Maximum Cut on Interval Graphs of Interval Count Four Is NP-Complete

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Abstract

The computational complexity of the MaxCut problem restricted to interval graphs has been open since the 80's, being one of the problems proposed by Johnson on his *Ongoing Guide to* NP*-completeness*, and has been settled as NP-complete only recently by Adhikary, Bose, Mukherjee and Roy. On the other hand, many flawed proofs of polynomiality for MaxCut on the more restrictive class of unit/proper interval graphs (or graphs with interval count 1) have been presented along the years, and the classification of the problem is still not known. In this paper, we present the first NP-completeness proof for MaxCut when restricted to interval graphs with bounded interval count, namely graphs with interval count 4.

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1 Introduction

A *cut* is a partition of the vertex set of a graph into two disjoint parts and the *maximum cut problem* (denoted MaxCut for short) aims to determine a cut with the maximum number of edges for which each endpoint is in a distinct part. The decision problem MaxCut is known to be NP-complete since the seventies [15], and only recently its restriction to interval graphs has been announced to be hard [1], settling a long-standing open problem that appeared in the 1985 column of the *Ongoing Guide to* NP*-completeness* by David S. Johnson [17]. We refer the reader to a revised version of the table in [12], where one can also find a parameterized complexity version of said table.

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An *interval model* is a family of closed intervals of the real line. A graph is an *interval graph* if there exists an interval model, for which each interval corresponds to a vertex of the graph, such that distinct vertices are adjacent in the graph if and only if the corresponding intervals intersect. Ronald L. Graham proposed in the 80's the study of the *interval count* of an interval graph as the smallest number of interval lengths used by an interval model of the graph. Interval graphs having interval count 1 are called *unit intervals* (can also be called proper interval, or indifference). Understanding the interval count, besides being an interesting and challenging problem by itself, can be also of value for the investigation of problems that are hard for general interval graphs, and easy for unit interval graphs (e.g. geodetic number [8,13], optimal linear arrangement [9,16], sum coloring [20,21]). The positive results for unit interval graphs usually take advantage of the fact that a representation for these graphs can be found in linear time [10,11]. Surprisingly, the recognition of interval graphs with interval count k is open, even for $k = 2$ [7]. Nevertheless, another generalization of unit interval graphs has been recently introduced which might be more promising in this aspect. These graphs are called *k-nested interval graphs*, introduced in [18], where the authors, among other things, give a linear time recognition algorithm.

In the same way that MaxCut on interval graphs has evaded being solved for so long, the community has been puzzled by the restriction to unit interval graphs. Indeed, two attempts at solving it in polynomial time were proposed in [4,6] just to be disproved closely after [3,19]. In this paper, we give the first classification that bounds the interval count, namely, we prove that MaxCut is NP-complete when restricted to interval graphs of interval count 4. This also implies NP-completeness for the newly generalized class of 4-nested graphs, and opens the search for a full polynomial/NP-complete dichotomy classification in terms of the interval count. It can still happen that the problem is hard even on graphs of interval count 1. We contribute towards filling the complexity gap between interval and unit interval graphs.

Next, we establish basic definitions and notation. Section 2 describes our reduction and Section 3 discusses the interval count of the interval graph constructed in [1].

1.1 Preliminaries

In this work, all graphs considered are simple. For missing definitions and notation of graph theory, we refer to [5]. For a comprehensive study of interval graphs, we refer to [14].

Let G be a graph. Let X and Y be two disjoint subsets of $V(G)$. We let $E_G(X, Y)$ be the set of edges of G with an endpoint in X and the other endpoint in Y. For every subset $S \subseteq V(G)$, we let $S^X = S \cap X$ and $S^Y = S \cap Y$. A *cut* of G is a partition of $V(G)$ into two parts $A, B \subseteq V(G)$, denoted by $[A, B]$; the edge set $E_G(A, B)$ is called the *cut-set* of G associated with $[A, B]$. For each two vertices $u, v \in V(G)$, we say that u and v *are in a same part of* $[A, B]$ if either $\{u, v\} \subseteq A$ or $\{u, v\} \subseteq B$; otherwise, we say that u and v *are in opposite parts of* $[A, B]$. Denote by $\mathsf{mc}(G)$ the maximum size of a cut-set of G. The MaxCut problem has as input a graph G and a positive integer k, and it asks whether $\mathsf{mc}(G) \geq k$.

Let $I \subseteq \mathbb{R}$ be a closed interval of the real line. We let $\ell(I)$ and $r(I)$ denote respectively the minimum and maximum points of I, which we call the *left* and the *right endpoints* of I, respectively. We denote a closed interval I by $[\ell(I), r(I)]$. The *length* of an interval I is defined as $|I| = r(I) - \ell(I)$. An *interval model* is a finite multiset $\mathcal{M}$ of intervals. The *interval count* of an interval model $\mathcal{M}$, denoted by $\mathsf{ic}(\mathcal{M})$, is defined as the number of distinct lengths of the intervals in $\mathcal{M}$. Let G be a graph and $\mathcal{M}$ be an interval model. An *$\mathcal{M}$-representation* of G is a bijection $\phi \colon V(G) \to \mathcal{M}$ such that, for every two distinct vertices $u, v \in V(G)$, we have that $uv \in E(G)$ if and only if $\phi(u) \cap \phi(v) \neq \emptyset$. If such an $\mathcal{M}$-representation exists, we say that $\mathcal{M}$ is an *interval model of* G. We note that a graph may have either no interval model or arbitrarily many distinct interval models. A graph is called an *interval graph* if it has an interval model. The *interval count of an interval graph* G, denoted by $\mathsf{ic}(G)$, is defined as $\mathsf{ic}(G) = \min\{\mathsf{ic}(\mathcal{M})\colon \mathcal{M}$ is an interval model of $G\}$. An interval graph is called a *unit interval graph* if its interval count is equal to 1.

Note that, for every interval model $\mathcal{M}$, there exists a unique (up to isomorphism) graph that admits an $\mathcal{M}$-representation. Thus, for every interval model $\mathcal{M} = \{I_1, \ldots, I_n\}$, we let $\mathbb{G}_{\mathcal{M}}$ be the graph with vertex set $V(\mathbb{G}_{\mathcal{M}}) = \{1, \ldots, n\}$ and edge set $E(\mathbb{G}_{\mathcal{M}}) = \{ij\colon I_i, I_j \in \mathcal{M},\ I_i \cap I_j \neq \emptyset,\ i \neq j\}$. Since $\mathbb{G}_{\mathcal{M}}$ is uniquely determined (up to isomorphism) from $\mathcal{M}$, in what follows we may make an abuse of language and use graph terminologies to describe properties related to the intervals in $\mathcal{M}$. Two intervals $I_i, I_j \in \mathcal{M}$ are said to be *true twins* in $\mathbb{G}_{\mathcal{M}}$ if they have the same close neighborhood in $\mathbb{G}_{\mathcal{M}}$, i.e. $N_{\mathbb{G}_{\mathcal{M}}}(I_i) \cup \{I_i\} = N_{\mathbb{G}_{\mathcal{M}}}(I_j) \cup \{I_j\}$.

For each three positive integers $a, b, c \in \mathbb{N}_+$, we write $a \equiv_b c$ to denote that a modulo b is equal to c modulo b.

