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PEBBLING IN HYPERCUBES*

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Abstract. This paper considers the following game on a hypercube, first suggested by Lagarias and Saks. Suppose 2^n pebbles are distributed onto vertices of an n-cube (with 2^n vertices). A pebbling step is to remove two pebbles from some vertex and then place one pebble at an adjacent vertex. The question of interest is to determine if it is possible to get one pebble to a specified vertex by repeatedly using the pebbling steps from any starting distribution of 2^n pebbles. This question is answered affirmatively by proving several stronger and more general results.

Key words. distributive lattice, graphs, retracts

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1. Introduction. An n-dimensional cube, or n-cube for short, consists of 2^n vertices labelled by $(0, 1)$-tuples of length n. Two vertices are adjacent if their labels are different in exactly one entry. Because of its highly parallel structure, the n-cube possesses many nice properties and is an ideal model for games of various distributive types. In this paper we investigate the following game that was first proposed by Lagarias and Saks [4], [7].

Suppose 2^n pebbles are distributed onto vertices of an n-cube. A pebbling step consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. We say a pebble can be moved to a vertex if we can apply pebbling steps repeatedly (if necessary) so that in the resulting configuration the vertex has one pebble. The question of interest is to determine if it is always possible to move one pebble to a specified vertex from any starting distribution of 2^n pebbles.

In this paper we answer this problem affirmatively. Independently, Guzman also solved the same problem by a different proof.

THEOREM 1. *For any distribution of 2^n pebbles to vertices of the n-cube, one pebble can be moved to any specified vertex.*

Theorem 1 turns out to be an immediate consequence of some stronger and more general results that lead to an alternative proof for the following result (due to Lemke and Kleitman [5] through a different method).

For any given integers $a_1, a_2, \cdots, a_d$ there is a nonempty subset

$$X \subseteq \{1, 2, \cdots, d\}$$

such that

$$d \mid \sum_{i \in X} a_i \quad \text{and} \quad \sum_{i \in X} gcd(a_i, d) \leqq d.$$

For any partially ordered set, the set of order ideals (downward closed subsets) ordered by inclusion is a distributive lattice. The Hasse diagram of the lattice can be viewed as a graph where the ideals u and v are adjacent if u contains v and u is exactly one larger than v. One of many variations of our result is the following.

Consider a given partially ordered set S in which each element is assigned an integer weight. In the corresponding finite distributive lattice, an admissible move involves two adjacent vertices u and v, say $u < v$. That is, to remove w pebbles from v (where w is

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the weight of the element in v but not in u) and to place one pebble on u. If $\prod_{x \in S} w(x)$ pebbles are assigned to vertices in the distributive lattice, then by repeatedly applying the admissible steps, one pebble can be placed on the empty set in the lattice.

2. Stronger and more general versions. We will first prove the following theorem.

THEOREM 2. *In an n-cube with a specified vertex v, the following are true:*

(i) *If* 2^n *pebbles are assigned to vertices of the n-cube, one pebble can be moved to v.*

(ii) *Let q be the number of vertices that are assigned an odd number of pebbles. If there are all together more than* $2^{n+1} - q$ *pebbles, then two pebbles can be moved to v.*

Proof. The proof is by induction on n. It is trivially true for $n = 0$. Suppose it is true for $n' < n$. The n-cube can be partitioned into two $(n-1)$-cubes, say M_1 and M_2, where v is in M_1. Let v' denote the vertex in M_2 adjacent to v. The edges between M_1 and M_2 form a perfect matching. Suppose M_i contains p_i pebbles with q_i vertices having an odd number of pebbles, for $i = 1, 2$.

Suppose there are $p \geqq 2^n$ pebbles assigned to vertices of the n-cube. We will first show (i) holds. If $p_1 \geqq 2^{n-1}$, then by induction, in M_1, one pebble can be moved to v. We may assume $p_1 < 2^{n-1}$ and we consider the following two cases.

Case (a1). $q_2 > p_1$.

