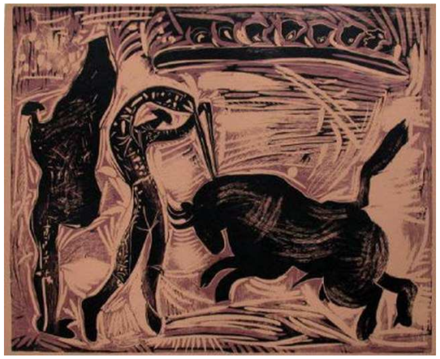

Centenary of Celina + Frédéric

Complexity-separating graph classes for vertex, edge and total coloring

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Universidade Federal do Rio de Janeiro

Grenoble, September 2010

Overview

Classification into P or NP-complete of challenging problems in graph theory

Full dichotomy: class of problems where each problem is classified into P or NP-complete

Coloring problems: vertex, edge, total

NP-completeness ongoing guide

Identification of an interesting problem, of an interesting graph class

Categorization of the problem according to its complexity status

Problems and complexity-separating graph classes

Graph classes and complexity-separating problems

Johnson's NP-completeness column 1985

Spinrad's book 2003

Complexity-separating graph classes

	VERTEXCOL	EDGECOL
perfect	P	N
chordal	P	O
split	P	O
strongly chordal	P	O
comparability	P	N
bipartite	P	P
permutation	P	O
cographs	P	O
indifference	P	O
split-indifference	P	P

N: NP-complete P: polynomial O: open

Johnson's NP-completeness column 1985

I. Holyer – *SIAM J. Comput.* 1981

Complexity-separating problems

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L. Cai, J. Ellis – *Discrete Appl. Math.* 1991

Spinrad's book 2003

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C. Ortiz Z., N. Maculan, J. Szwarcfiter – *Discrete Appl. Math.* 1998

C. Simone, C. Mello – *Theoret. Comput. Sci.* 2006

Full dichotomies

Classes of problems for which every problem is classified into P or NP-complete

Problems: EDGE COLORING, TOTAL COLORING

Graph classes: unichord-free, split-indifference, chordless

Unichord-free graphs

χ-bounded graph class: $\chi \leq f(\omega)$

Perfect graph: $\chi = \omega$

Line graph: $\chi \leq \omega + 1$, the Vizing bound

A. Gyárfás – *Zastos. Mat.* 1987

Unichord-free graphs

χ-bounded graph class: $\chi \leq f(\omega)$

Perfect graph: $\chi = \omega$

Line graph: $\chi \leq \omega + 1$, the Vizing bound

Which choices of forbidden induced subgraphs give χ-bounded class?