2 Our reduction

The following theorem is the main contribution of this work:

▶ **Theorem 1.** *MaxCut is* NP*-complete on interval graphs of interval count* 4*.*

This result is a stronger version of that of Adhikary et al. [1]. To prove Theorem 1, we present a polynomial-time reduction from MaxCut on cubic graphs, which is known to be NP-complete [2]. In order to explain the technical effort needed to push the construction of Adhikary et al. enabling our construction of a reduction graph that uses only four different lengths of intervals, we present our construction in three sections. First, we explain how the key gadget of Adhikary et al. relates the number of intervals of each size to the part where they are placed in a maximum cut. Second, we present our new gadget that organizes copies of the original key gadget into an escalator grid, which constitutes our key gadget to obtain a reduction graph that admits a model with an interval count bounded by a constant. Third, an outline of the proof explains how our use of the base gadgetry due to Adhikary et al. through the escalator allows us to relate maximum cuts of the input graph to maximum cuts of the reduction graph.

2.1 Grained gadget

The interval graph constructed in the reduction of [1] is strongly based on two types of gadgets, called *V-gadgets* and *E-gadgets*. In fact, these gadgets are the same, except for the amount of intervals of certain kinds contained in each of them. In this subsection, we present a generalization of such gadgets, rewriting their key properties to suit our purposes. In order to discuss the interval count of the reduction of [1], we describe it in details in Section 3.

Let x and y be two positive integers. An (x, y)*-grained gadget* is an interval model $\mathcal{H}$ formed by y long intervals (called *left long*) intersecting in their right endpoint with other y long intervals (called *right long)*, together with $2x$ short intervals, x of which intersect exactly the y left long ones (called *left short*), and x of which intersect exactly the y right long ones (called *right short*); see Figure 1. We write $\mathcal{LS}(\mathcal{H})$, $\mathcal{LL}(\mathcal{H})$, $\mathcal{RS}(\mathcal{H})$ and $\mathcal{RL}(\mathcal{H})$ to denote the left short, left long, right short and right long intervals of $\mathcal{H}$, respectively, and we omit $\mathcal{H}$ when it is clear from the context.

Note that, if $\mathcal{H}$ is an (x, y)-grained gadget, then $\mathbb{G}_{\mathcal{H}}$ is a split graph such that $\mathcal{LS} \cup \mathcal{RS}$ is an independent set of size $2x$, $\mathcal{LL} \cup \mathcal{RL}$ is a clique of size $2y$, $N_{\mathbb{G}_{\mathcal{H}}}(\mathcal{LS}) = \mathcal{LL}$ and $N_{\mathbb{G}_{\mathcal{H}}}(\mathcal{RS}) = \mathcal{RL}$. Moreover, the intervals in $\mathcal{LL}$ are true twins in $\mathbb{G}_{\mathcal{H}}$; similarly, the intervals in $\mathcal{RL}$ are true twins in $\mathbb{G}_{\mathcal{H}}$.

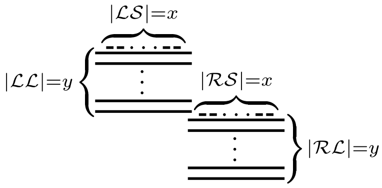
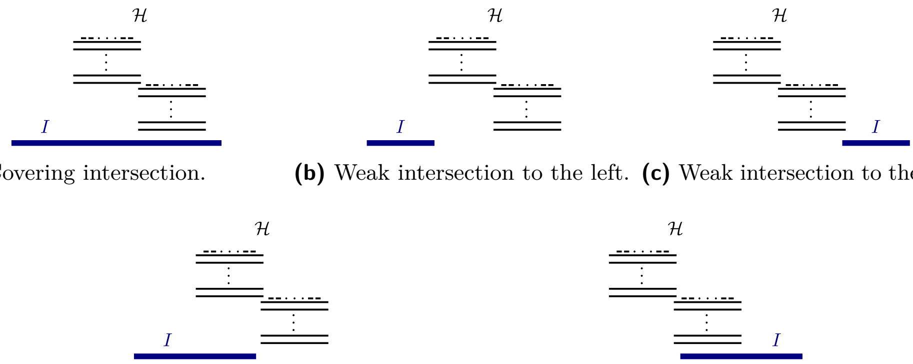


Figure 1 General structure of an (x, y)-grained gadget.

Let $\mathcal{M}$ be an interval model containing an (x, y)-grained gadget $\mathcal{H}$. The possible types of intersections between an interval $I \in \mathcal{M} \setminus \mathcal{H}$ and $\mathcal{H}$ in our construction are depicted in Figure 2, using our notation. More specifically, the *cover intersection* intersects all the intervals, the *weak intersection to the left* (*right*) intersects exactly the left (right) long intervals, while the *strong intersection to the left* (*right*) intersects exactly the left (right) long and short intervals. We say that $\mathcal{M}$ *respects the structure* of $\mathcal{H}$ if I either does not intersect $\mathcal{H}$ at all, or intersects $\mathcal{H}$ as depicted in Figure 2.



(a) Covering intersection. **(b)** Weak intersection to the left. **(c)** Weak intersection to the right.

(d) Strong intersection to the left. **(e)** Strong intersection to the right.

Figure 2 (a) Interval $I \in \mathcal{M} \setminus \mathcal{H}$ covering $\mathcal{H}$, (b-c) weakly intersecting $\mathcal{H}$ to the left and to the right, and (d-e) strongly intersecting $\mathcal{H}$ to the left and to the right.

The advantage of this gadget is that, by manipulating the values of x and y, we can ensure that, in a maximum cut, the left long and right short intervals are placed in the same part, opposite to the part containing the left short and right long intervals. The next lemma is a step in this direction. Denote by $\mathsf{c}_{\mathcal{M}}(\mathcal{H})$ the number of intervals of $\mathcal{M}$ that cover $\mathcal{H}$; by $\mathsf{wkl}_{\mathcal{M}}(\mathcal{H})$ (resp. $\mathsf{wkr}_{\mathcal{M}}(\mathcal{H})$) the number of intervals of $\mathcal{M}$ that weakly intersect $\mathcal{H}$ to the left (resp. right); and by $\mathsf{stl}_{\mathcal{M}}(\mathcal{H})$ (resp. $\mathsf{str}_{\mathcal{M}}(\mathcal{H})$) the number of intervals of $\mathcal{M}$ that strongly intersect $\mathcal{H}$ to the left (resp. right).

