \leq \Delta + 2.$$

We refer to the Equitable Total Coloring Conjecture as ETCC. As with total coloring, in order to state that a graph G has $\chi''_e(G) = \Delta+1$, we just need to display an equitable total coloring with $\Delta + 1$ colors. In order to state $\chi''_e(G) = \Delta + 2$ we must first prove that there is no equitable total coloring with $\Delta+1$ colors and later, show an equitable total coloring with $\Delta + 2$ colors. However, if ETCC holds for the graph class in question, this second step is not necessary.

The next theorem is a result that ensures that the Equitable Total Coloring Conjecture ETCC holds for cubic graphs.

Theorem 21 ([14])**.** *If G is a multigraph with $\Delta \leq 3$, then G has a equitable 5-total coloring.*

Furthermore, the ETCC holds for cycle graphs and complete graphs. This follows from the Theorems 15 and 16 which can be found in [16]. They determine the equitable total chromatic number of these two classes of graphs since the colorings presented in them are equitable. The ETCC also holds for bipartite graphs [9].

The next theorem guarantees that the equitable total coloring problem is difficult.

Theorem 22 ([7])**.** *The problem of deciding whether a cubic bipartite graph has an equitable 4-total coloring is $\mathcal{NP}$-complete.*

Now we are able to enunciate important properties about equitable total coloring of cubic graphs.

Proposition 23 ([11])**.** *The number of elements that must be colored in a 4-total coloring C of a cubic graph G is $n + \frac{3n}{2} = \frac{5n}{2}$.*

Proposition 24 ([11])**.** *Let C be a 4-total coloring of a cubic graph G. The total coloring C is equitable if and only if the number of vertices of each color class differs from at most 2.*

The following theorems were very important in our research. The Theorem 25 served as inspiration for us to build our results that will be presented in the next section. The Theorem 26 helped us find counterexamples to the conjecture presented in the next section as well.

Theorem 25 ([12])**.** *All 4-total colorings of the Petersen graph are equitable.*

Proof. The Petersen graph has 10 vertices and 15 edges. Then, to obtain an equitable 4-total coloring, we must use one color in 7 elements and the other three colors in 6 elements. Therefore, to generate a non-equitable coloring, we would need to color 8 elements with one color or use two colors to color 7 elements each. This way, we would have a color coloring only 5 elements. First, let's consider how many edges a single color can be used on. Suppose we color 4 of the 15 edges with color 1. We know that we cannot use this color in the vertices incident to these 4 edges, so we can assign color 1 to only two vertices. So, if we want to use a single color 7 or more times, this color can color at most 3 edges. Let's do the same analysis for the vertices of the graph. Consider in Figure 4.2 a drawing of the Petersen graph, where we have an external 5-cycle joined by a matching to an inner 5-cycle. We can only color with 1 two vertices in each cycle, that is, we can color with one color at most four vertices. Thus, a single color can be used on at most four vertices and at most three edges. Therefore, we could only have non-equitable coloring using two colors in 7 elements each. To exhaust the last possible case, it remains for us to investigate the assignment of these colors to the vertices of the graph. We know that each color can be used on at most four vertices. Let's explore this possibility with the colors 1 and 2. As already mentioned, the same color can only be used twice in the inner cycle and twice in the outer cycle. Thus, the only way of using a color labeled by 1 in four vertices can be seen in the Figure 4.2.

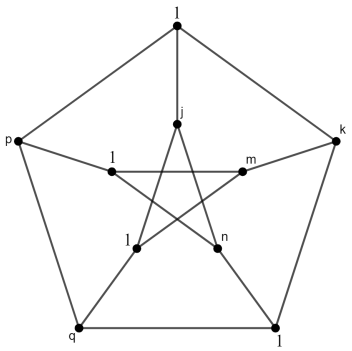

Figure 4.2: A maximum independent set consisting of 4 vertices labeled by color 1.

It remains six vertices to be colored by the other colors. In order to have another color class with seven elements, we need four of these six remaining vertices to define an independent set. This is not possible since an independent set can contain at most three vertices: k or m, j or n, and p or q. Thus, the color 3 can only be used on at most three vertices, so it cannot be used in 7 elements. We conclude that two colors cannot color 7 elements each in the Petersen graph and no color can be used more than 7 times, either. This means that the only 4-total coloring of the Petersen graph is using one color in 7 elements and the other colors in 6 elements each. Hence, all the 4-total colorings of the Petersen graph are equitable. □

By Theorem 18, every circular ladder graph L_n, with $n \geq 6$, is Type 1, except for L_{10}, which is Type 2. Actually, the next theorem establishes that every circular ladder graph L_n, with $n \geq 6$ and $n \neq 10$, admits an equitable 4-total coloring.

Theorem 26 ([7])**.** *For every $n \geq 6$ and $n \neq 10$, the circular ladder graph L_n has equitable total chromatic number 4.*

Note that since ETCC holds for cubic graphs, the Type 2 graphs Möbius ladders and L_{10} have equitable total chromatic number 5.

4.2 Stemock's conjecture

The following conjecture stablished by Stemock [12] was the main motivation for our work. After finding this conjecture, we started to investigate evidence for it in cubic graph classes where total coloring had already been studied. We consider the conjecture quite relevant, as it refers to more than 40000 graphs. See Table 3.1. The order of a graph G is its number of vertices, denoted by n.

