Pebbling in Kneser Graphs

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Abstract. Graph pebbling is a game played on graphs with pebbles on their vertices. A pebbling move removes two pebbles from one vertex and places one pebble on an adjacent vertex. The pebbling number $\pi(G)$ is the smallest t so that from any initial configuration of t pebbles it is possible, after a sequence of pebbling moves, to place a pebble on any given target vertex. We consider the pebbling number of Kneser graphs, and give positive evidence for the conjecture that every Kneser graph has pebbling number equal to its number of vertices.

Keywords: graph pebbling · Kneser graphs · odd graphs · weight function method

1 Introduction

Graph pebbling is a network model for studying whether or not a given supply of discrete pebbles can satisfy a given demand via pebbling moves. A pebbling move across an edge of a graph takes two pebbles from one endpoint and places one pebble at the other endpoint; the other pebble is lost in transit as a toll. The pebbling number of a graph is the smallest t such that every supply of t pebbles can satisfy every demand of one pebble by a vertex. The number of vertices is a sharp lower bound, and graphs where the pebbling number equals the number of vertices is a topic of much interest [7,9].

Pebbling numbers of many graphs are known: cliques, trees, cycles, cubes, diameter 2 graphs, graphs of connectivity exponential in its diameter, and others [11]. The pebbling number has also been determined for subclasses of chordal graphs: split graphs [2], semi-2-trees [3], and powers of paths [4], among others. Other well-known families of graphs (e.g. flower snarks [1]) have been investigated; here we continue the study on Kneser graphs. In order to state our main results in Sect. 2, we first introduce graph theoretic definitions, followed by graph pebbling terminology, and then present some context for these results.

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1.1 General Definitions

In this paper, $G = (V, E)$ is always a simple connected graph. The numbers of vertices and edges of G as well as its diameter, are denoted by $n(G)$, $e(G)$, and $D(G)$, respectively, or simply n, e, and D, when it is clear from the context. For a vertex w and positive integer d, denote by $N_d(w)$ the set of all vertices that are at distance exactly d from w, with $N_d[w] = \cup_{i=0}^{d} N_i(w)$ being the set of all vertices that are at distance at most d from w.

Given two positive integers m and t, the *Kneser graph* $K(m,t)$ is the graph whose vertices represent the t-subsets of $\{1, \ldots, m\}$, with two vertices being adjacent if, and only if, they correspond to disjoint subsets. Thus, $K(m,t)$ has $\binom{m}{t}$ vertices and is regular, with degree $\deg(K(m,t)) = \binom{m-t}{t}$. When $m = 2t$, each vertex is adjacent to just one other vertex and the Kneser graph $K(2t,t)$ is a perfect matching. Therefore we assume that $m \geq 2t+1$ so that $K(m,t)$ is connected. For $m \geq 1$, $K(m,1)$ is the complete graph on m vertices, so we assume that $t > 2$. The special case $K(2t+1,t)$ is known as the *odd graph* O_t; in particular, $O_2 = K(5,2)$ is the Petersen graph. The odd graphs constitute the sparsest case of connected Kneser graphs. A graph G is k-connected if it contains at least $k+1$ vertices but does not contain a set of $k-1$ vertices whose removal disconnects the graph, and the connectivity $\kappa(G)$ is the largest k such that G is k-connected.

Since Kneser graphs are regular and edge-transitive, their vertex connectivity equals their degree $\binom{m-t}{t} \geq t+1 \geq 3$ (see [13]). The diameter of $K(m,t)$ is given in [15] to be $\lceil \frac{t-1}{m-2t} \rceil + 1$. Notice that this value equals t for $m = 2t+1$ and equals 2 for $m \geq 3t-1$.

1.2 Graph Pebbling Definitions

A *configuration* C on a graph G is a function $C : V(G) \to \mathbb{N}$. The value $C(v)$ represents the number of pebbles at vertex v. The size $|C|$ of a configuration C is the total number of pebbles on G. A *pebbling move* consists of removing two pebbles from a vertex and placing one pebble on an adjacent vertex. For a target vertex r, a configuration C is r*-solvable* if one can place a pebble on r after a sequence of pebbling moves, and is r*-unsolvable* otherwise. Also, C is *solvable* if it is r-solvable for all r. The *pebbling number* $\pi(G,r)$ is the minimum number t such that every configuration of size t is r-solvable. The pebbling number of G equals $\pi(G) = \max_r \pi(G,r)$.

The basic lower and upper bounds for every graph are as follows.

Fact 1 **([5,10]).** *For every graph G we have* $\max\{n(G),\, 2^{D(G)}\} \leq \pi(G) \leq (n(G) - D(G))(2^{D(G)} - 1) + 1$.

A graph is called *Class 0* if $\pi(G) = n(G)$. For example, complete graphs, hypercubes, and the Petersen graph are known to be Class 0 [10].

1.3 Context

The upper bound in Fact 1 is due to the pigeonhole principle. The simplest pigeonhole argument yields an upper bound of $(n(G) - 1)(2^{D(G)} - 1) + 1$: a configuration of this size guarantees that either the target vertex r has a pebble on it or some other vertex has at least $2^{D(G)}$ pebbles on it, which can then move a pebble to r without assistance from pebbles on other vertices. The improvement of Chan and Godbole [5] combines the vertices on a maximum length induced path from r into one "pigeon hole", recognizing that $2^{D(G)}$ pebbles on that path is enough to move one of them to r. Generalizing further, one can take any spanning tree T of G and realize that the same pigeonhole argument yields the upper bound $|L(T)|(2^{D(G)}-1)+1$, where $L(T)$ is the set of leaves of T. Then Chung [6] found that the paths from the leaves to r, which typically overlap, could instead be shortened in a special way so as to partition the edges of T, thereby decreasing the exponent of 2 for most of the leaves. (The proof of her result needed double induction, however, rather than the pigeonhole principle.) In short, she defined the *maximum path partition* of T and used it to derive the exact formula for $\pi(T,r)$. We will not need to use this formula here, but we will record the resulting upper bound.