▶ **Lemma 2.** *Let x and y be positive integers, $\mathcal{H}$ be an (x, y)-grained gadget and $\mathcal{M}$ be an interval model that respects the structure of $\mathcal{H}$. For every maximum cut $[A, B]$ of $\mathbb{G}_{\mathcal{M}}$, the following conditions hold:*

1. *if $y + \mathsf{stl}_{\mathcal{M}}(\mathcal{H}) + \mathsf{c}_{\mathcal{M}}(\mathcal{H}) \equiv_2 1$ and $x > 2y - 1 + \mathsf{wkl}_{\mathcal{M}}(\mathcal{H}) + \mathsf{stl}_{\mathcal{M}}(\mathcal{H}) + \mathsf{c}_{\mathcal{M}}(\mathcal{H})$, then $\mathcal{LS}(\mathcal{H}) \subseteq A$ and $\mathcal{LL}(\mathcal{H}) \subseteq B$, or vice versa;*
2. *if $y + \mathsf{str}_{\mathcal{M}}(\mathcal{H}) + \mathsf{c}_{\mathcal{M}}(\mathcal{H}) \equiv_2 1$ and $x > 2y - 1 + \mathsf{wkr}_{\mathcal{M}}(\mathcal{H}) + \mathsf{str}_{\mathcal{M}}(\mathcal{H}) + \mathsf{c}_{\mathcal{M}}(\mathcal{H})$, then $\mathcal{RS}(\mathcal{H}) \subseteq A$ and $\mathcal{RL}(\mathcal{H}) \subseteq B$, or vice versa.*

Now, we want to add conditions that, together with the ones from the previous lemma, ensure that the left long intervals will be put opposite to the right long intervals. Based on Lemma 2, we say that $(\mathcal{H}, \mathcal{M})$ is *well-valued* if Conditions (1) and (2) hold, in addition to the following inequality

$$y^2 > y \cdot \mathsf{wkr}_{\mathcal{M}}(\mathcal{H}) + (x - y) \cdot \big(\mathsf{str}_{\mathcal{M}}(\mathcal{H}) + \mathsf{c}_{\mathcal{M}}(\mathcal{H})\big). \tag{1}$$

Let $[A, B]$ be a maximum cut of $\mathbb{G}_{\mathcal{M}}$. We say that $\mathcal{H}$ is *A-partitioned* by $[A, B]$ if $\mathcal{LS}(\mathcal{H}) \cup \mathcal{RL}(\mathcal{H}) \subseteq A$, and $\mathcal{RS}(\mathcal{H}) \cup \mathcal{LL}(\mathcal{H}) \subseteq B$. Define *B-partitioned* analogously. The next lemma finally ensures what we wanted.

▶ **Lemma 3.** *Let x and y be positive integers, $\mathcal{H}$ be an (x, y)-grained gadget, $\mathcal{M}$ be an interval model and $[A, B]$ be a maximum cut of $\mathbb{G}_{\mathcal{M}}$. If $\mathcal{M}$ respects the structure of $\mathcal{H}$ and $(\mathcal{H}, \mathcal{M})$ is well-valued, then $\mathcal{H}$ is either A-partitioned or B-partitioned by $[A, B]$.*

We have rewritten above in a more technical form the lemmas presented in [1], so that we are able to explicitly give the conditions that ensure the key property of their gadgets.

2.2 Reduction graph

In this subsection, we formally present our construction. Recall that we are making a reduction from MaxCut on cubic graphs. So, consider a cubic graph G on n vertices and m edges. Intuitively, we consider an ordering of the edges of G, and we divide the real line into m regions, with the j-th region holding the information about whether the j-th edge is in the cut-set. For this, each vertex u will be related to a subset of intervals traversing all the m regions, bringing the information about which part u belongs to. Let $\pi_V = (v_1, \ldots, v_n)$ be an ordering of $V(G)$, $\pi_E = (e_1, \ldots, e_m)$ be an ordering of $E(G)$, and $\mathfrak{G} = (G, \pi_V, \pi_E)$.

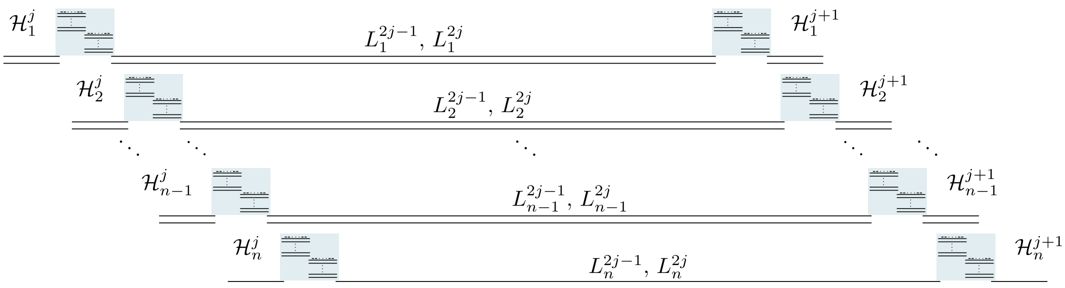


Figure 3 General structure of a region of the (n, m)-escalator. The rectangles represent the (p, q)-grained gadgets $\mathcal{H}_i^j$.

We first describe the gadgets related to the vertices. Please refer to Figure 3 to follow the construction. The values of p, q used next will be defined later. An (n, m)-*escalator* is an interval model $\mathcal{D}$ formed by $m + 1$ (p, q)-grained gadgets for each v_i, denoted by $\mathcal{H}_i^1, \ldots, \mathcal{H}_i^{m+1}$, together with $2m$ *link intervals*, $L_i^1, \ldots, L_i^{2m}$, such that L_i^{2j-1} and L_i^{2j} weakly intersect $\mathcal{H}_i^j$ to the right and weakly intersect $\mathcal{H}_i^{j+1}$ to the left. Additionally, all the grained gadgets are mutually disjoint, and given $j \in \{1, \ldots, m+1\}$ and $i, i' \in \{1, \ldots, n\}$ with $i < i'$, the grained gadget $\mathcal{H}_i^j$ occurs to the left of $\mathcal{H}_{i'}^j$.

Now, we add the gadgets related to the edges. Please refer to Figure 4 to follow the construction. The values of p', q' used next will be defined later. For each edge $e_j = v_i v_{i'} \in E(G)$, with $i < i'$, create a (p', q')-grained gadget $\mathcal{E}_j$ and intervals $C_j^1, C_j^2, C_j^3, C_j^4$ in such a way that $\mathcal{E}_j$ is entirely contained in the j-th region (i.e., in the open interval between the right endpoint of $\mathcal{H}_n^j$ and the left endpoint of $\mathcal{H}_1^{j+1}$), C_j^1 and C_j^2 weakly intersect $\mathcal{H}_i^j$ to the right and weakly intersect $\mathcal{E}_j$ to the left, and C_j^3 and C_j^4 weakly intersect $\mathcal{H}_{i'}^j$ to the right and strongly intersect $\mathcal{E}_j$ to the left. Denote the constructed model by $\mathcal{M}(\mathfrak{G})$.

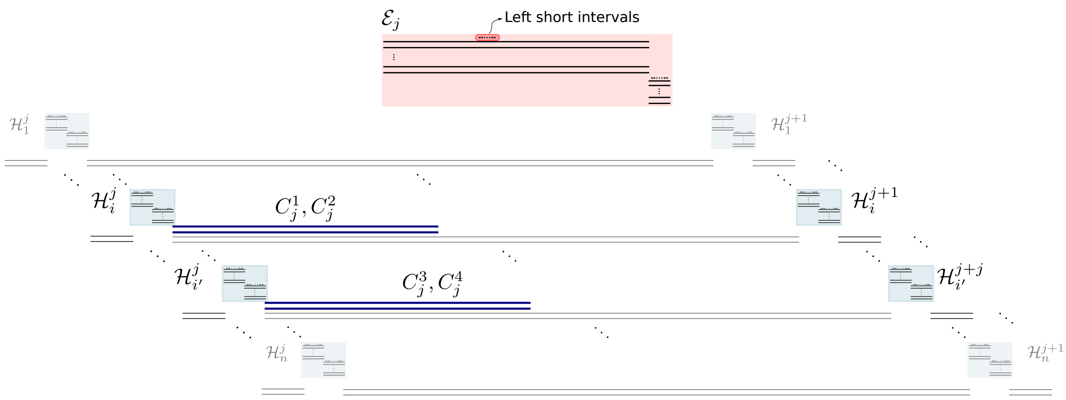


Figure 4 General structure of the constructed interval model $\mathcal{M}(\mathfrak{G})$ highlighting the intersections between the intervals of the (n, m)-escalator $\mathcal{D}$, the intervals of the (p', q')-grained gadget $\mathcal{E}_j$, and the intervals $C_j^1, C_j^2, C_j^3, C_j^4$.