Conjecture 3 (Stemock's Conjecture [12])**.** *Every 4-total coloring of a cubic graph G is equitable, given that the order of G is less than 20.*

This upper bound was motivated by a Type 1 graph with 20 vertices that can be found in [7] that does not have an equitable total coloring with 4 colors. See Figures 4.3 and 4.4.

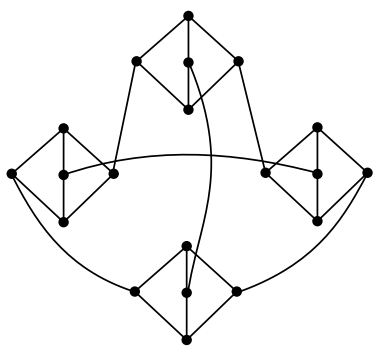

Figure 4.3: The graph R with 20 vertices that motivated Stemock's conjecture.

Note that the graph R in Figure 4.3 has four copies of $K_{2,3}$ as a subgraph. The complete bipartite graph $K_{2,3}$ graph has a unique 4-total coloring. Thus, the edges connecting these copies of $K_{2,3}$ receive colors that force every 4-total coloring of this graph with 20 vertices non-equitable.

After some research, we concluded that Conjecture 3 is false. We reached this conclusion by consulting two articles. In [6], the authors present a method to obtain the total coloring of circular ladder graphs. Refer to Theorem 18 to see their method. By this method, the cubic graphs L_{12} and L_{18} have a 4-total coloring that is not equitable. However, in [7] we have that the class L_n has an equitable 4-total coloring. These two results make us believe that the conjecture can be true if we consider

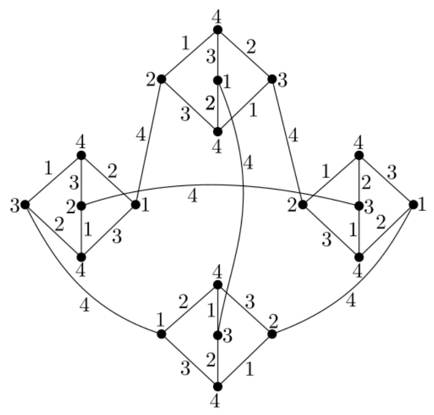

Figure 4.4: An non-equitable 4-total coloring of R.

that the cubic graph with $n < 20$ has at least one equitable 4-total coloring. Moving forward in this direction, we will enunciate a new question with the proposed added constraints.

Question 1. *Does every Type 1 cubic graph, with $n < 20$, have at least one equitable 4-total coloring?*

Another relevant question is to verify whether the graph L_{12} is the least counterexample to Stemock's conjecture.

Question 2. *Is the graph L_{12} the smallest counterexample to the Conjecture 3?*

In the next section, we will answer Question 2.

4.2.1 Our results

By analyzing the number of elements of a cubic graph, for each $n < 20$ in accordance with Stemock's conjecture, we get the following results. Note that K_4 and M_6 are both Type 2 and both have an equitable 5-total coloring. The graph L_6 has an equitable 4-total coloring by Theorem 26. Thus, we will start the proofs for $n = 6$ to answer the Question 2. In all the theorems in this section we use the Lemma 14, that is, all the total independent sets considered in the proofs have an even number of vertices.

Theorem 27. *All 4-total colorings of Type 1 cubic graphs with 6 vertices are equitable.*

Proof. Note that a cubic graph of 6 vertices has 9 edges. Thus, the equitable 4-total coloring of these graphs will look like this: each color class colors exactly 4, 4, 4, and 3 elements respectively. Furthermore, the number of vertices in each color class will necessarily be even, by Lemma 14. If there is a total independent set of

5 elements it would be possible to assign a color to this set and the coloring might not be equitable. However, we will prove that a total independent set of 5 elements does not exist in a cubic graph of 6 vertices.

Let's divide the proof into cases according to the number of elements of each kind in a total independent set of 5 elements.

- 4 vertices and 1 edge.

  Consider an independent set S with 4 vertices. There would be only 2 more vertices left in the graph. Select a vertex $v \in S$. Note that v can only be adjacent to 2 vertices that do not belong to S. Thus, $d(v) \leq 2$ and the graph would not be cubic.

- 2 vertices and 3 edges.

  Note that if we consider a matching with 3 edges, this will prevent us from selecting 6 vertices. Thus, there are no other vertices left to be selected for the total independent set.

- 0 vertices and 5 edges.

  Note that for us to have a graph with a matching of 5 edges, it must have at least 10 vertices. However, the graph has only 6 vertices.

We conclude that all total colorings of Type 1 cubic graphs with 6 vertices must be equitable. □

Theorem 28. *All 4-total colorings of Type 1 cubic graphs with* 8 *vertices are equitable.*

Proof. Note that a cubic graph of 8 vertices has 12 edges. Thus, the equitable 4-total coloring of these graphs will look like this: each color class colors exactly 5 elements. Furthermore, the number of vertices in each color class will necessarily be even, by Lemma 14. If there is a total independent set of 6 elements it would be possible to assign a color to this set and the coloring might not be equitable. However, we will prove that a total independent set of 6 elements does not exist in a cubic graph of 8 vertices.

Let's divide the proof into cases according to the number of elements of each kind in a total independent set of 6 elements.

- 6 vertices and 0 edges.