Fact 2. *If T is a spanning tree of G and r is a vertex of G, then $\pi(G,r) \leq \pi(T,r)$.*

Moreover, Fact 2 holds if T is any spanning subgraph of G. However, it is mostly used when T is a tree because we have Chung's formula for trees. In Sect. 3.1 we describe a powerful generalization from [12] that uses many (not necessarily spanning) trees instead of just one, and utilizes linear optimization as well.

2 Results

Here we briefly present known results on the pebbling numbers of Kneser graphs, followed by our new theorems, which we will prove in Sect. 4 after describing the tools used for them in Section 3.

2.1 Historical Contributions

It was proved in [14] that every diameter two graph G has pebbling number at most $n(G)+1$, and in [7] the authors characterize which diameter two graphs are Class 0. As a corollary they derive the following result.

Theorem 3 ([7]). *If $D(G)=2$ and $\kappa(G)\geq 3$, then G is Class 0.*

As those authors pointed out, since almost every graph is 3-connected with diameter 2, it follows that almost all graphs are Class 0. Additionally, since $K(m,t)$ is 3-connected with diameter two for every $m \geq 3t-1$ and $t \geq 2$, one obtains the following corollary.

Corollary 4. *If $t \geq 2$ and $m \geq 3t-1$, then $K(m,t)$ is Class 0.*

A much better asymptotic result was obtained in [9].

Theorem 5 **([9]).** *For any constant c there is a t_0 such that, for all $t \geq t_0$ and $s \geq c(t/\lg t)^{1/2}$ and $m = 2t + s$, we have that $K(m,t)$ is Class 0.*

Based on this evidence, the following was raised as a question in [7], which has since been conjectured in numerous talks on the subject by Hurlbert.

Conjecture 6. *If $m \geq 2t + 1$, then $K(m,t)$ is Class 0.*

2.2 Our Contributions

From Corollary 4 we see that the smallest three open cases for Kneser graphs are $K(7,3)$, $K(9,4)$, and $K(10,4)$. In every case, the lower bound of $K(m,t) \geq \binom{m}{t}$ comes from Fact 1. Conjecture 4 posits that these graphs have pebbling numbers equal to their number of vertices, namely $\binom{7}{3} = 35$, $\binom{9}{4} = 126$, and $\binom{10}{4} = 210$, respectively. Our main results in this paper address the upper bounds for these cases.

Fact 1 delivers upper bounds of 224, 1830, and 1449, respectively. By using breadth-first-search spanning trees, Fact 2, and Chung's tree formula, it is not difficult to derive the improved upper bounds of 54, 225, and 247, respectively. However, our Theorems 7, 8, and 9, below, are significantly stronger. Besides the infinite family $K(m,2)$ and the Kneser graphs satisfying Theorem 5, Theorem 9 gives further positive evidence to Conjecture 6.

Theorem 7. *For $K(7,3)$ we have $35 \leq \pi(K(7,3)) \leq 36$.*

Theorem 8. *For $K(9,4)$ we have $126 \leq \pi(K(9,4)) \leq 141$.*

Theorem 9. *For $K(10,4)$ we have $\pi(K(10,4)) = 210$; i.e., $K(10,4)$ is Class 0.*

Additionally, the most obvious infinite family of open cases for Kneser graphs are the odd graphs $K(2t+1,t)$ for $t \geq 3$. We note that the number of vertices of $G = K(2t+1,t)$ is $n = \binom{2t+1}{t}$, which Stirling's formula implies is asymptotic to $4^{t+1}/\sqrt{\pi t}$, so that t is roughly (in fact greater than) $(\lg n)/2$. Observe also that for odd graphs, we have $D(G) = t$. Thus Fact 1 yields an upper bound on $\pi(G)$ on the order of $n^{1.5}$. Here we improve this exponent significantly.

Theorem 10. *For any $t \geq 3$, let $n = n(K(2t+1,t))$ and $\alpha = \log_4((5e)^{2/3}) \approx 1.25$. Then we have $n \leq \pi(K(2t+1,t)) \leq .045n^{\alpha}(\lg n)^{\alpha/2} < .045n^{1.26}(\lg n)^{0.63}$.*

We will also prove in Theorem 19 below that a well-known lower bound technique (Lemma 13) will not produce a lower bound for odd graphs that is higher than that of Fact 1.

3 Techniques

3.1 Upper Bound

Here we describe a linear optimization technique invented in [12] to derive upper bounds on the pebbling numbers of graphs.

Let T be a subtree of a graph G rooted at the vertex r, with at least two vertices. For a vertex $v \in V(T)$, a *parent* of v, denoted by v^+, is the unique neighbor of v in T whose distance to r is one less than that of v. Moreover v is called a *child* of v^+). We say that T is an *r-strategy* if we assign to it a non-negative *weight function* w having the properties that $w(r) = 0$ and $w(v^+) \geq 2w(v)$ for every vertex $v \in V(T)$ that is not a neighbor of r. In addition, $w(v) = 0$ for vertices not in T.

Now set $\boldsymbol{T}$ to be the configuration defined by $\boldsymbol{T}(r) = 0$, $\boldsymbol{T}(v) = 1$ for all $v \in V(T) - \{r\}$, and $\boldsymbol{T}(v) = 0$ for all $v \in V(G) - V(T)$. Then the *weight* of any configuration C, including $\boldsymbol{T}$, is defined to be $w(C) = \sum_{v \in V} w(v)C(v)$. The following Lemma 11 provides an upper bound on $\pi(G)$.

Lemma 11 (Weight Function Lemma, [12]). *Let T be an r-strategy of G with associated weight function w. Suppose that C is an r-unsolvable configuration of pebbles on $V(G)$. Then $w(C) \leq w(\boldsymbol{T})$.*

The main use of Lemma 11 is as follows. Given a collection of r-strategies, the Weight Function Lemma delivers a corresponding set of linear equations. From these, one can use linear optimization to maximize the size of a configuration, subject to those constraints. If α is the result of that optimization, then the size of every r-unsolvable configuration is at most $\lfloor \alpha \rfloor$ and so $\pi(G, r) \leq \lfloor \alpha \rfloor + 1$.