Since $p_2 = p - p_1 > 2^n - q_2$, by induction from (ii) in M_2 two pebbles can be moved to v'. Therefore one pebble can be moved to v.

Case (a2). $q_2 \leqq p_1$.

We apply pebbling steps to all vertices in M_2 and we can move at least $(p_2 - q_2)/2$ pebbles to vertices of M_1. Therefore, in M_1 we have all together $p_1 + (p_2 - q_2)/2 \geqq p_1 + (p_2 - p_1)/2 = (p_1 + p_2)/2 = 2^{n-1}$ pebbles. By induction, we can then move one pebble to v. This establishes (i).

It suffices now to prove (ii). Suppose there are $p = p_1 + p_2 > 2^{n+1} - q_1 - q_2$ pebbles assigned to vertices of the n-cube. We want to show that two pebbles can be moved to v. We consider the following three possibilities:

Case (b1). $p_1 > 2^n - q_1$.

By induction from (ii) in M_1, two pebbles can be moved to v.

Case (b2). $2^n - q_1 \geqq p_1 \geqq 2^{n-1}$.

Since $p_1 \geqq 2^{n-1}$, by induction from (i) in M_1 one pebble can be moved to v. Since $p_2 = p - p_1 > 2^{n+1} - q_1 - q_2 - p_1 \geqq 2^n - q_2$, two pebbles can be moved to v' using induction from (ii) in M_2. Therefore an additional pebbling step results in moving one more pebble to v.

Case (b3). $p_1 < 2^{n-1}$.

For any integer t satisfying $p_2 \geqq q_2 + 2t$, t pebbles can be moved to vertices of M_1 while $p_2 - 2t$ pebbles remain in M_2. Note $p_2 > 2^{n+1} - q - p_1 = (2^n - q_2) + (2^n - q_1 - p_1) \geqq q_2 + (2^n - q_1 - p_1)$, where the last inequality follows since q_2 is at most 2^{n-1}. Thus taking t to be $2^{n-1} - \lceil (q_1 + p_1)/2 \rceil$, t pebbles can be moved to M_1 leaving more than $2^n - q_2$ pebbles in M_2. In M_1 there are $p_1 + 2^{n-1} - \lceil (q_1 + p_1)/2 \rceil = 2^{n-1} + \lfloor (p_1 - q_1)/2 \rfloor \geqq 2^{n-1}$ pebbles. We can then move one pebble to v in M_1 and at the same time move two pebbles to v' in M_2, which will result in one additional pebble to v.

This completes the proof of Theorem 2.

We remark that Theorem 2 provides an efficient algorithm for pebbling in the n-cube. Furthermore in all the pebbling steps, pebbles are moved toward the specified vertex.

Theorem 2 can be slightly strengthened in Theorem 2$'$. This variation is useful for proving several generalized versions.

We say i pebbles are extracted from an n-cube if i pebbling steps are performed in a way that $2i$ pebbles are removed, but the placement of the i new pebbles to the neighbors will be suspended until a later time.

THEOREM 2′. *If $2^{n+1} - r + w + 1$ pebbles are assigned to vertices of an n-cube while r vertices have at least one pebble, two pebbles can be moved to v after $\lfloor w/2 \rfloor$ pebbles are extracted from the cube.*

Proof. Again we will prove by induction on n and view the n-cube as the union of two $(n-1)$-cubes M_1 and M_2 as described in the proof of Theorem 2, where M_i contains p_i pebbles with r_i vertices having at least one pebble for $i = 1, 2$. Suppose $p = p_1 + p_2 = 2^{n+1} - r_1 - r_2 + w + 1$ pebbles are assigned to vertices of the n-cube.

We consider the following three possibilities:

Case (c1). $p_1 > 2^n - r_1$.

By induction in M_1, two pebbles can be moved to v after $\lfloor (p_1 - 2^n + r_1 - 1)/2 \rfloor$ pebbles are extracted from M_1. In M_2, $\lceil (p_2 - r_2)/2 \rceil$ pebbles can be extracted. Therefore, altogether two pebbles can be moved to v after $\lfloor (p_1 - 2^n + r_1 - 1 + p_2 - r_2)/2 \rfloor \geqq \lfloor w/2 \rfloor$ pebbles are extracted.