  Consider an independent set S with 6 vertices. There would be only 2 more vertices left in the graph. Select a vertex $v \in S$. Note that v can only be adjacent to 2 vertices that do not belong to S. Thus, $d(v) \leq 2$ and the graph would not be cubic.

- 4 vertices and 2 edges.

  Consider the vertices $v_1, v_2, v_3, v_4 \in S$ and the edges $e_1, e_2 \in M$. Note that we need to construct a graph of 8 vertices and 12 edges with this total independent set. Since the graph must be cubic, each vertex of S must have $d(v) = 3$. For that, each vertex of S should have 3 edges incident to it. Thus, we would need another 12 edges, in addition to the 2 edges of M, totaling 14 edges. However, the graph has only 12 edges.

- 2 vertices and 4 edges.

  Note that if we consider a matching with 4 edges, this will prevent us from selecting 8 vertices. Thus, there are no other vertices left to be selected for the total independent set.

- 0 vertices and 6 edges.

  Note that for us to have a graph with a matching of 6 edges, it must have at least 12 vertices. However, the graph has only 8 vertices.

We conclude that all total colorings of Type 1 cubic graphs with 8 vertices must be equitable. □

Note that our next theorem generalizes Theorem 25 which proves the result for the Petersen graph.

Theorem 29. *All 4-total colorings of Type 1 cubic graphs with* 10 *vertices are equitable.*

Proof. Note that a cubic graph of 10 vertices has 15 edges. Thus, the equitable 4-total coloring of these graphs will look like this: each color class colors exactly 7, 6, 6, and 6 elements respectively. Furthermore, the number of vertices in each color class will necessarily be even, by Lemma 14. If there is a total independent set of 8 elements or two total independent sets of 7 elements it would be possible to assign a color to these sets and the coloring might not be equitable. However, we will prove that a total independent set of 8 elements does not exist on a cubic graph of 10 vertices and that two total independent sets of 7 elements can't exist simultaneously on a cubic graph of 10 vertices.

Let's divide the proof into cases according to the number of elements of each kind in a total independent set of 8 elements or two total independent sets of 7 elements.

- 8 vertices and 0 edges.

  Consider an independent set S with 8 vertices. There would be only 2 more vertices left in the graph. Select a vertex $v \in S$. Note that v can only be adjacent to 2 vertices that do not belong to S. Thus, $d(v) \leq 2$ and the graph would not be cubic.

- 6 vertices and 2 edges.

  Consider the vertices $v_1, v_2, v_3, v_4, v_5, v_6 \in S$ and the edges $e_1, e_2 \in M$. Note that we need to construct a graph of 10 vertices and 15 edges with this total independent set. Since the graph must be cubic, each vertex of S must have $d(v) = 3$. For that, each vertex of S should have 3 edges incident to it. Thus, we would need another 18 edges, in addition to the 2 edges of M, totaling 20 edges. However, the graph has only 15 edges.

- 4 vertices and 4 edges.

  Note that if we consider a matching with 4 edges, this will prevent us from selecting 8 vertices. As the graph has 10 vertices, there will only be 2 vertices left to select for a total independent set.

- 2 vertices and 6 edges.

  Note that for us to have a graph with a matching of 6 edges it should have at least 12 vertices. However, the graph has only 10 vertices.

- 0 vertices and 8 edges.

  Note that for us to have a graph with matching of 8 edges, it should have at least 16 vertices. However, the graph has only 10 vertices.

- two total independent sets with 7 elements: 4 vertices and 3 edges each.

  Consider the vertices $v_1, v_2, v_3, v_4 \in S$ and the edges $e_1, e_2, e_3 \in M$ where the vertices a_1, a_2, a_3, a_4, a_5 and a_6 are the extremes of the edges e_1, e_2 and e_3 respectively. These elements form a total independent set. See Figure 4.5.

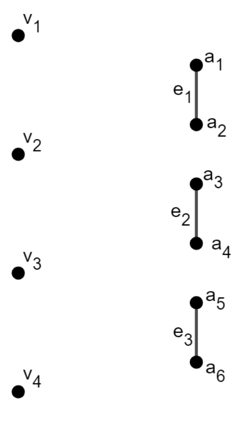

Figure 4.5: Diagram of case with 4 vertices and 3 edges in a total independent set of 7 elements, considering a graph with $n = 10$ vertices.

Note that we could only select the vertices of e_1, e_2, and e_3 for the other total independent set. So we could only select three vertices, one from each edge. Then, a graph of 10 vertices does not have two total independent sets of 4 vertices and 3 edges each.

We conclude that all total colorings of Type 1 cubic graphs with 10 vertices must be equitable. □

The only cubic graph with 4 vertices is K_4 which is Type 2. The two graphs, with 6 vertices, are L_6 and M_6 which are Type 1 and Type 2, respectively. All these colorings are equitable. These results and the Theorems 28 and 29 guarantee that Question 2 is true. Then, we continue to use this technique of analyzing the total independent sets of the cubic graphs for the remaining ones with $n < 20$.

Theorem 30. *All 4-total colorings of Type 1 cubic graphs with* 14 *vertices are equitable.*

Proof. Note that a cubic graph of 14 vertices has 21 edges. Thus, the equitable 4-total coloring of these graphs will look like this: each color class colors exactly 9, 9, 9, and 8 elements. Furthermore, the number of vertices in each color class will necessarily be even, by Lemma 14. If there is a total independent set of 10 elements it would be possible to assign a color to this set and the coloring might not be equitable. However, we will prove that a total independent set of 10 elements does not exist in a cubic graph of 14 vertices.