A special instance of Lemma 11 yields the following result.

Lemma 12 (Uniform Covering Lemma, [12]). *Let $\mathcal{T}$ be a set of strategies for a root r of a graph G. If there is some q such that, for each vertex $v \neq r$, we have $\sum_{T \in \mathcal{T}} T(v) = q$, then $\pi(G, r) = n(G)$.*

3.2 Lower Bound

Now we turn to a technique introduced in [8] to derive lower bounds for the pebbling numbers of graphs.

Lemma 13 (Small Neighborhood Lemma [8]). *Let G be a graph and $u, v \in V(G)$. If $N_a[u] \cap N_b[v] = \emptyset$ and $|N_a[u] \cup N_b[v]| < 2^{a+b+1}$, then G is not Class 0.*

The idea behind Lemma 13 is that one considers the configuration that places $2^{a+b+1} - 1$ pebbles on u, 1 pebble on each vertex of $V(G) - (N_a[u] \cup N_b[v])$, and no pebbles elsewhere. It is not difficult to argue that this configuration is v-unsolvable and, under the hypotheses of Lemma 13, has size at least $n(G)$. Thus, what the idea behind the Small Neighborhood Lemma delivers is slightly stronger: if $N_a[u] \cap N_b[v] = \emptyset$ then $\pi(G, u) \geq n(G) + 2^{a+b+1} - |N_a[u] \cup N_b[v]|$.

With this in mind, when attempting to prove that a graph is not Class 0, one always checks if the Small Neighborhood Lemma applies. We show in Theorem 19 below that this lemma cannot apply to odd graphs. Thus, if one attempts to prove that some odd graph is not Class 0, a different method would be required.

4 Proofs

We begin with an important result that describes the distance structure of Kneser graphs. This result and its consequent corollary will be used in both the upper and lower bound arguments that follow.

Lemma 14 ([15]). *Let A and B be two different vertices of $K(m,t)$, where $t \geq 2$ and $m \geq 2t+1$. If $|A \cap B| = s$, then $\mathsf{dist}(A,B) = \mathsf{min}\{2\lceil \frac{t-s}{m-2t} \rceil, 2\lceil \frac{s}{m-2t} \rceil + 1\}$. In particular, $D(K(m,t)) = \lceil \frac{t-1}{m-2t} \rceil + 1$.*

For odd graphs, this yields the following characterization of vertices at a fixed distance from any given vertex, a corollary that is easily proved by induction.

Corollary 15. *Let $A \in V(K(2t+1,t))$. For each $0 \leq d \leq t$ we have $B \in N_d(A)$ if and only if $|B \cap A| = t - d/2$ for even d and $|B \cap A| = \lfloor d/2 \rfloor$ for odd d. Consequently $|N_d(A)| = \binom{t}{\lfloor d/2 \rfloor}\binom{t+1}{\lceil d/2 \rceil}$ for all d.*

4.1 Upper Bounds

Because Kneser graphs are vertex-transitive, we know that for every vertex r we have $\pi(K(m,t)) = \pi(K(m,t),r)$. Thus we may set $r = \{1,\ldots,t\}$ in each case.

Proof of Theorem 7

Proof. Let $G = K(7,3)$. We describe a particular r-strategy $\boldsymbol{T}$ (see Fig. 1, with weights in red). From this, we set $\mathcal{T}$ to be the set of all r-strategies determined by the set of automorphisms of G that fix r. The result of summing together all the corresponding inequalities given by Lemma 11 is that every pair of vertices having the same distance from r will have the same coefficient.

Thus, note that $\mathcal{T}$ is a set of $3!4! = 144$ r-strategies, one for each permutation of $\{1,\ldots,7\}$ that fixes r. As $D(G) = 3$, and considering the structure of G from Corollary 15, we see that $|N_1(r)| = 4$, $|N_2(r)| = 12$, and $|N_3(r)| = 18$. For each d define c_d to be the average of the coefficients in $N_d(r)$: $c_1 = 16/4 = 4$, $c_2 = [3(8) + 6(2)]/12 = 3$, and $c_3 = [9(4) + 9(2)]/18 = 3$. We now consider the sum of all these inequalities and then re-scale by dividing the result by 144. The result is that if $v \in N_d(r)$ then the coefficient of $C(v)$ in the re-scaled inequality equals c_d. Thus we derive

$$\begin{aligned}
3|C| &= \sum_{v \neq r} 3C(v) \\
&\leq \sum_{v \in N_1(r)} 4C(v) + \sum_{v \in N_2(r)} 3C(v) + \sum_{v \in N_3(r)} 3C(v) \\
&\leq \sum_{v \in N_1(r)} 4 + \sum_{v \in N_2(r)} 3 + \sum_{v \in N_3(r)} 3 \qquad \text{(by Lemma 11)} \\
&= 3(n(K(7,3) - 1) + |N_1(r)| \\
&= 3(35) + 1.
\end{aligned}$$