Case (c2). $2^n - r_1 \geqq p_1 \geqq 2^{n-1}$.

Since $p_1 \geqq 2^{n-1}$, one pebble can be moved to v in M_1. Since $p_2 = p - p_1 > 2^{n+1} - r_1 - r_2 - p_1 + w \geqq 2^n - r_2 + w$, two pebbles can be moved to v' after $\lfloor w/2 \rfloor$ pebbles are extracted by induction in M_2. Therefore one pebble can be moved to v.

Case (c3). $p_1 < 2^{n-1}$.

Since $p_2 > 2^{n+1} - r_1 - r_2 - p_1 + w = (2^n - r_2) + (2^n - r_1 - p_1) + w$, by induction two pebbles can be moved to v' after $\lfloor (2^n - r_1 - p_1 + w)/2 \rfloor$ pebbles are extracted from M_2, among which $(2^n - r_1 - p_1)/2$ pebbles will be placed in M_1. In M_1 there are $p_1 + 2^{n-1} - \lceil (r_1 + p_1)/2 \rceil = 2^{n-1} + \lfloor (p_1 - r_1)/2 \rfloor \geqq 2^{n-1}$ pebbles. We can then move one pebble to v in M_1 and at the same time move two pebbles to v' in M_2, which will result in one additional pebble to v after $\lfloor w/2 \rfloor$ pebbles are extracted from the cube.

This completes the proof of Theorem 2′.

One of the original versions of the problem proposed by Lagarias is the following theorem.

THEOREM 3. *For integers $p_1, p_2, \cdots, p_n \geqq 2$, suppose $p_1 p_2 \cdots p_n$ pebbles are assigned to the vertices of an n-cube Q_n. Each admissible pebbling step is to remove p_i pebbles from a vertex $(a_1, \cdots, a_{i-1}, 1, a_{i+1}, \cdots, a_n)$ and place one pebble on $(a_1, \cdots, a_{i-1}, 0, a_{i+1}, \cdots, a_n)$. One can now repeatedly use the admissible pebbling steps to place one pebble on $(0, 0, \cdots, 0)$.*

Proof. The proof is very similar to that of Theorem 2 except that (ii) should read somewhat differently. We say an admissible step is of direction i and cost p_i if p_i pebbles are removed from a vertex $(a_1, \cdots, a_{i-1}, 1, a_{i+1}, \cdots, a_n)$ and place one pebble on $(a_1, \cdots, a_{i-1}, 0, a_{i+1}, \cdots, a_n)$. For a fixed constant k, an admissible step of direction 0 if k pebbles are removed from a vertex and one pebble will be kept to be placed later in a (possible future) direction of cost k. Let q be the number of vertices that are assigned at least one pebble. If there are altogether more than $kp_1 \cdots p_n - q + w$ pebbles, then k pebbles can be moved to v after $\lfloor w_0/k \rfloor$, $\lfloor w_1/p_1 \rfloor$, $\lfloor w_2/p_2 \rfloor$, $\cdots$ $\lfloor w_n/p_n \rfloor$, where $w_0 + w_1 + \cdots + w_n = w$, are extracted from the cube in the ith direction of cost p_i, for $i = 0, 1, \cdots, n$, respectively.

3. The pebbling number. For a graph G, we define the pebbling number $f(G)$ to be the smallest integer m such that for any distribution of m pebbles to the vertices of G, one pebble can be moved to a specified vertex. Theorem 2 states that $f(Q_n) = 2^n$. We here include some facts about $f(G)$, most of which are quite straightforward (the proofs are left for the reader).

FACT 1. $f(G) \geqq |V(G)|$.

FACT 2. $f(G) \geqq 2^D$ *where* $D = D(G)$ *is the diameter of* G.

FACT 3. *If* G' *is a spanning subgraph of* G, *then* $f(G') \geqq f(G)$.