Let's divide the proof into cases according to the number of elements of each kind in a total independent set of 10 elements.

- 10 vertices and 0 edges.

  Consider a independent set S with 10 vertices and a set $V - S$ with 4 vertices. Note that each vertex of $V - S$ must have 3 edges incident to it. Since the vertices of S do not have edges between them, we would have at most 12 edges in the graph. However, the graph has 21 edges.

- 8 vertices and 2 edges.

  Consider the vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \in S$ and the edges $e_1, e_2 \in M$. Note that we need to construct a graph of 14 vertices and 21 edges with this total independent set. Since the graph must be cubic, each vertex of S must have $d(v) = 3$. For that, each vertex of S should have 3 edges incident to it. Thus, we would need another 24 edges, in addition to the 2 edges of M, totaling 26 edges. However, the graph has only 21 edges.

- 6 vertices and 4 edges.

  Consider the vertices $v_1, v_2, v_3, v_4, v_5, v_6, \in S$ and the edges $e_1, e_2, e_3, e_4 \in M$. Note that we need to construct a graph of 14 vertices and 21 edges with this total independent set. Since the graph must be cubic, each vertex of S must have $d(v) = 3$. For that, each vertex of S should have 3 edges incident to it. Thus, we would need another 18 edges, in addition to the 4 edges of M, totaling 22 edges. However, the graph has only 21 edges.

- 4 vertices and 6 edges.

  Note that if we consider a matching with 6 edges, this will prevent us from selecting 12 vertices. Thus, there would only be 2 vertices left in the graph and we would need another 4 for the total independent set.

- 2 vertices and 8 edges.

  Note that for us to have a graph with a matching of 8 edges, it must have at least 16 vertices. However, the graph has 14 vertices.

- 0 vertices and 10 edges.

  Note that for us to have a graph with a matching of 10 edges, it must have at least 20 vertices. However, the graph has 14 vertices.

We conclude that all total colorings of Type 1 cubic graphs with 14 vertices must be equitable. □

We then conclude that all 4-total colorings of cubic graphs with 6, 8, 10 or 14 vertices are necessarily equitable. We will deal with the other values of $n < 20$ in the next section.

4.2.2 A closer look at counterexamples

We will make a study of cubic graphs, with 12 and 18 vertices, in light of the counterexamples L_{12} and L_{18} of the Stemock's conjecture. See two of those graphs depicted in Figure 4.6.

Note that a cubic graph of 12 vertices has 30 elements. Thus, the graph would have a non-equitable 4-total coloring, if there are three independent sets of 8 elements. Hence, the fourth color would color 6 elements. See such a non-equitable 4-total coloring depicted in Figure 4.7.

Similarly, a cubic graph of 18 vertices has 45 elements. Thus, the graph would have a non-equitable 4-total coloring, if there are three independent sets of 12 elements. Hence, the fourth color would color 9 elements. See such a non-equitable 4-total coloring depicted in Figure 4.7.

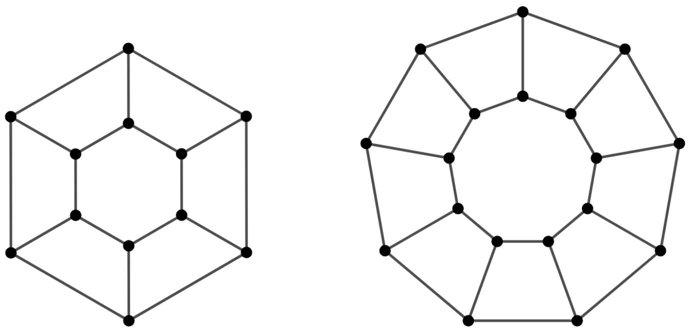

Figure 4.6: The graphs L_{12} and L_{18}, respectively.

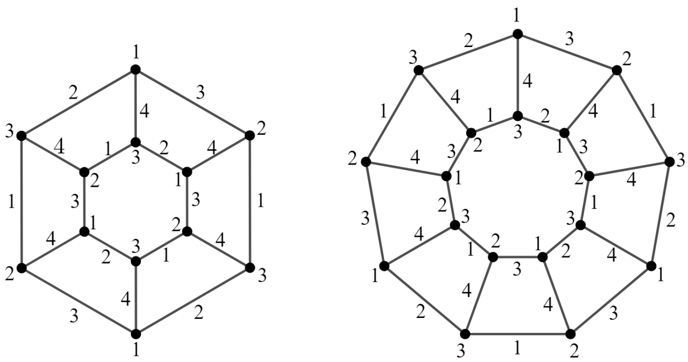

Figure 4.7: L_{12} and L_{18} and their non-equitable 4-total colorings.

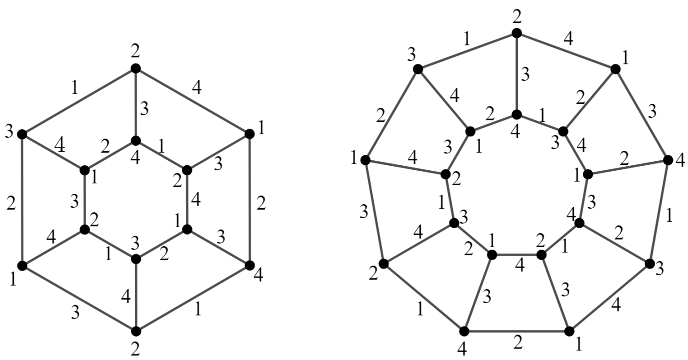

Figure 4.8: L_{12} and L_{18} and their equitable 4-total colorings.