Hence $|C| \leq 35$ and so $\pi(K(7,3)) \leq 36$. $\square$

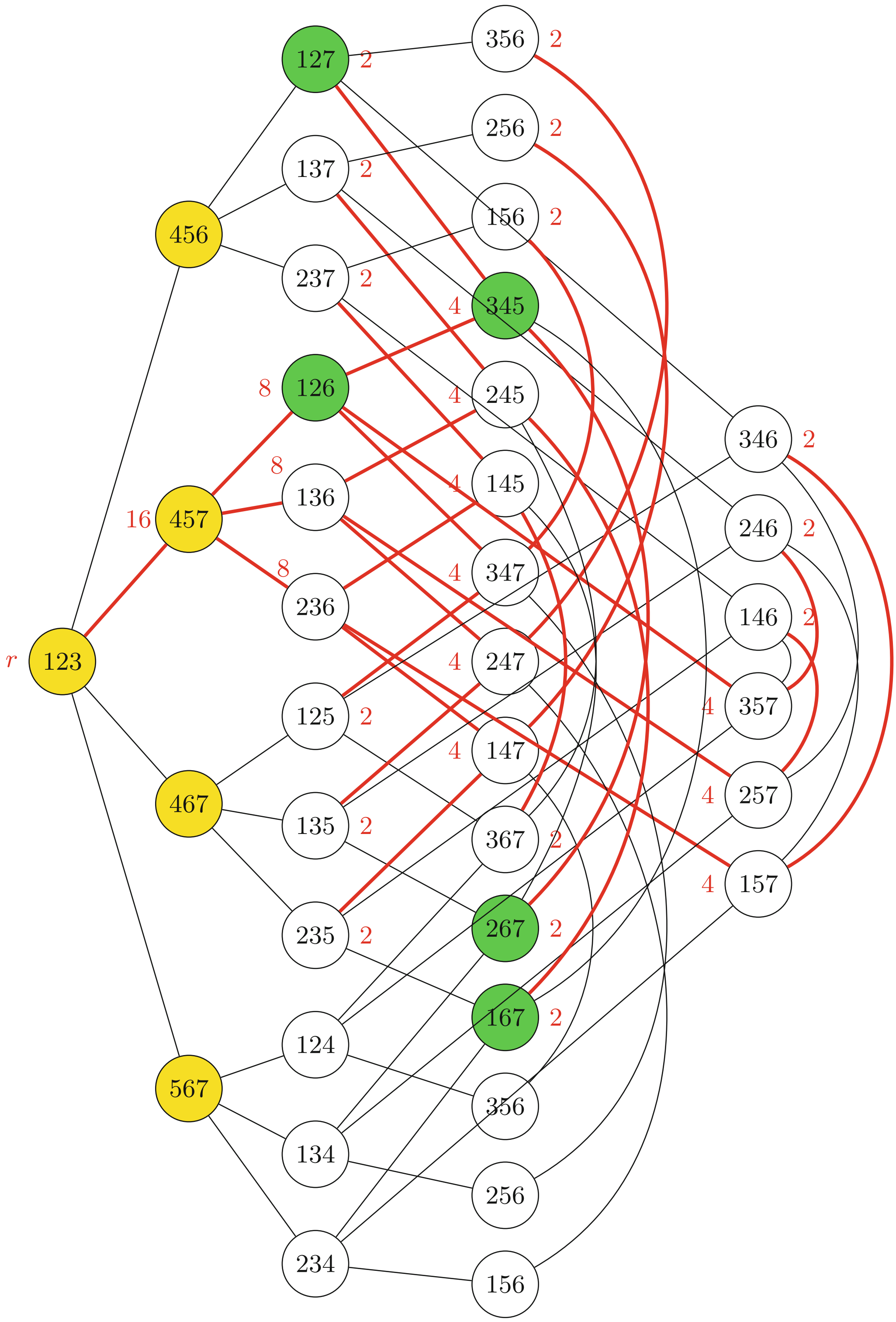

Fig. 1. The Kneser graph $K(7,3)$, with red edges showing the strategy $\boldsymbol{T}$ defined in the proof of Theorem 7, and yellow and green vertices illustrating $N_1[123] \cup N_1[345]$ in the proof of Theorem 19. It is easy to see that $N_2[123] \cup N_0[345]$ is a much larger set, containing 18 vertices instead of 10. Note that vertices 356, 256, and 156 have been drawn twice (near the top and the bottom) for ease in drawing their edges and that $N_3(123)$ has been drawn in the rightmost two columns for similar reasons.

Proof of Theorem 8

Proof. Let $G = K(9,4)$. We describe a particular r-strategy $\boldsymbol{T}$ (see Fig. 2), using the tree T defined as follows: choose a vertex $v \in N_1(r)$ and set $T_0 = \{r\}$ and $T_1 = \{v\}$; for each $d \in \{2,3,4\}$ set $T_d = \{u \in N_d(r) \cap N_1(w) \mid w \in T_{d-1}\}$; then set $T_5 = \{u \in (N_5(r) - T_4) \cap N_1(w) \mid w \in T_4\}$. Note that $|T_1| = 1$, $|T_2| = \binom{4}{3}\binom{1}{1} = 4$, $|T_3| = \binom{4}{1}\binom{4}{3} = 16$, $|T_4| = \binom{4}{2}\binom{4}{1} = 24$, and $|T_5| = \binom{4}{2}\binom{4}{2} = 36$. Indeed, these calculations are derived from observing that the distance from a vertex $u \in T_d$ to r is d, while its distance to v is $d-1$, and using Lemma 14 for both instances. Now define $\boldsymbol{T}$ by giving weight $160/2^d$ to each vertex in T_d.

From this, we set $\mathcal{T}$ to be the set of all r-strategies determined by the set of automorphisms of $K(9,4)$ that fix r. The result of summing together all the corresponding inequalities given by Lemma 11 is that every pair of vertices having the same distance from r will have the same coefficient.

Thus, note that $\mathcal{T}$ is a set of $4!5! = 2880$ r-strategies, one for each permutation of $\{1,\ldots,9\}$ that fixes r. As $D(G) = 4$, and considering the structure of G from Corollary 15, we see that $|N_1(r)| = 5$, $|N_2(r)| = 20$, $|N_3(r)| = 40$, and $|N_4(r)| = 60$ (see Table 1). For each d define c_d to be the average of the coefficients in $N_d(r)$: $c_1 = 80/5 = 16$, $c_2 = [4(40)]/20 = 8$, $c_3 = [16(20)]/40 = 8$, and $c_4 = [24(10) + 36(5)]/60 = 7$. We now consider the sum of all these inequalities and then re-scale by dividing the result by 2880. The result is that if $v \in N_d(r)$ then the coefficient of $C(v)$ in the re-scaled inequality equals c_d. Thus we derive

$$\begin{aligned}
7|C| &= \sum_{v \neq r} 7C(v) \\
&\leq \sum_{v \in N_1(r)} 16C(v) + \sum_{v \in N_2(r)} 8C(v) + \sum_{v \in N_3(r)} 8C(v) + \sum_{v \in N_4(r)} 7C(v) \\
&\leq \sum_{v \in N_1(r)} 16 + \sum_{v \in N_2(r)} 8 + \sum_{v \in N_3(r)} 8 + \sum_{v \in N_3(r)} 7 \qquad \text{(by Lemma 11)} \\
&= (5)(16) + (20)(8) + (40)(8) + (60)(7) \\
&= 980.
\end{aligned}$$