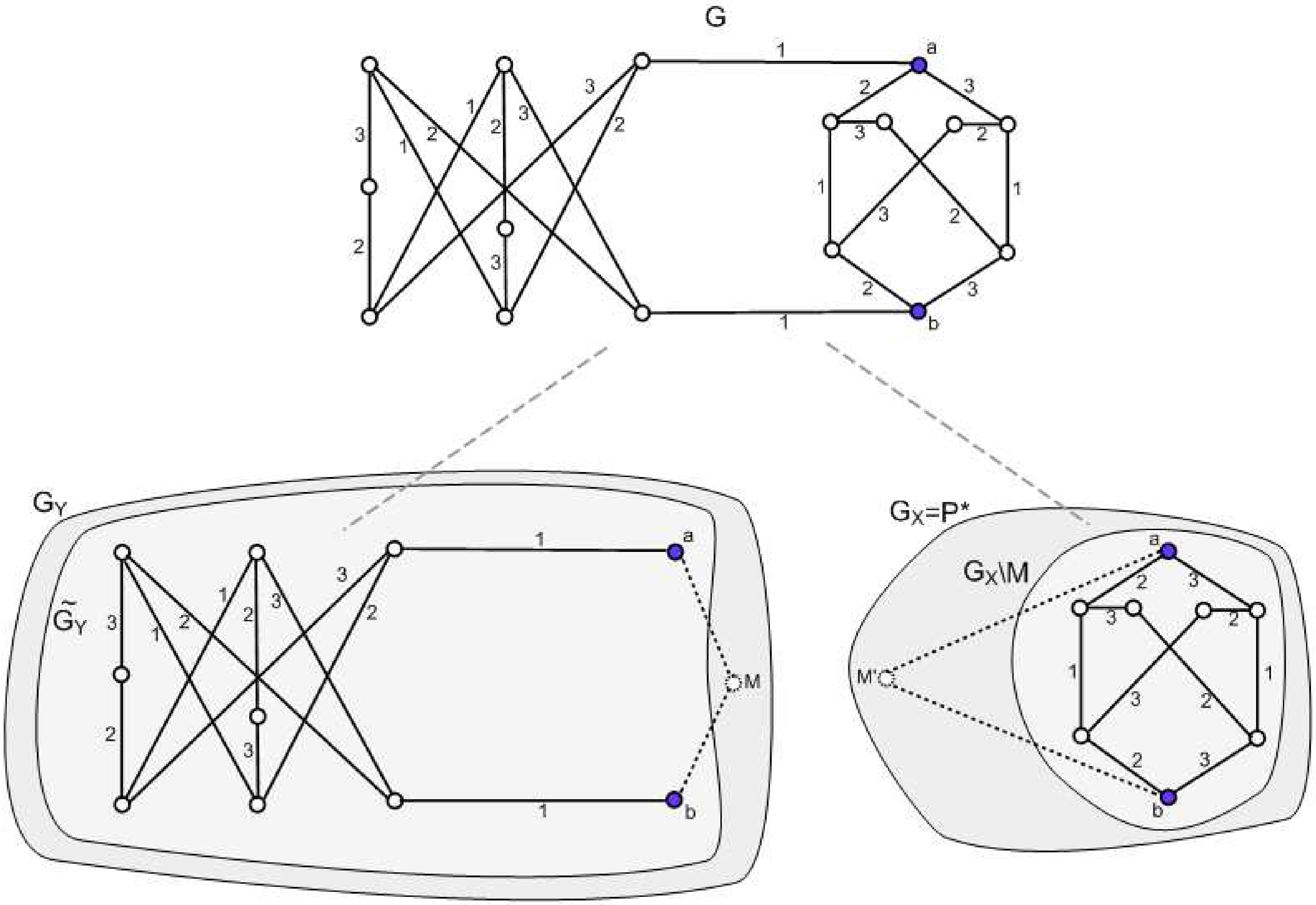
Unichord-free graphs: $\chi \leq \omega + 1$

Structure theorem:

every graph in the class can be built from basic graphs

N. Trotignon, K. Vušković – *J. Graph Theory* 2009

Combining edge-colorings with respect to 2-cutset



Decomposition with respect to a proper 2-cutset $\{a, b\}$
G is Class 1: Δ colors suffice, but $G_X = P^*$ is Class 2: $\Delta + 1$ colors needed

Edge-coloring unichord-free graphs

Class $\mathcal{C}$ = unichord-free graphs

	$\Delta = 3$	$\Delta \geq 4$	regular
graphs of $\mathcal{C}$	N	N	N
4-hole-free graphs of $\mathcal{C}$	N	P	P
6-hole-free graphs of $\mathcal{C}$	N	N	N
$\{$4-hole, 6-hole$\}$-free graphs of $\mathcal{C}$	P	P	P

"Chromatic index of graphs with no cycle with a unique chord"

Theoret. Comput. Sci. 2010 (with Raphael Machado, Kristina Vušković)

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EDGECOL is N for k-partite r-regular, for each $k \geq 3$, $r \geq 3$

	$k \leq 2$	$k \geq 3$
k-partite graphs	P	N

“Chromatic index of graphs with no cycle with a unique chord”

Theoret. Comput. Sci. 2010 (with Raphael Machado, Kristina Vušković)