However, we present in Figure 4.8 equitable 4-total colorings for the same graphs, with a different color class configuration. In the next two theorems, we will show the color class configurations that allow a cubic graph of 12 or 18 vertices to have a non-equitable 4-total coloring.

Theorem 31. *The only color class configuration that might allow a non-equitable 4-total coloring of a cubic graph of* 12 *vertices is* 8, 8, 8, *and* 6.

Proof. Note that a cubic graph of 12 vertices has 18 edges. Thus, the equitable 4-total coloring of these graphs will look like this: each color class colors exactly 8, 8, 7, and 7 elements. Furthermore, the number of vertices in each color class will necessarily be even, by Lemma 14. If there is a total independent set of 9 elements it would be possible to assign a color to this set and the coloring might not be equitable. However, we will prove that a total independent set of 9 elements does not exist in a cubic graph of 12 vertices. Another possibility would be three total independent sets of 8 elements. This possibility has already been verified by the color classes of the non-equitable 4-total coloring of L_{12} which guarantees that it is possible for a cubic graph with 12 vertices to present this structure.

Let's divide the proof into cases according to the number of elements of each kind in a total independent set of 9 elements.

- 8 vertices and 1 edge.

  Consider a independent set S with 8 vertices and a set $V - S$ with 4 vertices. Note that each vertex of $V - S$ must have 3 edges incident to it. Since the vertices of S do not have edges between them, we would have at most 12 edges in the graph. However, the graph has 18 edges.

- 6 vertices and 3 edges.

  Consider the vertices $v_1, v_2, v_3, v_4, v_5, v_6, \in S$ and the edges $e_1, e_2, e_3 \in M$. Note that we need to construct a graph of 12 vertices and 18 edges with this total independent set. Since the graph must be cubic, each vertex of S must have $d(v) = 3$. For that, each vertex of S should have 3 edges incident to it. Thus, we would need another 18 edges, in addition to the 3 edges of M, totaling 21 edges. However, the graph has only 18 edges.

- 4 vertices and 5 edges.

  Consider the vertices $v_1, v_2, v_3, v_4 \in S$ and the edges $e_1, e_2, e_3, e_4, e_5 \in M$. Note that we need to construct a graph of 12 vertices and 18 edges with this total independent set. Since the graph must be cubic, each vertex of S must have $d(v) = 3$. For that, each vertex of S should have 3 edges incident to it. Thus, we would need another 12 edges, in addition to the 5 edges of M, totaling 17 edges. We are left with only one edge. Note that the 12 edges connecting the vertices of S to the vertices of M ensure that all vertices of S have degree 3. However, with just one more edge, it will not be possible to guarantee that all vertices of M are cubic.

- 2 vertices and 7 edges.

  Note that for us to have a graph with a matching of 7 edges, it must have at least 14 vertices. However, the graph has only 12 vertices.

- 0 vertices and 9 edges.

  Note that for us to have a graph with a matching of 9 edges, it must have at least 18 vertices. However, the graph has only 12 vertices.

We conclude that the only color class configuration that might allow a non-equitable 4-total coloring of a cubic graph of 12 vertices is 8, 8, 8, and 6. □

We show next a generalized Petersen graph with 12 vertices with a structure similar to the graph L_{12} and its equitable total coloring and non-equitable total coloring. See Figure 4.9.

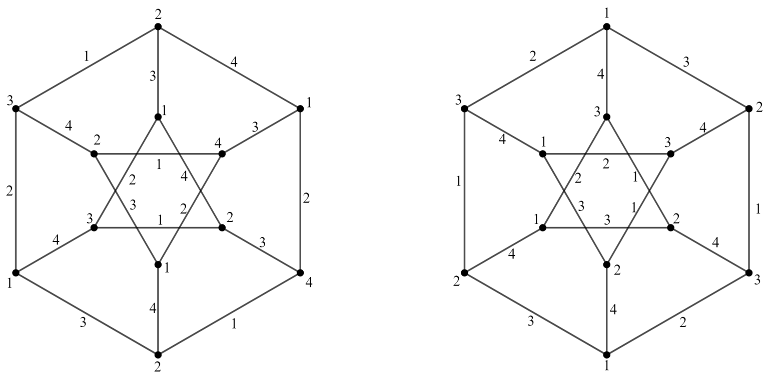

Figure 4.9: The graph $G(6,2)$ and its equitable 4-total coloring and non-equitable 4-total coloring, respectively.

Theorem 32. *There are two color class configurations that might allow a non-equitable* 4*-total coloring of a cubic graph of* 18 *vertices:* 12*,* 12*,* 12*, and* 9*, or* 12*,* 12*,* 10*, and* 10*.*

Proof. Note that a cubic graph of 18 vertices has 27 edges. Thus, the equitable 4-total coloring of these graphs will look like this: each color class colors exactly 12, 11, 11, and 11 elements. Furthermore, the number of vertices in each color class will necessarily be even, by Lemma 14. If there is a total independent set of 13 elements it would be possible to assign a color to this set and the coloring might not be equitable. However, we will prove that a total independent set of 13 elements does not exist in a cubic graph of 18 vertices. Another possibilities would be three or two total independent sets of 12 elements. The first possibility has already been verified by the color classes of the non-equitable 4-total coloring of L_{18} which guarantees that it is possible for a cubic graph with 12 vertices to present this structure. See such a non-equitable Type 1 coloring depicted in Figure 4.7.