Hence $|C| \leq 140$ and so $\pi(K(9,4)) \leq 141$. $\square$

Proof of Theorem 9

Proof. Let $G = K(10,4)$. We describe a particular r-strategy $\boldsymbol{T}$ (see Fig. 3), using the tree T defined as follows: choose vertex $v = \{5,6,7,8\} \in N_1(r)$ and define the set $Z = \{9,0\}$. We assign the label (x,y,z) to a vertex u if u shares x, y, and z elements with r, v, and Z, respectively; $V(x,y,z)$ will denote the set of vertices with such a label. We add edges in T from v to all its neighbors in $N_2(r)$; i.e. to $V(3,0,1) \cup V(2,0,2)$. Because $V(2,2,0) \subset N_2(r) \cap N_1(u)$ for some $u \in V(2,0,2)$, we extend T with edges from $V(2,0,2)$ to $V(2,2,0)$. Finally, we add edges in T from $V(3,0,1)$ to $V(1,2,1) \cup V(1,3,0)$. Note that $|V(3,0,1)| = 8$,

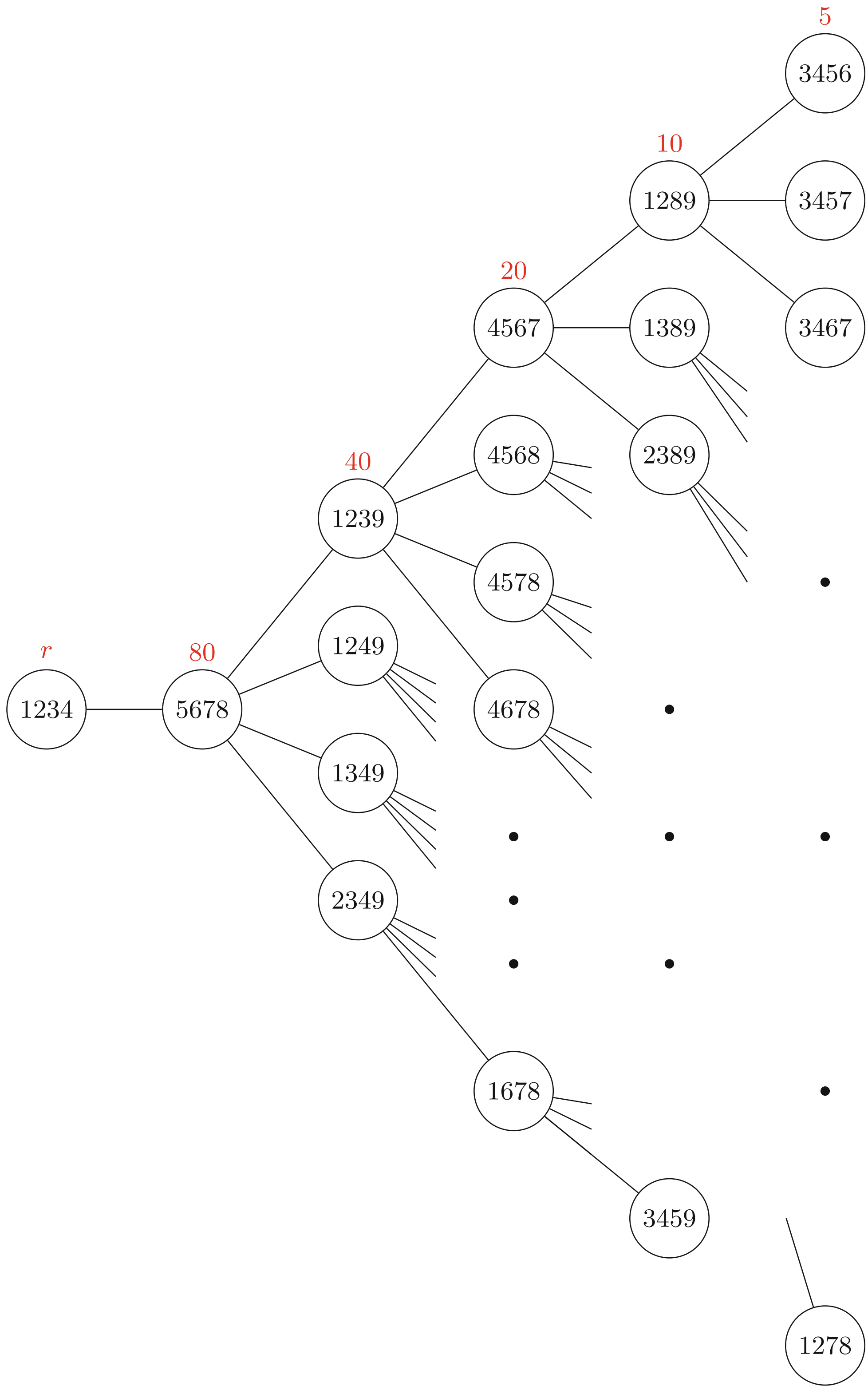

Fig. 2. A schematic diagram of the strategy $\boldsymbol{T}$ in $K(9,4)$ defined in the proof of Theorem 8, with weights in red. As in Fig. 1, the vertices in both of the rightmost two columns have maximum distance (4) from r (1234); the two columns differ, however, in their distances (3 and 4, respectively) from 5678.