Class 2 = overfull implies EDGECOL is P

Overfull graph: $|E| > \Delta \left\lfloor \frac{|V|}{2} \right\rfloor$

Complete multipartite: Class 2 = overfull

Graphs with a universal vertex: Class 2 = overfull

Split-indifference graphs: Class 2 = subgraph overfull

$\{$4-hole,unichord$\}$-free graphs, with $\Delta \neq 3$: Class 2 = subgraph overfull

D. Hoffman, C. Rodger – *J. Graph Theory* 1992

M. Plantholt – *J. Graph Theory* 1981

C. Ortiz Z., N. Maculan, J. Szwarcfiter – *Discrete Appl. Math.* 1998

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Conjecture for edge-coloring chordal graphs:
Class 2 = subgraph overfull

“On edge-colouring indifference graphs”

Theoret. Comput. Sci. 1997 (with João Meidanis, Célia Mello)

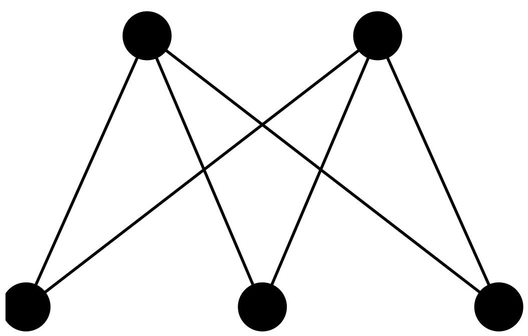
Total coloring conjecture

Vizing's edge coloring theorem: every graph is $(\Delta + 1)$-edge colorable

Total coloring conjecture: every graph is $(\Delta + 2)$-total colorable
Type 1 = $(\Delta + 1)$-total colorable, Type 2 = $(\Delta + 2)$-total colorable

M. Molloy, B. Reed – *Combinatorica* 1998

Natural to consider classes of graphs for which TCC is established



TCC for bipartite: 2-color vertices, Δ-color edges

Total coloring is hard

NP-hard for k-regular bipartite

Reduction from edge-coloring

Consider classes of graphs for which edge-coloring is polynomial

Edge-coloring is polynomial for split-indifference graphs

C. McDiarmid, A. Sánchez-Arroyo – *Discrete Math.* 1994

C. Ortiz Z., N. Maculan, J. Szwarcfiter – *Discrete Appl. Math.* 1998

Type 2 = Hilton condition implies TOTALCOL is P

	Δ even	Δ odd
complete	Type 1	Type 2 (Hilton condition)
univ. vertex	Type 1	Hilton condition
split	Type 1	open
indifference	Type 1	open
split-indifference	Type 1	Hilton condition
3 max cliques	Type 1	open

A. Hilton – *Discrete Math.* 1989

What is the largest class of graphs for which:
G Type 2 iff Hilton condition holds for closed neighborhood of Δ vertex

Necessary condition:

Δ even implies Type 1

“The total chromatic number of split-indifference graphs”

FCC 2010 (with Christiane Campos, Raphael Machado, Célia Mello)

Total chromatic number of unichord-free graphs

	VERTEXCOL	EDGECOL	TOTALCOL
unichord-free	P	N	N
{4-hole,unichord}-free, $\Delta \geq 4$	P	P	P
{4-hole,unichord}-free, $\Delta = 3$	P	N	P

Surprising full-dichotomy wrt EDGECOL:

$\Delta \geq 4$ is polynomial whereas $\Delta = 3$ is NP-complete

Surprising complexity-separating graph class:

EDGECOL is NP-complete whereas TOTALCOL is polynomial

"Total chromatic number of {square,unichord}-free graphs"

ISCO 2010 (with Raphael Machado)

Edge coloring chordless graphs

G is chordless iff $L(G)$ is wheel-free

Chordless, with $\Delta = 3$ is Class 1 implies
$\{$wheel,$ISK_4\}$-free is 3 vertex colorable