Let's divide the proof into cases according to the number of elements of each kind in a total independent set of 13 elements.

- 12 vertices and 1 edge.

  Consider the vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12} \in S$ and the edges $e_1 \in M$. Note that we need to construct a graph of 18 vertices and 27 edges with this total independent set. Since the graph must be cubic, each vertex of S must have $d(v) = 3$. For that, each vertex of S should have 3 edges incident to it. Thus, we would need another 36 edges, in addition to the 1 edge of M, totaling 37 edges. However, the graph has only 27 edges.

- 10 vertices and 3 edges.

  Consider the vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10} \in S$ and the edges $e_1, e_2, e_3 \in M$. Note that we need to construct a graph of 18 vertices and 27 edges with this total independent set. Since the graph must be cubic, each vertex of S must have $d(v) = 3$. For that, each vertex of S should have 3 edges incident to it. Thus, we would need another 30 edges, in addition to the 3 edges of M, totaling 33 edges. However, the graph has only 27 edges.

- 8 vertices and 5 edges.

  Consider the vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \in S$ and the edges $e_1, e_2, e_3, e_4, e_5 \in M$. Note that we need to construct a graph of 18 vertices and 27 edges with this total independent set. Since the graph must be cubic, each vertex of S must have $d(v) = 3$. For that, each vertex of S should have 3 edges incident to it. Thus, we would need another 24 edges, in addition to the 5 edges of M, totaling 29 edges. However, the graph has only 27 edges.

- 6 vertices and 7 edges.

  Note that if we consider a matching with 7 edges, this will prevent us from selecting 14 vertices.Thus, there would only be 4 vertices left in the graph and we would need another 6 vertices for the total independent set with 13 elements.

- 4 vertices and 9 edges.

  Note that if we consider a matching with 9 edges, this will prevent us from selecting 18 vertices. Thus, there would not be vertices left in the graph for the total independent set with 13 elements.

- 2 vertices and 11 edges.

  Note that for us to have a graph with a matching of 11 edges, it must have at least 22 vertices. However, the graph has only 18 vertices.

- 0 vertices and 13 edges.

  Note that for us to have a graph with a matching of 13 edges, it must have at least 26 vertices. However, the graph has only 18 vertices.

We conclude that there are two color class configurations that might allow a non-equitable 4-total coloring of a cubic graph of 18 vertices: 12, 12, 12, and 9, or 12, 12, 10, and 10. □

Additionally, we show next a generalized Petersen graph with 18 vertices with a structure similar to the graph L_{18} and its equitable total coloring and non-equitable total coloring. See Figure 4.10.

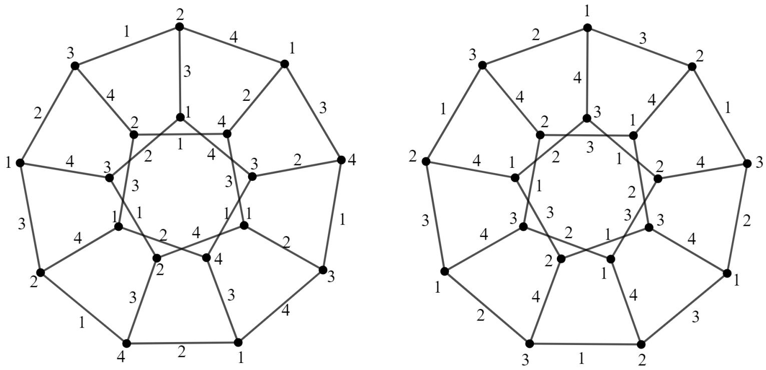

Figure 4.10: The graph $G(9,2)$ and its equitable 4-total coloring and non-equitable 4-total coloring, respectively.

It is important to mention that the existence of these total independent sets in cubic graphs with 12 and 18 vertices allows the possibility of obtaining a non-equitable 4-total coloring, but does not guarantee its existence.

Reading the article [6] gave us these counterexamples to the Stemock's conjecture with $n = 12$ and $n = 18$ vertices. However, the 4-total coloring given in [6] for L_{16} is equitable. Next, for $n = 16$, we will present a counterexample obtained by our method.

Theorem 33. *The only color class configuration that might allow a non-equitable 4-total coloring of a cubic graph of 16 vertices is 11, 10, 10 and 9.*

Proof. Note that a cubic graph of 16 vertices has 24 edges. Thus, the equitable 4-total coloring of these graphs will look like this: each color class colors exactly 10 elements. Furthermore, the number of vertices in each color class will necessarily be even, by Lemma 14. If there is a total independent set of 11 elements it would be possible to assign a color to this set and the coloring might not be equitable. We will show that there is only one possible configuration of a total independent set with 11 elements in a cubic graph of 16 vertices. This configuration will generate the only possible way a cubic graph with 16 vertices may have a non-equitable 4-total coloring.

Let's divide the proof into cases according to the number of elements of each kind in a total independent set of 11 elements.

- 10 vertices and 1 edge.

  Consider an independent set S with 10 vertices. There would be only 2 more vertices left in the graph. Select a vertex $v \in S$. Note that v can only be adjacent to 2 vertices that do not belong to S. Thus, $d(v) \leq 2$ and the graph would not be cubic.