2.3 Outline of the proof

As above, consider a cubic graph G on n vertices and $m = \frac{3n}{2}$ edges, and let $\pi_V = (v_1, \dots, v_n)$ be an ordering of $V(G)$, $\pi_E = (e_1, \dots, e_m)$ be an ordering of $E(G)$ and $\mathfrak{G} = (G, \pi_V, \pi_E)$. We give an outline of the proof that $\mathsf{mc}(G) \geq k$ if and only if $\mathsf{mc}(\mathbb{G}_{\mathcal{M}(\mathfrak{G})}) \geq f(G, k)$, where f is defined at the end of the subsection. As it is usually the case in this kind of reduction, constructing an appropriate cut of the reduction graph $\mathbb{G}_{\mathcal{M}(\mathfrak{G})}$, given a cut of G, is an easy task. On the other hand, constructing an appropriate cut $[X, Y]$ of G, from a given a cut $[A, B]$ of the reduction graph $\mathbb{G}_{\mathcal{M}(\mathfrak{G})}$, requires that the intervals in $\mathcal{M}(\mathfrak{G})$ behave in a way with respect to $[A, B]$ so that $[X, Y]$ can be inferred, a task achieved with the help of Lemmas 2 and 3. In order to use these lemmas, we choose next suitable values for p, q, p', q', and we observe that $\mathcal{M}(\mathfrak{G})$ respects the structure of the involved grained gadgets. After ensuring that each grained gadget behaves well individually, we also need to ensure that $\mathcal{H}_i^1$ can be used to decide in which part of $[X, Y]$ we should put v_i, and for this it is necessary that all gadgets related to v_i agree with one another. In other words, for each v_i, we want that the behaviour of the first gadget $\mathcal{H}_i^1$ influence the behaviour of the subsequent gadgets $\mathcal{H}_i^2, \dots, \mathcal{H}_i^{m+1}$, as well as the behaviour of the gadgets related to edges incident to v_i. This is done by choosing the following values for our floating variables:

$$q = 60n^3 + 1, \;\; p = 2q + 7n, \;\; q' = 18n^3 + 1 \;\text{ and }\; p' = 2q' + 5n. \tag{2}$$

These values indeed satisfy Conditions (1) and (2) of Lemma 2, as well as Inequality (1). As previously said, the idea behind this choice of values is to store information about v_i in the gadgets $\mathcal{H}_i^1, \dots, \mathcal{H}_i^{m+1}$. Now, given $e_j = v_i v_{i'}$, $i < i'$, a final ingredient is to ensure that $\mathcal{E}_j$ is influenced only by the intervals C_j^3 and C_j^4, which in turn are influenced by the vertex $v_{i'}$, in a way that the number of edges in the cut-set of $\mathbb{G}_{\mathcal{M}(\mathfrak{G})}$ increases when the edge $v_i v_{i'}$ is in the cut-set of G. All these ideas are captured in the definitions below.

Given $v_i \in V(G)$ and a cut $[A, B]$ of $\mathbb{G}_{\mathcal{M}(\mathfrak{G})}$, we say that *the gadgets of v_i alternate in* $[A, B]$ if, for every $j \in [m]$, we get that $\mathcal{H}_i^j$ is A-partitioned if and only if $\mathcal{H}_i^{j+1}$ is B-partitioned. Also, we say that $[A, B]$ is *alternating partitioned* if the gadgets of v_i alternate in $[A, B]$, for every $v_i \in V(G)$, and the following conditions hold for every $e_j = v_i v_{i'} \in E(G)$, with $i < i'$:

(i) If $\mathcal{H}_i^j$ is A-partitioned by $[A, B]$, then $\{C_j^1, C_j^2\} \subseteq B$; otherwise, $\{C_j^1, C_j^2\} \subseteq A$; and

(ii) If $\mathcal{H}_{i'}^j$ is A-partitioned by $[A, B]$, then $\{C_j^3, C_j^4\} \subseteq B$ and $\mathcal{E}_j$ is A-partitioned by $[A, B]$; otherwise, $\{C_j^3, C_j^4\} \subseteq A$ and $\mathcal{E}_j$ is B-partitioned by $[A, B]$.

The following lemma is a key element in our proof.

▶ **Lemma 4.** *If $[A, B]$ is a maximum cut of $\mathbb{G}_{\mathcal{M}(\mathfrak{G})}$, then $[A, B]$ is an alternating partitioned cut.*

Sketch. The proof of Conditions (i) and (ii) is similar to the proof of Lemma 3, and in fact the same ideas are also part of the proof in [1]. Our ability to bound the interval count is due mainly to the fact that the vertex gadgets alternate in $[A, B]$, so we focus on this part of the proof. Another skipped detail is the fact that the pairs of link intervals, and the pairs of *intervals of type* C always go together. More formally, for every $j \in \{1, \cdots, m\}$, we have that C_j^1, C_j^2 are in the same part, as well as C_j^3, C_j^4. Similarly, for every $j \in \{1, \cdots, m\}$ and $i \in \{1, \cdots, n\}$, the intervals L_i^{2j-1}, L_i^{2j} are in the same part. Just to give an idea for the latter types of intervals, this is due to the fact that the intervals in $\mathcal{H}_i^j$ or $\mathcal{H}_{i+1}^j$ outweigh the total number of *relevant* intervals intersecting L_i^{2j-1}, L_i^{2j} that are outside such vertex gadgets.

Denote $\mathcal{M}(\mathfrak{G})$ by $\mathcal{M}$ for simplicity, and let $\mathcal{M}_i$ be the set of all the intervals related to vertex v_i; more formally, it contains the grained gadget $\mathcal{H}_i^j$, for every $j \in [m+1]$, the link interval L_i^j, for every $j \in \{1, \cdots, 2m\}$, every interval of type C_j^h that intersects $\mathcal{H}_i^j$ to the right (this happens if e_j has v_i as endpoint), and every interval in $\mathcal{E}_j$ for e_j incident to v_i. We count the number f_i of edges of the cut incident to some interval in $\mathcal{M}_i$ and argue that, if the gadgets of v_i do not alternate in $[A, B]$, then we can obtain a bigger cut by rearranging $\mathcal{M}_i$, thus getting a contradiction.