Table 1. Number and structure of $K(9,4)$ vertices per distance to the root $\{1,2,3,4\}$, according to Corollary 15. Columns 3 and 4 show the numbers of elements chosen from the sets $\{1,2,3,4\}$ and $\{5,6,7,8,9\}$, respectively, for vertices at each distance; e.g., $A \in N_2(\{1,2,3,4\})$ if and only if $|A \cap \{1,2,3,4\}| = 3$ and $|A \cap \{5,6,7,8,9\}| = 1$.

Distance i	$\lvert N_i(\{1,2,3,4\})\rvert$	$\{1,2,3,4\}$	$\{5,6,7,8,9\}$
0	$\binom{4}{4}\cdot\binom{5}{0} = 1$	4	0
1	$\binom{4}{0}\cdot\binom{5}{4} = 5$	0	4
2	$\binom{4}{3}\cdot\binom{5}{1} = 20$	3	1
3	$\binom{4}{1}\cdot\binom{5}{3} = 40$	1	3
4	$\binom{4}{2}\cdot\binom{5}{2} = 60$	2	2

$|V(2,0,2)| = 6$, $|V(2,2,0)| = 36$, $|V(1,2,1)| = 48$, and $|V(1,3,0)| = 16$, while $|N_1(r)| = 15$, $|N_2(r)| = 114$, and $|N_3(r)| = 80$.

Now we define $\boldsymbol{T}$ by giving weight 60 to v, weight 30 to each vertex of $V(3,0,1) \cup V(2,0,2)$, weight 5 to each vertex of $V(1,2,1) \cup V(1,3,0)$, and weight 1 to each vertex of $V(2,2,0)$. From this, we set $\mathcal{T}$ to be the set of all r-strategies determined by the set of $4!6!$ automorphisms of $K(10,4)$ that fix r. The result of summing together all the corresponding inequalities given by Lemma 11 is that every pair of vertices having the same distance from r will have the same coefficient.

For each d define c_d to be the average of the coefficients in $N_d(r)$: $c_1 = [60(1)]/15 = 4$, $c_2 = [30(8)+30(6)+1(36)]/114 = 4$, and $c_3 = [5(48)+5(16)]/80 = 4$. We now consider the sum of all these inequalities and then re-scale by dividing the result by $4!6!$. The result is that if $v \in N_d(r)$ then the coefficient of $C(v)$ in the re-scaled inequality equals c_d. By Lemma 12 we have that $\pi(G,r) = n(G) = 210$; i.e. $K(10,4)$ is Class 0. $\square$

Proof of Theorem 10

Proof. Let $G = K(2t+1,t)$. Theorems 7 and 8 already have better bounds, so we will assume that $t \geq 5$. As in the proof of Theorem 8, we set $r = \{1,\ldots,t\}$, choose some $v \in N_1(r)$, and define the tree T by $T_0 = \{r\}$, $T_1 = \{v\}$, and for each $d \in \{2,\ldots,t\}$ set $T_d = \{u \in N_d(r) \cap N_1(w) \mid w \in T_{d-1}\}$, with $T_{t+1} = \{u \in (N_t(r) - T_t) \cap N_1(w) \mid w \in T_t\}$. We note that $|N_d(r)| = \binom{t}{\lfloor d/2 \rfloor}\binom{t+1}{\lceil d/2 \rceil}$ for $1 \leq d \leq t$ and that $|T_d| = \binom{t}{\lfloor d/2 \rfloor}\binom{t}{\lceil d/2 \rceil - 1}$ for $1 \leq d \leq t-1$, with $|T_t| = \binom{t}{\lfloor t/2 \rfloor}\binom{t}{\lceil t/2 \rceil - 1}$ and $|T_{t+1}| = \binom{t}{\lfloor t/2 \rfloor}\binom{t}{\lceil t/2 \rceil}$ when t is even and $|T_t| = \binom{t}{\lfloor t/2 \rfloor}\binom{t}{\lceil t/2 \rceil}$ and $|T_{t+1}| = \binom{t}{\lfloor t/2 \rfloor}\binom{t}{\lceil t/2 \rceil - 1}$ when t is odd. Now define $\boldsymbol{T}$ by giving weight $w_d = (t+1)2^{t+1-d}$ to each vertex in T_d for all $d > 0$.

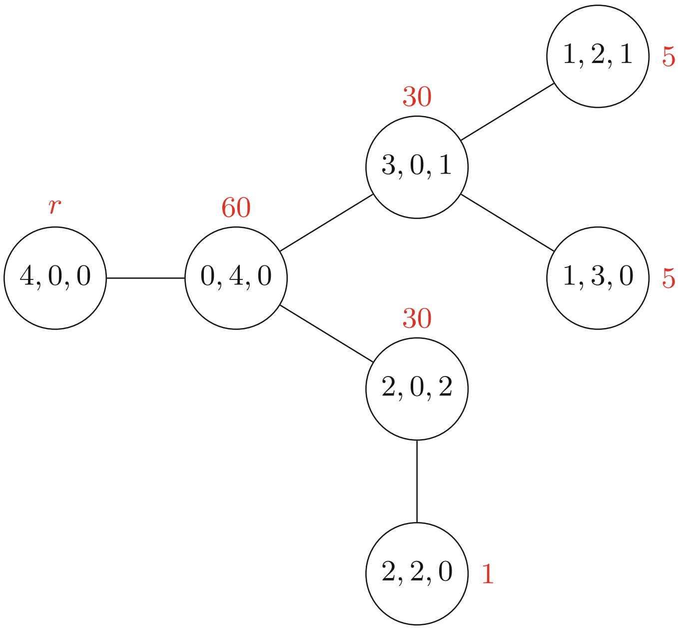

Fig. 3. A schematic diagram of the strategy $\boldsymbol{T}$ in $K(10, 4)$ defined in the proof of Theorem 9, with weights in red. Each vertex of the tree represents a set of vertices of the form (x, y, z), where x, y, and z are the numbers of digits chosen from $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}$, and $\{9, 0\}$, respectively.