FACT 4. *For a path* P_{t+1} *on* $t+1$ *vertices*, $f(P_{t+1}) = 2^t$.

FACT 5. *For a complete graph* K_t, $f(K_t) = t$.

A graph H is said to be a retract of a graph G if there is a mapping from $V(G)$ to $V(H)$ which preserves edges, i.e., which maps adjacent vertices in G to adjacent vertices in H. The reader is referred to [1], [3], [6] for various facts about retracts. The following simple fact turns out to be very useful.

FACT 6. *If* H *is a retract of* G, *then* $f(H) \leqq f(G)$.

Duffus and Rival [2] showed that a finite distributive lattice of height n is a retract of the n-dimensional cube. By using Theorem 3, we immediately have the following:

THEOREM 4. *Suppose* S *is a partially ordered set in which each element is assigned an integer weight. In the corresponding finite distributive lattice, an admissible move is to remove* w *pebbles from a vertex* v *and place one pebble on* u *where* w *is the weight of the element in* $v - u$. *If* $\prod_{x \in S} w(x)$ *pebbles are assigned to vertices in the distributive lattice, then by repeatedly applying the admissible steps, one pebble can be moved to the empty set in the lattice.*

We remark that when S consists on n pairwise incomparable points, the distributive lattice is exactly the n-cube. Theorems 1–3 are all special cases of Theorem 4.

For any two graphs G_1 and G_2, we define the product $G_1 \square G_2$ to be the graph with vertex set $= \{(v_1, v_2) : v_1 \in V(G_1), v_2 \in V(G_2)\}$ and there is an edge between (v_1, v_2) and (v_1', v_2') if and only if ($v_1 = v_1'$ and $\{v_2, v_2'\} \in E(G_2)$) or ($\{v_1, v_1'\} \in E(G_1)$ and $v_2 = v_2'$). It is easy to see that the 1-cube is K_2; the 2-cube is $K_2 \square K_2$; and the n-cube Q_n is $Q_{n-1} \square K_2$.

We say a graph G satisfies the 2-pebbling property if two pebbles can be moved to a specified vertex when the total starting number of pebbles is $2f(G) - q + 1$, where q is the number of vertices with at least one pebble. Clearly the n-cube satisfies the 2-pebbling property and the paths also have the 2-pebbling property.

THEOREM 5. *Suppose* G *satisfies the* 2-*pebbling property. Then the following holds*:

(i) $f(G \square K_t) \leqq tf(G)$.

(ii) If $f(G \square K_t) = tf(G)$, $G \square K_t$ *satisfies the* 2-*pebbling property*.

The proof of Theorem 5 is extremely similar to the proof of Theorem 2 and 2′ and will be omitted here.

Using Theorem 5 together with Fact 1 yields the pebbling number for all products of cliques.

FACT 7. $f(K_{t_1} \square K_{t_2} \square \cdots \square K_{t_s}) = t_1 t_2 \cdots t_s$.

FACT 8. $f(P_{t_1+1} \square P_{t_2+1} \square \cdots \square P_{t_s+1}) = 2^{t_1+t_2+\cdots t_s}$.

Proof. On one hand, we have $f(P_{t_1+1} \square P_{t_2+1} \square \cdots P_{t_s+1}) \geqq 2^{t_1+\cdots+t_s}$ since the diameter is $t_1 + t_2 + \cdots + t_s$. On the other hand, since $P_{t_1+1} \square P_{t_2+1} \square \cdots \square P_{t_s+1}$ is a retract of the $(t_1 + t_2 + \cdots + t_s)$-cube, Fact 8 follows from Fact 6.

FACT 9. *For integers* $p_1, p_2, \cdots, p_n \geqq 2$, $\alpha_1, \alpha_2, \cdots, \alpha_n \geqq 1$. *Suppose* $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ *pebbles are assigned to the vertices of* $P_{\alpha_1+1} \square P_{\alpha_2+1} \square \cdots \square P_{\alpha_n+1}$. *Each admissible pebbling step is to remove* p_i *pebbles from a vertex*

$$(a_1, \cdots, a_{i-1}, x, a_{i+1}, \cdots, a_n)$$

and place one pebble on $(a_1, \cdots, a_{i-1}, x-1, a_{i+1}, \cdots, a_n)$. *One can repeatedly apply the admissible pebbling step to move a pebble to* $(0, \cdots, 0)$.