Denote by $\overline{\mathcal{M}}_i$ the set of intervals $\mathcal{M} \setminus \mathcal{M}_i$, and by $\mathcal{L}$ the set of all link intervals. In what follows, we do the counting in terms of m, n, p, q, p', q' for simplicity, and we do not make an exact counting, since it would be tedious and not help so much in the understanding of the ideas behind the proof. Also, there will be some values that should be added to f_i that remain the same, independently from how $\mathcal{M}_i$ is partitioned; we call these values *irrelevant* and do not add them to f_i. Recall that every (x, y)-grained gadget has exactly $x + y$ intervals in A and $x + y$ in B. Thus, for each $j \in \{1, \cdots, 2m\}$, we know that the number of edges between L_i^j and intervals in $\overline{\mathcal{M}}_i$ that are within a grained gadget do not change if we switch L_i^j from A to B or vice-versa; in other words, these values are irrelevant. Additionally, because we are considering that Conditions (i) and (ii) hold, the number of edges of the cut within each grained gadget of $\mathcal{M}_i$, and between grained gadgets of type $\mathcal{H}_i^j$ and intervals of $\mathcal{M}_i$ of type C can also be considered irrelevant. So now, for each $j \in \{1, \cdots, m\}$, denote by ℓ_A^j the number of intervals in $\overline{\mathcal{M}}_i \cap \mathcal{L} \cap A$ that intersect L_i^{2j}; define ℓ_B^j similarly. Observe that $\ell_A^j + \ell_B^j \leq 4n$ since it includes all the link intervals in the j-th region, plus at most the link intervals of the $(j-1)$-th region related to $v_{i'}$ for $i' > i$, and the link intervals of the $(j+1)$-th region related to $v_{i'}$ for $i' < i$. Additionally, let a_j be equal to 1 if L_i^{2j} is opposite to the right long intervals of $\mathcal{H}_i^j$, and 0 otherwise; similarly, let b_j be equal to 1 if L_i^{2j} is opposite to the left long intervals of $\mathcal{H}_i^{j+1}$, and 0 otherwise. Now, let e_{j_1}, e_{j_2}, e_{j_3} be the edges incident to v_i, and for each $h \in \{1, 2, 3\}$, write e_{j_h} as $v_i v_{i_h}$. For each $h \in \{1, 2, 3\}$, observe that the non-irrelevant number of edges of the cut incident to $\mathcal{E}_{j_h}$ is $2(p'+q')$ if $\mathcal{H}_i^j$ and $\mathcal{H}_{i_h}^j$ are partitioned differently, and that it is equal to $2p'$ otherwise. Therefore, if we let c_h be equal to 1 if $\mathcal{H}_i^j$ and $\mathcal{H}_{i_h}^j$ are partitioned differently, and 0 otherwise, we get that there are $2p' + 2q'c_h$ edges in the cut incident to $\mathcal{E}_{j_h}$. Because $2p'$ is added for each $h \in \{1,2,3\}$, there is an irrelevant value of $6p'$ that we ignore. Therefore, we get that (recall that L^{2j-1} and L^{2j} are true twins):

$$f_i \leq \sum_{j=1}^{m}(2q(a_j+b_j) + \ell_A^j + \ell_B^j) + \sum_{h=1}^{3} 2q'c_h. \tag{3}$$

If L_i^{2j} is on the same side as the right long intervals of $\mathcal{H}_i^j$ and the left long intervals of $\mathcal{H}_i^{j+1}$, we can increase f_i simply by switching its side (together of course with L_i^{2j-1}). Indeed, in this case we would lose at most $\max\{\ell_A^j, \ell_B^j\} \leq 4n$ edges, while gaining $4q$, a positive exchange since $q > n$. Observe that this implies that $a_j + b_j \geq 1$. Note also that this type of argument can be always applied, i.e., whenever in what follows we switch sizes of some subset of intervals, we can suppose that this property still holds. Now, let j be the minimum value for which $a_j + b_j = 1$ (j is well defined since otherwise we get that the gadgets of v_i alternate in $[A,B]$ and there is nothing to prove). Observe that this means that either both $\mathcal{H}_i^j$ and $\mathcal{H}_i^{j+1}$ are A-partitioned, or both are B-partitioned. Suppose the former, without loss of generality, and note that this means that $\mathcal{RL}(\mathcal{H}_i^j) \subseteq A$, while $\mathcal{LL}(\mathcal{H}_i^{j+1}) \subseteq B$. Also, let $j' > j$ be the minimum value for which the left long intervals of $\mathcal{H}_i^{j'+1}$ are on the opposite side of the right long intervals of $\mathcal{H}_i^{j'}$; if it does not exist, let $j' = m+1$. We switch sides of the following intervals: $\mathcal{H}_i^h$, for every $h \in \{j+1, \cdots, j'\}$; L_i^{2j-1}, L_i^{2j} if they are also in A; L_i^{2h-1}, L_i^{2h} for each $h \in \{j+1, \cdots, j'-1\}$; and $L_i^{2j'-1}, L_i^{2j'}$ if $j' < m+1$ and they are on the same side as $\mathcal{LL}(\mathcal{H}_i^{j'+1})$. Also switch the intervals of type C and intervals in edge gadgets appropriately in order to maintain the desired properties. We prove that we gain at least $2q - 4n$ edges, while losing at most $4nm + 6q' = 6(n^2 + q')$ (recall that $m = \frac{3n}{2}$). As previously said, this is not the exact count but gives an idea as how to choose the values for p, q, p', q'. Indeed, it suffices to choose values in a way as to ensure that the number of gained edges is bigger than the number of lost edges.

Observe that if we did not need to switch L_i^{2j-1}, L^{2j}, then, concerning these intervals, we gain at least $2q$ edges and lose none; otherwise, we gain $2q$ edges but lose at most $\ell_B^j \leq 4n$; thus we gain at least $2q - 4n$. As for the intervals L_i^{2h-1}, L_i^{2h} for $h \in \{j+1, \cdots, j'-1\}$, by the definition of j' we know that we lose at most $\max\{\ell_A^h, \ell_B^j\} \leq 4n$, while maintaining the same number between them and the vertex gadgets. And if $j' < m+1$, then we either gain $2q$ more edges if we did not need to change the side of $L_i^{2j'-1}, L_i^{2j'}$, or we gain $2q$ more edges while losing at most $\max\{\ell_A^{j'}, \ell_B^{j'}\} \leq 4n$. Hence, concerning the link intervals in $\mathcal{M}_i$, in total we lose at most $4nm = 6n^2$. As for the $6q'$ value, it suffices to see that, in the worst case scenario, $\{j_1, j_2, j_3\} \subseteq \{j+1, \cdots, j'\}$ and all the values c_h were previously equal to 1, and are now equal to 0 (observe again Inequality 3). ◀