- 8 vertices and 3 edges.

  Consider the vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \in S$ and the edges $e_1, e_2, e_3 \in M$. Note that we need to construct a graph of 16 vertices and 24 edges with this total independent set. Since the graph must be cubic, each vertex of S must have $d(v) = 3$. For that, each vertex of S should have 3 edges incident to it. Thus, we would need another 24 edges, in addition to the 3 edges of M, totaling 27 edges. However, the graph has only 21 edges.

- 4 vertices and 7 edges.

  Note that if we consider a matching with 7 edges, this will prevent us from selecting 14 vertices.Thus, there would only be 2 vertices left in the graph and we would need another 4 vertices for the total independent set with 11 elements.

- 2 vertices and 9 edges.

  Note that for us to have a graph with a matching of 9 edges, it must have at least 18 vertices. However, the graph has only 16 vertices.

- 0 vertices and 11 edges.

  Note that for us to have a graph with a matching of 11 edges, it must have at least 22 vertices. However, the graph has only 16 vertices.

- 6 vertices and 5 edges.

  Consider the vertices $v_1, v_2, v_3, v_4, v_5, v_6, \in S$ and the edges $e_1, e_2, e_3, e_4, e_5 \in M$. Note that we need to construct a graph of 16 vertices and 24 edges with this total independent set. Since the graph is cubic, every vertex of S must have $d(v) = 3$. For that, each vertex of S should have 3 edges incident to it. Thus, we would need another 18 edges, in addition to the 5 edges of M, totaling 23 edges. Thus, it is still necessary to assign an edge to this graph. First, let's analyze the possibilities to assign the 18 edges that depart the vertices of S. These must necessarily connect the vertices that are extremes of the edges of M.

  The first possibility is to leave a vertex of an edge in M with no neighbors in S. In this way, all vertices of S and all vertices of M, except the ones of e_5 will have $d(v) = 3$. Thus, it will not be possible to add one more edge. See Figure 4.11.

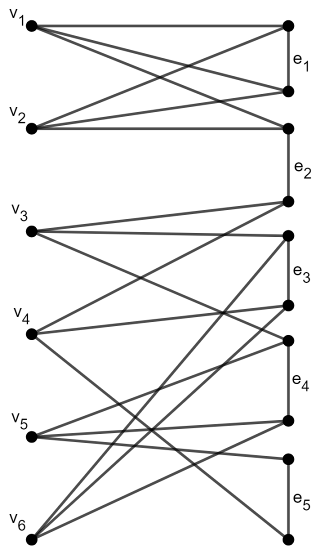

Figure 4.11: First possibility of the case with 6 vertices and 5 edges, considering a graph with $n = 16$ vertices.

The second possibility is to assign the 18 edges starting from S so that two vertices of distinct edges of M are such that each is adjacent to only two vertices in S.

In this possibility, we can assign an edge between these two vertices, please refer to Figure 4.12, completing the 24 edges of the graph.

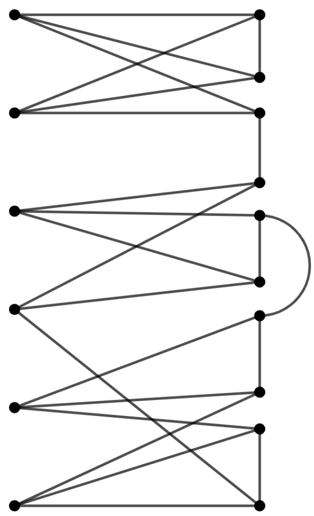

Figure 4.12: Second possibility of the case with 6 vertices and 5 edges, considering a graph with $n = 16$ vertices.

Note that, we cannot have two disjoint total independent sets each with 11 elements. If there were another disjoint total independent set, by the above analysis, it consists of 6 vertices and 5 edges. However, note that we cannot select 6 independent vertices among the 10 that are extreme edges of M.

We conclude that the only color class configuration that might allow a non- equitable

4-total coloring of a cubic graph of 16 vertices is 11, 10, 10, and 9. □

Since in a cubic graph with 16 vertices we have 40 elements, we may assume that there are four disjoint total independent sets with respective sizes 11, 10, 10, 9. Since the graph contains an independent set S of size 6, and by Observation 13, each of the remaining three colors must be represented at each vertex s of S by an edge between s and an extreme vertex of an edge of M, each of the remaining three colors must have respectively 6, 6, and 7 edges. See Figure 4.13 for an example of such a coloring.

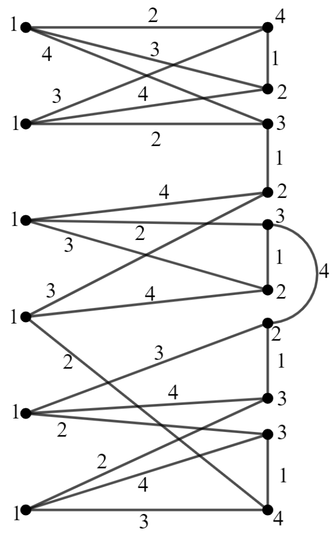

Figure 4.13: A non-equitable 4-total coloring of a cubic graph H with 16 vertices.

A natural question when obtaining this coloring is whether or not this graph, which we call H, has an equitable 4-total coloring. If it did not, Question 1 would be false. However, we get an equitable 4-total coloring for this graph. See Figure 4.14.