Fact 9 follows from Fact 6 and Theorem 3. We can now use Fact 9 to give a different proof to the result of Lemke and Kleitman [5]. That Theorem 3 implies Theorem 6 was Lagarias' motivation for formulating the problem.

THEOREM 6. *For any given integers* $a_1, a_2, \cdots, a_d$, *there is a nonempty subset* $X \subseteq \{1, \cdots, d\}$ *such that* $d \mid \sum_{i \in X} a_i$ *and* $\sum_{i \in X} gcd(a_i, d) \leqq d$.

Proof. Suppose $d = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$. We consider $P_{\alpha_1+1} \square P_{\alpha_2+1} \square \cdots \square P_{\alpha_n+1}$. For each integer a_i, we place a pebble at $(b_1, \cdots, b_n)$ where $d/gcd(a_i, d) = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$. Suppose there are p_i pebbles corresponding to numbers $x_1, \cdots, x_{p_i}$ at the vertex $(b_1, \cdots, b_n)$. There is a subset $S \subseteq \{1, \cdots, p_i\}$ such that $p_i \mid \sum_{i \in S} x_i = y$. We note that

$$\sum_{i \in S} gcd(x_i, d) \leqq p_i \cdot gcd(x_1, d)$$

$$= gcd(y, d).$$

We can then replace numbers $x_1, \cdots, x_{p_i}$ by the number y. If we can repeat this process and eventually move a pebble to $(0, \cdots, 0)$, then this implies that there is a subset $X \subseteq \{1, \cdots, d\}$ such that $d \mid \sum_{i \in X} a_i$ and $\sum_{i \in X} gcd(a_i, d) \leqq d$.

This completes the proof of Theorem 6.

Suppose T is a tree with a specified vertex. T can be viewed as a directed tree denoted by T_v^* with edges directed toward the specified vertex v, also called the root. A path partition is a set of nonoverlapping directed paths the union of which is T. A path-partition is said to majorize another if the nonincreasing sequence of the path size majorizes that of the other. (That is $(a_1, a_2, \cdots, a_t) > (b_1, b_2, \cdots, b_t)$ if and only if $a_i > b_i$ where $i = \min\{j : a_j \neq b_j\}$.) A path-partition of a tree T is said to be maximum if it majorizes all other path-partitions.

We define the pebbling number $f(T, v)$ to be the smallest integer m such that if m pebbles are assigned to the vertices of T, then one pebble can be moved to v.

FACT 10. *The pebbling number* $f(T, v)$ *for a vertex* v *in a tree* T *is* $2^{a_1} + 2^{a_2} + \cdots + 2^{a_t} - t + 1$ *where* $a_1, a_2, \cdots, a_t$ *is the sequence of the path size* (*i.e., the number of vertices in the path*) *in a maximum path-partition of* T_v^*.

Fact 10 is a special case of Fact 11 that considers the following general formulation. Let $f_k(T, v)$ denote the smallest integer m such that if m pebbles are assigned to the vertices of T then k pebbles can be moved to v.

FACT 11. *The pebbling number* $f_k(T, v)$ *for a vertex* v *in a tree* T *is* $k2^{a_1} + 2^{a_2} + \cdots + 2^{a_t} - t + 1$ *where* $a_1, a_2, \cdots, a_t$ *is the sequence of the path size in a maximum path-partition of* T_v^*.