Now, if $[A,B]$ is an alternating partitioned cut of $\mathbb{G}_{\mathcal{M}(\mathfrak{G})}$, we let $\Phi(A,B) = [X,Y]$ be the cut of G such that, for each vertex $v_i \in V(G)$, we have $v_i \in X$ if and only if $\mathcal{H}_i^1$ is A-partitioned by $[A,B]$. Note that $[X,Y]$ is well-defined and uniquely determined by $[A,B]$. On the other hand, given a cut $[X,Y]$ of G, there is a unique alternating partitioned cut $[A,B] = \Phi^{-1}(X,Y)$ of $\mathbb{G}_{\mathcal{M}(\mathfrak{G})}$ such that $[X,Y] = \Phi(A,B)$. Therefore, it remains to relate the sizes of these cut-sets. Basically we can use the good behaviour of the cuts in $\mathbb{G}_{\mathcal{M}(\mathfrak{G})}$ to prove that the size of $[A,B]$ grows as a well-defined function on the size of $\Phi(A,B)$. More formally, we can prove that the function f previously referred to is given by (recall that k is part of the input on the original problem):

$$f(G,k) = \left(\frac{3n^2}{2}+n\right)(2pq+q^2) + \frac{3n}{2}(2p'q'+(q')^2) + 6nq(n+1) \\ + \left(3n^2+3n\right)(n-1)(p+q) + 3n^2(p'+q') + 3n((k+1)q'+p') + 4k. \quad (4)$$

The above value is obtained using counting arguments much similar to the ones given in the proof of Lemma 4. The only downside is that we are not able to give an exact value for $|E_{\mathbb{G}_{\mathcal{M}(\mathfrak{G})}}(A,B)|$ as a function of $|E_G(X,Y)|$ and n,p,q,p',q'. For instance, note that the number of edges between link intervals within a region depend on the size of A and B, instead of the size of $E_G(X,Y)$. Nevertheless, we know that the range of values that $|E_{\mathbb{G}_{\mathcal{M}(\mathfrak{G})}}(A,B)|$ can assume, given that $|E_G(X,Y)| = k$, is distinct for each value of k, as the following lemma states.

▶ **Lemma 5.** *Let G be a cubic graph on n vertices, $\pi_V = (v_1,\ldots,v_n)$ be an ordering of $V(G)$, $\pi_E = (e_1,\ldots,e_{\frac{3n}{2}})$ be an ordering of $E(G)$, $\mathfrak{G} = (G,\pi_V,\pi_E)$, $[A,B]$ be an alternating partitioned cut of $\mathbb{G}_{\mathcal{M}(\mathfrak{G})}$ and $[X,Y] = \Phi(A,B)$. If $k = |E_G(X,Y)|$, then $f(G,k) \le |E_{\mathbb{G}_{\mathcal{M}(\mathfrak{G})}}(A,B)| < f(G,k')$ for any integer $k' > k$.*

Sketch. Since $[A,B]$ is an alternating partitioned cut of $\mathbb{G}_{\mathcal{M}(\mathfrak{G})}$, we shall count the edges in the cut-set $E_{\mathbb{G}_{\mathcal{M}(\mathfrak{G})}}(A,B)$ according to the following three types of intervals incident to these edges: the edges in the cut-set that have an endpoint in a (p,q)-grained gadget; the edges in the cut-set that have an endpoint in a (p',q')-grained gadget; and the edges in the cut-set that have both endpoints in a link interval and/or an interval of the type C_j^ℓ.

First, we count the edges in the cut-set that have an endpoint in a (p,q)-grained gadget. The possible combinations are as follows.

(1.1) Edges within (p,q)-grained gadgets related to vertices. There are exactly $(\frac{3n^2}{2}+n)(2pq+q^2)$ such edges.
(1.2) Edges between link intervals L_i^{2j-1} and L_i^{2j}, and the (p,q)-gadgets related to vertices. There are exactly $m\cdot n\cdot(2q+2q) = 6n^2q$ such edges.
(1.3) Edges between intervals $C_j^1,\ldots,C_j^4$ and the (p,q)-grained related to the vertices incident to edge e_j. There are exactly $\frac{3n}{2}(2q+2q) = 6nq$ such edges.
(1.4) Edges between (p,q)-grained gadgets related to vertices, and link intervals covering them. There are exactly $mn(n-1)(2p+2q) = 3n^2(n-1)(p+q)$ such edges.
(1.5) Edges between intervals $C_j^1,\ldots,C_j^4$ and (p,q)-grained gadgets covered by them. There are exactly $\sum_{i\in[n]} 6(n-i)(p+q) = 3n(n-1)(p+q)$ such edges.

Second, we count the edges in the cut-set that have an endpoint in a (p',q')-grained gadget. The possible combinations are as follows.

(2.1) Edges within (p',q')-grained gadgets related to edges. There are exactly $\frac{3n}{2}(2p'q'+(q')^2)$ such edges.
(2.2) Edges between (p',q')-grained gadgets related to edges and the link intervals covering them. There are exactly $3n^2(p'+q')$ such edges.
(2.3) Edges between (p',q')-grained gadget $\mathcal{E}_j$ and intervals $C_j^1,\ldots,C_j^4$. There are exactly $\frac{3n}{2}(2kq'+2(p'+q')) = 3n((k+1)q'+p')$ such edges (recall that $k = |E_G(X,Y)|$).

Third, we count the edges in the cut-set that have both endpoints in a link interval and/or an interval of the type C_j^ℓ for some $\ell \in \{1,\ldots,4\}$ and $j \in [m]$.

(3.1) Edges between intervals C_j^1, C_j^2 and C_j^3, C_j^4. There are exactly $4k$ such edges.
(3.2) Edges between pairs of intervals $L_1^{2j-1}, L_1^{2j}, \ldots, L_n^{2j-1}, L_n^{2j}$. There are *at most* $\sum_{j\in[m]} n^2 = mn^2 = \frac{3n^3}{2}$ such edges.
(3.3) Edges between intervals $L_1^{2j-1}, L_1^{2j}, \ldots, L_n^{2j-1}, L_n^{2j}$ and intervals in $C_j^1, \ldots, C_j^4$. There are *at most* $\sum_{j\in[m]} 8(n-1) = 8m(n-1) = 12n(n-1) = 12n^2 - 12n$ such edges.
(3.4) Edges between link intervals in consecutive regions of the escalator. There are *at most*

$$\sum_{j\in\{2,\ldots,m\}} \sum_{i\in[n]} 4(n-i) = \sum_{j\in\{2,\ldots,m\}} 2n(n-1) = 2(m-1)n(n-1)$$
$$= 3n^2(n-1) - 2n(n-1) = 3n^3 - 5n^2 + 2n$$

such edges.
(3.5) Finally, edges between intervals $C_j^1, \ldots, C_j^4$ and link intervals in the previous regions of the escalator. There are *at most* $\sum_{i\in[n]} 12(n-i) = 6n^2 - 6n$ such edges.

Therefore, summing up the number of edges in the cut-set $E_{\mathbb{G}_{\mathcal{M}(\mathfrak{G})}}(A, B)$ according to three types described above, except for the edges described in Cases (3.2)–(3.5) which, as we have seen, do not give exact values, we obtain that

$$\begin{aligned}|E_{\mathbb{G}_{\mathcal{M}(\mathfrak{G})}}(A, B)| \geq{} & \left(\tfrac{3n^2}{2} + n\right)(2pq + q^2) + \tfrac{3n}{2}(2p'q' + (q')^2) + 6nq(n+1) \\ & + (3n^2 + 3n)(n-1)(p+q) + 3n^2(p' + q') \\ & + 3n((k+1)q' + p') + 4k \\ ={} & f(G, k).\end{aligned}$$

On the other hand, note that the number of edges in Cases (3.2)–(3.5) is upper bounded by $\frac{9n^3}{2} + 13n^2 - 16n$. Thus, since $q' > \frac{9n^3}{2} + 13n^2 - 16n$, we have:

$$f(G, k) \leq |E_{\mathbb{G}_{\mathcal{M}(\mathfrak{G})}}(A, B)| \leq f(G, k) + \frac{9n^3}{2} + 13n^2 - 16n < f(G, k) + q'.$$

As a result, because there is a factor kq' in $f(G, k)$, we obtain that $f(G, k') > |E_{\mathbb{G}_{\mathcal{M}(\mathfrak{G})}}(A, B)|$ for any $k' > k$. ◀

The next lemma together with Lemma 7 stated next in Section 2.4 complete the proof of Theorem 1.