B. Lévêque, F. Maffray, N. Trotignon – “On graphs with no subdivision of K_4” 2010

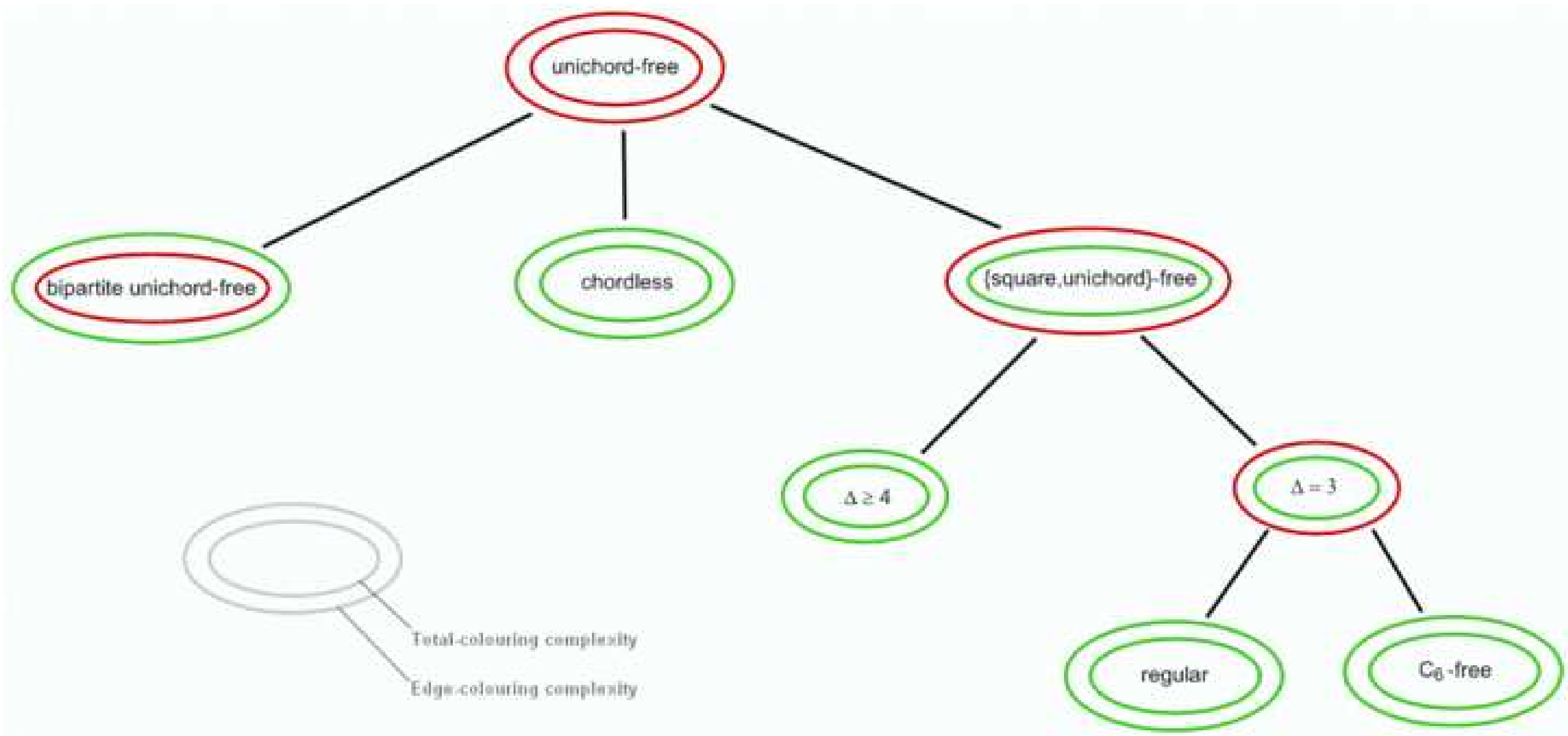
Chordless is a subclass of unichord-free
EDGECOL is NP-complete for unichord-free graphs

Every chordless, with $\Delta > 3$ is Class 1

“Chromatic index of chordless graphs”

CTW 2010 (with Raphael Machado and Nicolas Trotignon)

Edge and total coloring complexity-separating classes



When restricted to {square,unichord}-free graphs,
edge coloring is NP-complete whereas total coloring is polynomial