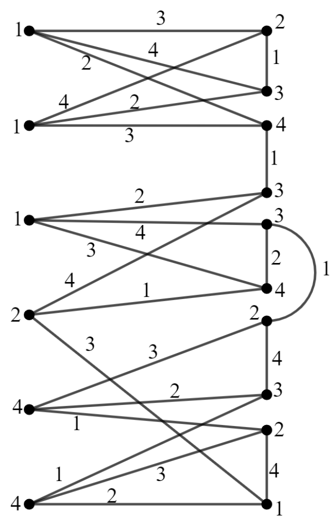

Figure 4.14: An equitable 4-total coloring of a cubic graph H with 16 vertices.

4.3 Concluding Remarks

The equitable total coloring problem is relatively recent since its main conjecture ETCC dates back to 2002. Thus, it is a very fertile problem with many results to be obtained. The study of the articles [12], [7] and [6] helped us to formulate the Questions 1 and 2.

We were able to satisfactorily answer Question 2 and identified a technique that could be useful in future work. We observe that although Question 2 refers to cubic graphs with less than 12 vertices, our work shows that besides $n = 6, 8$, and 10, cubic graphs with $n = 14$ must have all 4-total colorings equitable, which leads to another question:

Question 3. *What is the largest value of* n*, such that every Type 1 cubic graph with* n *vertices is such that all of its 4-total colorings are equitable?*

The Theorems 27, 28 and 29, respectively for $n = 6, 8$ and 10, although they cover a small number of graphs, allowed us to study the equitable total coloring by analyzing the total independent sets. Our Theorem 29 generalizes to an arbitrary graph with 10 vertices the previous Theorem 25 that proves the result only for the Petersen graph. We then use the technique of analyzing the total independent sets to formalize Theorems 30, 32 and 33, respectively for $n = 14, 18$ and 16 that encompass a much larger number of graphs, according to the Table 3.1.

We observe that the pasting strategy that provides the 4-total colorings of Theorem 18 considers suitable equitable 4-total colorings of L_6. By our Theorem 27 we know that in fact every 4-total coloring of L_6 must be equitable. We have presented the proof by Chetwynd and Hilton published in 1988, which considers three cases. For the smallest graphs for each case, the constructed 4-total coloring is equitable. These smallest graphs are: for the first case L_6, for the second case L_8 and L_{14}, and for the third case L_{16} and L_{22}. For the larger circular ladders, the constructed 4-total coloring is clearly not equitable, since at each pasting, the pasting strategy increases the difference between the sizes of the color classes.

Note that by the Theorem 18 coloring strategy we know that the L_{12} and L_{18} graphs have a non-equitable total coloring (see Figure 4.7). In agreement with our results, the colorings generated by the Theorem 18 of L_6, L_8, and L_{14} are equitable. Moreover, the colorings generated by the Theorem 18 of L_{16} and L_{22} are also equitable. However, the remaining colorings generated by the Theorem 18 of circular ladder graphs are not equitable.

We were able to determine the configurations of total independent sets that cubic graphs with 12 and 18 vertices must have to obtain a non-equitable 4-total coloring in these graphs. A relevant question for us now is whether cubic graphs with n multiple of 3, $n \neq 6$, always have a configuration of total independent sets that might allow a non-equitable 4-total coloring. Furthermore, in Theorem 33, we conclude that cubic graphs with 16 vertices also might allow a non-equitable 4-total coloring.

Our investigation to find the structural properties of the graphs to try to generalize our technique in function of some parameters such as the number of vertices and number of elements leads to many interesting related questions, that might be further studied.

We intend to continue investigating the technique of analyzing total independent sets in future works, seeking its improvement and additional results which might establish a positive answer to our Question 1 about whether every Type 1 cubic graph, with $n < 20$, does have at least one equitable 4-total coloring. In order to answer yes, we should provide, as we have done for L_{12} and L_{18}, and for the graph H with 16 vertices, an equitable 4-total coloring, for all graphs with $n = 12, 16$ and 18. Please refer to Figure 4.7 and Figure 4.10 where we have non-equitable 4-total colorings respectively for $G(9,1)$ (recall that L_{18} is isomorphic to $G(9,1)$) and $G(9,2)$, but both have the same color class configuration 12, 12, 12, and 9. We are currently looking for non-equitable 4-total colorings of $G(9,1)$ and $G(9,2)$, with color class configuration 12, 12, 10, and 10, whose existence is not excluded by Theorem 32. So far, we do not know of an example of a graph with $n = 18$ admitting a non-equitable 4-total coloring with color class configuration 12, 12, 10, and 10. Another interesting task is to provide for each graph admitting a non-equitable 4-total coloring with a certain color class configuration, another non-equitable 4-total coloring with a distinct color class configuration.

Regarding our Question 3 we seek to find the largest value of n, such that every Type 1 cubic graph with n vertices is such that all of its 4-total colorings are equitable. Observe that the graph R is a Type 1 cubic graph with 20 vertices such that all 4-total colorings are non equitable. See figures 4.3 and 4.4. So far, according to our Theorem 30 the largest value of n, such that every Type 1 cubic graph with n vertices is such that all of its 4-total colorings are equitable is $n = 14$. Towards answering Question 3, we should extend our Theorem 30 to larger values of n, for instance $n = 22$ for which the Theorem 18 coloring strategy provides an equitable 4-total coloring.

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