▶ **Lemma 6.** *Let G be a cubic graph on n vertices, $\pi_V = (v_1, \ldots, v_n)$ be an ordering of $V(G)$, $\pi_E = (e_1, \ldots, e_{\frac{3n}{2}})$ be an ordering of $E(G)$ and $\mathfrak{G} = (G, \pi_V, \pi_E)$. For each positive integer k, $\mathsf{mc}(G) \geq k$ if and only if $\mathsf{mc}(\mathbb{G}_{\mathcal{M}(\mathfrak{G})}) \geq f(G, k)$.*

Proof. First, suppose that $\mathsf{mc}(G) \geq k$. Then, there is a cut $[X, Y]$ of G such that $|E_G(X, Y)| \geq k$. Let $[A, B]$ be the unique alternating partitioned cut of $\mathbb{G}_{\mathcal{M}(\mathfrak{G})}$ that, for each $i \in [n]$, satisfies the following condition: if $v_i \in X$, then $\mathcal{H}_i^1$ is A-partitioned; otherwise, $\mathcal{H}_i^1$ is B-partitioned. One can verify that $[A, B] = \Phi^{-1}(X, Y)$. Therefore, it follows from Lemma 5 that $\mathsf{mc}(\mathbb{G}_{\mathcal{M}(\mathfrak{G})}) \geq |E_{\mathbb{G}_{\mathcal{M}(\mathfrak{G})}}(A, B)| \geq f(G, k)$. Conversely, suppose that $\mathsf{mc}(\mathbb{G}_{\mathcal{M}(\mathfrak{G})}) \geq f(G, k)$. Then, there exists a cut $[A, B]$ of $\mathbb{G}_{\mathcal{M}(\mathfrak{G})}$ such that $|E_{\mathbb{G}_{\mathcal{M}(\mathfrak{G})}}(A, B)| \geq f(G, k)$. Assume that $[A, B]$ is a maximum cut of $\mathbb{G}_{\mathcal{M}(\mathfrak{G})}$. It follows from Lemma 4 that $[A, B]$ is an alternating partitioned cut. Consequently, by Lemma 5, $[X, Y] = \Phi(A, B)$ is a cut of G such that $|E_G(X, Y)| \geq k$. Indeed, if $|E_G(X, Y)| < k$, then $|E_{\mathbb{G}_{\mathcal{M}(\mathfrak{G})}}(A, B)| < f(G, k)$. Therefore, $\mathsf{mc}(G) \geq k$. ◀

2.4 Bounding the interval count

Consider a cubic graph G on n vertices and $m = \frac{3n}{2}$ edges, and orderings π_V, π_E of the vertex set and edge set of G. Denote the triple (G, π_V, π_E) by $\mathfrak{G}$. We want to prove that the interval count of our constructed interval model $\mathcal{M}(\mathfrak{G})$ is at most 4. But observe that the construction of $\mathcal{M}(\mathfrak{G})$ is actually not unique, since the intervals are not uniquely defined; e.g., given such a model, one can obtain a model satisfying the same properties simply by adding $\epsilon > 0$ to all points defining the intervals. In this section, we provide a construction of a uniquely defined interval model related to $\mathfrak{G}$ that satisfies the desired conditions and has interval count 4.

Consider our constructed interval model $\mathcal{M}(\mathfrak{G})$, and denote, for each $j \in [m]$, $\mathcal{S}_j = \mathcal{E}_j \cup \bigcup_{\ell \in [4]} C_j^\ell \cup \bigcup_{i \in [n]} (\mathcal{H}_i^j \cup \{L_i^{2j} \cup L_i^{2j-1}\})$. We show how to accommodate $\mathcal{S}_1$ within $[0, 6n-2]$ in such a way that the same pattern can be adopted in the subsequent regions of $\mathcal{M}(\mathfrak{G})$ too, each time starting at multiples of $4n$. More specifically, letting $t = 4n$, we will accommodate $\mathcal{S}_j$ within $[t \cdot (j-1), 6n - 2 + t \cdot (j-1)]$. Assume $e_1 = v_h v_{h'}$, with $h < h'$. Below, we say exactly which closed interval of the line corresponds to each interval $I \in \mathcal{S}_1$.

- For each $i \in [n]$, the left long intervals of $\mathcal{H}_i^1$ are equal to $[2i-2, 2i-\frac{3}{2}]$ and the left short intervals are any choice of q distinct points within the open interval $(2i-2, 2i-\frac{3}{2})$, whereas the right long intervals of $\mathcal{H}_i^1$ are equal to $[2i-\frac{3}{2}, 2i-1]$ and the right short intervals are any choice of q distinct points within the open interval $(2i-\frac{3}{2}, 2i-1)$. Note that open intervals are used to locate the closed intervals of length zero, but that the short intervals themselves are not open.
- C_1^1 and C_1^2 are equal to $[2h-1, 2h+2n-2]$.
- C_1^3 and C_1^4 are equal to $[2h'-1, 2h'+2n-2]$.
- The left long intervals of $\mathcal{E}_1$ are equal to $[2n, 4n-1]$.
- The left short intervals of $\mathcal{E}_1$ are any choice of q' distinct points in the open interval $(2h+2n-2, 2h'+2n-2)$. Again, the open interval is used just to locate the closed intervals of length zero.
- The right long intervals of $\mathcal{E}_1$ are equal to $[4n-1, 4n-\frac{1}{2}]$ and the right short intervals are any choice of q' distinct points within the corresponding open interval.
- For each $i \in [n]$, intervals L_i^1, L_i^2 are equal to $[2i-1, 4n+2(i-1)]$.

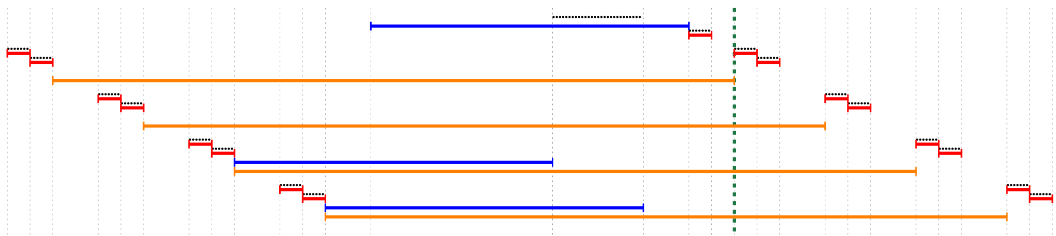


Figure 5 The closed intervals in $\mathcal{S}_1 \cup \bigcup_{i=1}^{4} \mathcal{H}_i^2$ of a graph on 4 vertices. We consider e_1 to be equal to v_3v_4. Each colour represents a different interval size. The short intervals are represented by the dots located inside the open interval. Vertical lines mark the endpoints of the intervals in $\mathcal{S}_1 \setminus \mathcal{L}$, while the green vertical line marks the beginning of the intervals in $\mathcal{S}_2$.