Proof. The proof is by induction on the number of vertices of T. If we remove v from T, the resulting graph is the union of subtrees $T_1, T_2, \cdots, T_s$ where T_i contains a neighbor of v, say u_i. It is easy to see that for any $\lfloor k_1/2 \rfloor + \cdots + \lfloor k_s/2 \rfloor < k$ we have

$$f_k(T, v) - 1 \geqq f_{k_1+1}(T_1, u_1) + f_{k_2+1}(T_2, u_2) + \cdots + f_{k_s+1}(T_s, u_s) - s.$$

In fact,

$$f_k(T, v) - 1 = \operatorname*{Max}_{k_i} \{f_{k_1+1}(T_1, u_1) + f_{k_2+1}(T_2, u_2) + \cdots + f_{k_s+1}(T_s, u_s) - s\}.$$

Using the fact that $2^a + 2^b \geqq 2^{a-1} + 2^{b+1}$ if $a > b$, the maximum is achieved when $k_1 = k_3 = \cdots = k_s = 1$, $k_1 = 2k - 1$ while T_1 contains a vertex furthest from v. It is then straightforward to check that $f_k(T, v)$ has the desired expression.

FACT 12. *A tree satisfies the* 2-*pebbling property*.

Proof. From Fact 11 we know that $f_2(T, v) = f(T, v) + 2^{a_1}$ where a_1 is the number of edges in a longest directed path in T with root v. It remains to show that

$$f(T, v) - |V(T)| + 1 \geqq 2^{a_1}$$

which follows from Fact 11.

4. Questions on the pebbling number. There are many problems on the pebbling number that we will mention here.

Question 1. Is it true that $f(G) = \max\{2^{D(G)}, |V(G)|\}$?

Answer. False. Consider the star of 3 edges. The pebbling number is 5 while $2^D = 4 = |V(G)|$.

Question 2. Is it true that $f(G_1 \square G_2) = f(G_1)f(G_2)$?

Answer. False. Consider $G_1 = K_3$ and $G_2 = P_3$.

$$f(K_3 \square P_3) = 9 \neq f(K_3) \cdot f(P_3) = 12.$$

There are many questions not resolved at this point.

Question 3 (RLG). Is it true that $f(G_1 \square G_2) \leqq f(G_1)f(G_2)$?

Question 4. Is it true that any graph has the 2-pebbling property?

If these two properties are true, the proof of Theorem 2 can be much simplified. Recently, Lemke constructed a counterexample to Question 4. His example does not provide a "no" answer to Question 3.

We remark that Theorem 5 can be used to determine $f(G)$ for a variety of graphs other than products of cliques or paths. For example, for the 5-cycle C_5, it is easy to see that $f(C_5) = 5$. Theorem 5 asserts $f(K_5 \square C_5) = 25$. It would be of interest to determine $f(C_5 \square C_5 \square \cdots \square C_5)$.

Question 5. Is it true that

$$f(\overbrace{C_5 \square C_5 \square \cdots \square C_5}^{n\, C_5\text{'s}}) = 5^n?$$

The following generalization of Theorem 6 was conjectured in [5].

CONJECTURE. *Any sequence of $|G|$ elements (not necessarily distinct) of the finite group G contains a nonempty subsequence $g_1, g_2, \cdots, g_k$ such that $g_1 g_2 \cdots g_k = e$ and* $\sum_{i=1}^{k} (1)/|g_i| \leqq 1$.

When G is cyclic, the conjecture is true as seen from Theorem 6 and [5].

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REFERENCES

[1] H. J. BANDELT, *Retracts of hypercubes*, J. Graph Theory, 8 (1984), pp. 501–510.

[2] D. DUFFUS AND I. RIVAL, *Graphs orientable as distributive lattices*, Proc. Amer. Math. Soc., 88 (1983), pp. 197–200.

[3] P. HELL, *Graph retractions*, Colloq. Intern. Teorie Combinatorie II, Roma, 1976, pp. 263–268.

[4] J. LAGARIAS, *personal communication*, (1987).

[5] P. LEMKE AND D. KLEITMAN, *An addition theorem on the integers modulo n*, J. Number Theory, 31 (1989), pp. 335–345.

[6] H. M. MULDER, *n-cubes and median graphs*, J. Graph Theory, 4 (1980), pp. 107–110.

[7] M. SAKS, *personal communication*, (1987).