From this, we set $\mathcal{T}$ to be the set of all r-strategies determined by the set of $t!(t + 1)!$ automorphisms of G that fix r. The result of summing together all the corresponding inequalities given by Lemma 11 is that every pair of vertices having the same distance from r will have the same coefficient. For each d define c_d to be the average of the coefficients in $N_d(r)$:

$$c_d = \begin{cases} |T_d|w_d/|N_d(r)| = \lceil d/2 \rceil 2^{t+1-d} \text{ for } d < t \text{ and} \\ \big(|T_t|w_t + |T_{t+1}|w_{t+1}\big)/|N_t(r)| = \begin{cases} 3t/2 + 1 \text{ for even } t \text{ and} \\ 3\lceil t/2 \rceil \text{ for odd } t. \end{cases} \end{cases}$$

We now consider the sum of all these inequalities and then re-scale by dividing the result by $t!(t+1)!$. The result is that if $v \in N_d(r)$ then the coefficient of $C(v)$ in the re-scaled inequality equals c_d. Because c_t is the smallest coefficient when $t \geq 6$ (it is c_4 when $t = 5$ but we can add some edges from T_5 into $N_4(r) - T_4$, as in the proof of Theorem 7, with sufficiently chosen weights to remedy this without effecting the calculations below), and using Lemma 11, we derive

$$\begin{aligned} c_t|C| = \sum_{v \neq r} c_t C(v) \ &\leq \ \sum_{d=1}^{t} \sum_{v \in N_d(r)} c_d C(v) \\ &\leq \sum_{d=1}^{t} \sum_{v \in N_d(r)} c_d \ = \ \sum_{d=1}^{t} |N_d(r)| c_d. \end{aligned} \tag{1}$$

By computing the ratios $r_d = |N_d(r)|c_d/|N_{d-1}(r)|c_{d-1}$ we find that $r_d = (t-a+1)/2a$ for $d \in \{2a, 2a+1\}$, showing that the sequence $|N_1(r)|c_1, \ldots, |N_t(r)|c_t$ is unimodal with its maximum occurring when $d = j := 2\lceil (t+1)/3 \rceil - 1$. Hence from Inequality 1 we obtain the upper bound

$$|C| < t|N_j(r)|c_j/c_t, \tag{2}$$

which yields that

$$\pi(G) \;\leq\; t|N_j(r)|c_j/c_t \;=\; t\binom{t}{\lfloor j/2 \rfloor}\binom{t+1}{\lceil j/2 \rceil}\lceil j/2 \rceil 2^{t+1-j}\Big/(3t/2). \tag{3}$$

Because the bound in Inequality 2 is so generous, we can dispense with floors and ceilings and addition/subtraction by one, approximating $j/2$ by $t/3$ and considering it to be an integer, thereby replacing the right side of Inequality 3 by the following. We use the notation $x^{\underline{h}} = x(x-1)\cdots(x-h+1)$, observe that $x^{\underline{h}} \leq (x-h/2)^h$, and make use of the lower bound $x! \geq (x/e)^x\sqrt{2ex}$, which works for all x, rather than using Stirling's asymptotic formula.

$$\begin{aligned} t\binom{t}{t/3}^2 \frac{(t/3)2^{2t/3}}{(3t/2)} &= \left(\frac{t^{\underline{t/3}}}{(t/3)!}\right)^2 \frac{(2t)2^{2t/3}}{9} \;\leq\; \left(\frac{5t}{6}\right)^{2t/3}\left(\frac{3e}{t}\right)^{2t/3}\frac{(2t)2^{2t/3}}{9(2et/3)} \\ &= \frac{(5e)^{2t/3}}{3e} \;<\; 5.7^t/8 \;<\; n\left(\frac{\pi^{\alpha/2}}{2^{3+2\alpha}}\right)t^{\alpha/2} < .045n^{\alpha}(\lg n)^{\alpha/2}, \end{aligned}$$

where $\alpha = \log_4((5e)^{2/3})$. This completes the proof. □

4.2 Lower Bound Attempt

The following two claims will be useful in proving Theorem 19.

Claim 16. *The following inequalities hold for all $k \geq 1$.*

1. $\binom{4k+1}{k}/\binom{4k-3}{k-1} > 4$.
2. $\binom{4k+2}{k}\binom{4k+4}{k+2}/\binom{4k-2}{k-1}\binom{4k}{k+1} > 16$.
3. $\binom{4k+1}{k}\binom{4k+2}{k}/\binom{4k-3}{k-1}\binom{4k-2}{k-1} > 16$.
4. $\binom{4k+3}{k}\binom{4k+4}{k+1}/\binom{4k-1}{k-1}\binom{4k}{k} > 16$.

Proof. We only display the proof for the first inequality, as the others use identical techniques. Indeed, we calculate

$$\begin{aligned} \frac{\binom{4k+1}{k}}{\binom{4k-3}{k-1}} &= \frac{(4k+1)\cdots(4k-k+2)(k-1)!}{(4k-3)\cdots(4k-k-1)k!} \\ &= \frac{(4k+1)\cdots(4k-2)}{(4k-k+1)\cdots(4k-k-1)k} \\ &= \left(\frac{4k+1}{k}\right)\left(\frac{4k}{3k+1}\right)\left(\frac{4k-1}{3k}\right)\left(\frac{4k-2}{3k-1}\right) \;>\; \frac{4k}{k} \;=\; 4 \end{aligned}$$

since $k \geq 1$. □

The next corollary follows from Claim 16 by induction.

Corollary 17. *The following inequalities hold for all $k \geq 1$.*

1. $\binom{4k+1}{k} > 4^k$.
2. $\binom{4k+2}{k}\binom{4k+4}{k+2} > 16^k$.
3. $\binom{4k+1}{k}\binom{4k+2}{k} > 16^k$.
4. $\binom{4k+3}{k}\binom{4k+4}{k+1} > 16^k$.

For any vertex r in $K(2t+1,t)$ and any $0 \leq d \leq t$ define $g_d(t) = |N_d(r)|$.

Claim 18. *The following inequalities hold for every $0 \leq d \leq t$.*

1. $g_d(2d+2) + g_{d+1}(2d+2) \geq 2^{2d+2}$.
2. $2g_d(2d+1) \geq 2^{2d+1}$.

Proof. Recall the formulas from Corollary 15: $g_d(t) = \binom{t}{d/2}\binom{t+1}{d/2}$ for even d and $g_d(t) = \binom{t}{\lfloor d/2 \rfloor}\binom{t+1}{\lceil d/2 \rceil}$ for odd d. We will only display the proof for the first inequality, as the second uses identical techniques.

When $d = 2s-1$ is odd we have by Corollary 17 that

$$\begin{aligned} g_{2s-1}(4s) + g_{2s}(4s) &= \binom{4s}{s-1}\binom{4s+1}{s} + \binom{4s}{s}\binom{4s+1}{s} \\ &= \binom{4s+1}{s}^2 > 4^{2s} = 2^{2d+2}. \end{aligned}$$

The case when $d = 2s$ is even is proven similarly. □

Theorem 19. *The hypotheses of Lemma 13 are not satisfied for any odd graph.*

Proof. Recall from above that $D(K(2t+1,t)) = t$ and define $f_d(t) = |N_d[A]| = \sum_{i=0}^{d} g_d(t)$, where A is any vertex. Then we must show that $f_a(t) + f_b(t) \geq 2^{a+b+1}$ for every $a \geq b$ and $t \geq a+b+1$.

We first reduce to the "balanced" case, in which $a \leq b+1$; that is, we prove that $f_a(t) + f_b(t) \geq f_x(t) + f_y(t)$, where $x = \lceil \frac{a+b}{2} \rceil$ and $y = \lfloor \frac{a+b}{2} \rfloor$. It is sufficient to show that $f_a(t) + f_b(t) \geq f_{a-1}(t) + f_{b+1}(t)$ whenever $a > b+1$. This inequality is equivalent to

$$g_a(t) = f_a(t) - f_{a-1}(t) \geq f_{b+1}(t) - f_b(t) = g_{b+1},$$

which is trivial since Corollary 15 states that $g_d(t) = \binom{t}{\lfloor d/2 \rfloor}\binom{t+1}{\lceil d/2 \rceil}$ and binomial coefficients increase up to $\lfloor t/2 \rfloor$.

Second, we reduce to the case in which $t = a+b+1$; that is, we prove that $f_x(t) + f_y(t) \geq f_x(a+b+1) + f_y(a+b+1)$. This inequality follows simply from the property that $\binom{t}{d}$ is an increasing function in t when d is fixed.

Third, we note the obvious relation that $f_x(a+b+1) + f_y(a+b+1) \geq g_x(a+b+1) + g_y(a+b+1)$, since each $g_d(t)$ is merely the final term of the summation $f_d(t)$.

Thus it suffices to show that

1. $g_y(2y+2) + g_{y+1}(2y+2) \geq 2^{2y+2}$ and
2. $2g_y(2y+1) \geq 2^{2y+1}$,

which follows from Claim 18. The above arguments yield

$$\begin{aligned} f_a(t) + f_b(t) &\geq f_x(t) + f_y(t) \\ &\geq f_x(a+b+1) + f_y(a+b+1) \\ &\geq f_x(x+y+1) + f_y(x+y+1) \\ &\geq g_x(x+y+1) + g_y(x+y+1) \ \geq \ 2^{a+b+1}. \end{aligned}$$

The final inequality follows from the identity $x + y = a + b$ and the two cases that $x = y$ or $x = y + 1$. □

5 Concluding Remarks

As discussed above, all diameter two Kneser graphs are Class 0, and we verified in Theorem 9 that the diameter three Kneser graph $K(10, 4)$ is also Class 0, while the diameter three Kneser graph $K(7, 3)$ remains undecided. By Lemma 14 we see that $D(K(m,t)) \leq 3$ for all $m \geq (5t-1)/2$. The following theorem shows that graphs with high enough connectivity are Class 0 (the value $2^{2D(G)+3}$ is not thought to be best possible, but cannot be smaller than $2^{D(G)}/D(G)$). It is this theorem that was used to prove Theorem 5.

Theorem 20 ([9]). *If G is $2^{2D(G)+3}$-connected, then G is Class 0.*

Accordingly, Theorem 20 implies that diameter three Kneser graphs with connectivity at least 2^9 are Class 0, which occurs when $\binom{m-t}{t} \geq 512$ because connectivity equals degree for Kneser graphs. This begs the following subproblem of Conjecture 6.

Problem 21. *For all $t \geq 5$ and $m \geq (5t-1)/2$, if $\binom{m-t}{t} \leq 511$ then is $K(m,t)$ Class 0? In particular, can the Weight Function Lemma be used to prove so?*

For example, if $t = 5$, then the interval of interest in Problem 21 is $12 \leq m \leq 16$.

We also see from our work both the power and the limitations of the usage of the Weight Function Lemma. For example, it did not produce a very close bound for $K(9, 4)$, which has diameter 4, but did produce the actual pebbling number for $K(10, 4)$, which has diameter 3. Indeed the power of strategies weakens as the diameter grows. Curiously, though, it did not yield a Class 0 result for $K(7, 3)$, which also has diameter 3. (Conceivably, it did give the right answer, but we do not believe this.) The trees we used in the proof for this case were very simple and structured and were all isomorphic. In trying to improve the result, we had a computer generate hundreds of thousands of tree strategies and fed them into linear programming software and even used integer programming. No results were better than the bound we presented in Theorem 7.

For fixed t the Kneser graphs $K(m,t)$ with the largest diameter (t) have $m = 2t + 1$; the odd graphs. We see that weight functions produce a fairly large upper bound in this case, with the multiplicative factor of $n^{.26}(\lg n)^{.63}$ attached, where $n = n(K(2t+1,t))$. Nonetheless, this is the best known bound. Along these lines we offer the following additional subproblem of Conjecture 6.

Problem 22. *Find a constant c such that $\pi(K(2t+1,t)) \leq cn$, where $n = n(K(2t+1,t)$.*

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