The suitable chosen lengths of the above defined closed intervals are (see Figure 5, where we denote by $\mathcal{L}$ the set of link intervals):

1. 0: short intervals of all grained gadgets (dots in Figure 5);
2. 1/2: left long and right long intervals of each $\mathcal{H}_i^1$, and right long intervals of $\mathcal{E}_1$ (red intervals in Figure 5);
3. $2n-1$: intervals $C_1^1, \ldots, C_1^4$, and left long intervals of $\mathcal{E}_1$ (blue intervals in Figure 5);
4. $4n-1$: intervals L_i^1 and L_i^2, for every $i \in [n]$ (orange intervals in Figure 5).

Now, let $\mathcal{M}'(\mathfrak{G})$ be the interval model where each $\mathcal{S}_j$ is defined exactly as $\mathcal{S}_1$, except that we shift all the intervals to the right in a way that point 0 now coincides with point $t \cdot (j-1)$. More formally, an interval I in $\mathcal{S}_j$ corresponding to the copy of an interval $[\ell, r]$ in $\mathcal{S}_1$ is defined as $[\ell + t \cdot (j-1), r + t \cdot (j-1)]$. Also, we assign the intervals in the $(m+1)$-th grained gadgets to be at the end of this model, using the same sizes of intervals as above; i.e., $\mathcal{H}_i^{m+1}$ is within the interval $[2i-2+t \cdot m, 2i-1+t \cdot m]$.

We have shown above that $\mathcal{M}'(\mathfrak{G})$ has interval count 4. The following lemma shows that the above chosen intervals satisfy the properties imposed in Subsections 2.1 and 2.2 on our constructed interval model $\mathcal{M}(\mathfrak{G})$.

▶ **Lemma 7.** *Let G be a cubic graph. Then, there exists an interval model $\mathcal{M}(\mathfrak{G})$ with interval count 4 for $\mathfrak{G} = (G, \pi_V, \pi_E)$, for every ordering π_V and π_E of the vertex set and edge set of G, respectively.*

3 The interval count of Adhikary et al.'s construction

We provided in Section 2 a reduction from the MaxCut problem having as input a cubic graph G into that of MaxCut in an interval graph G' having $\mathsf{ic}(G') \leq 4$. Although our reduction requires the choice of orderings π_V and π_E of respectively $V(G)$ and $E(G)$ in order to produce the resulting interval model, we have established that we are able to construct an interval model with interval count 4 regardless of the particular choices for π_V and π_E (Lemma 7). Our reduction was based on that of [1], strengthened in order to control the interval count of the resulting model.

This section is dedicated to discuss the interval count of the original reduction [1]. Although the interval count was not of concern in [1], in order to contrast the reduction found there with the presented in this work, we investigate how interval count varies in the original reduction considering different vertex/edge orderings. First, we establish that the original reduction yields an interval model corresponding to a graph G' such that $\mathsf{ic}(G') = O(\sqrt[4]{|V(G')|})$. Second, we exhibit an example of a cubic graph G for which a choice of π_V and π_E yields a model $\mathcal{M}'$ with interval count $\Omega(\sqrt[4]{|V(G')|})$, proving that this bound is tight for some choices of π_V and π_E. For bridgeless cubic graphs, we are able in Lemma 8 to decrease the upper bound by a constant factor, but to the best of our knowledge $O(\sqrt[4]{|V(G')|})$ is the tightest upper bound. Before we go further analysing the interval count of the original reduction, it is worthy to note that a tight bound on the interval count of a general interval graph G as a function of its number of vertices n is still open. It is known that $\mathsf{ic}(G) \leq \lfloor (n+1)/2 \rfloor$ and that there is a family of graphs G for which $\mathsf{ic}(G) = (n-1)/3$ [7,14]. That is, the interval count of a graph can achieve $\Theta(n)$.

In the original reduction, given a cubic graph G, an interval graph G' is defined through the construction of one of its models $\mathcal{M}$, described as follows:

1. let $\pi_V = (v_1, v_2, \ldots, v_n)$ and $\pi_E = (e_1, e_2, \ldots, e_m)$ be arbitrary orderings of $V(G)$ and $E(G)$, respectively;
2. for each $v_i \in V(G)$, $e_j \in E(G)$, let $\mathcal{G}(v_i)$ and $\mathcal{G}(e_j)$ denote respectively a (p,q)-grained gadget and a (p',q')-grained gadget, where:
   - $q = 200n^3 + 1$, $p = 2q + 7n$, and
   - $q' = 10n^2 + 1$, $p' = 2q' + 7n$;

3. for each $v_k \in V(G)$, insert $\mathcal{G}(v_k)$ in $\mathcal{M}$ such that $\mathcal{G}(v_i)$ is entirely to the left of $\mathcal{G}(v_j)$ if and only if $i < j$. For each $e_k \in E(G)$, insert $\mathcal{G}(e_k)$ in $\mathcal{M}$ entirely to the right of $\mathcal{G}(v_n)$ and such that $\mathcal{G}(e_i)$ is entirely to the left of $\mathcal{G}(e_j)$ if and only if $i < j$;
4. for each $e_j = (v_i, v_{i'}) \in E(G)$, with $i < i'$, four intervals $I^1_{i,j}, I^2_{i,j}, I^1_{i',j}, I^2_{i',j}$ are defined in $\mathcal{M}$, called *link* intervals, such that:
   - $I^1_{i,j}$ and $I^2_{i,j}$ (resp. $I^1_{i',j}$ and $I^2_{i',j}$) are true twin intervals that weakly intersect $\mathcal{G}(v_i)$ (resp. $\mathcal{G}(v_{i'})$) to the right;
   - $I^1_{i,j}$ and $I^2_{i,j}$ (resp. $I^1_{i',j}$ and $I^2_{i',j}$) weakly intersect (resp. strongly intersect) $\mathcal{G}(e_j)$ to the left.

   By construction, therefore, $I^1_{i,j}$ and $I^2_{i,j}$ (resp. $I^1_{i',j}$ and $I^2_{i',j}$) cover all intervals in grained gadgets associated to a vertex v_ℓ with $\ell > i$ (resp. $\ell > i'$) or an edge e_ℓ with $\ell < j$.

Note that the number of intervals in $\mathcal{M}$ is invariant under the particular choices of π_V and π_E and, therefore, so is the number of vertices of G'. Let $n' = |V(G')|$. Since G is cubic, $m = \frac{3n}{2}$. By construction,

$$n' = n(2p + 2q) + m(2p' + 2q') + 4m = 1200n^4 + 90n^3 + 25n^2 + 21n$$

and thus $n = \Theta(\sqrt[4]{n'})$. Since the set of intervals covered by any link interval depends on π_V and π_E, distinct sequences yield distinct resulting graphs G' having distinct interval counts.

We show next that $\mathsf{ic}(G') = O(\sqrt[4]{n'})$. Note that

- the intervals of all gadgets $\mathcal{G}(v_i)$ and $\mathcal{G}(e_j)$ can use only two interval lengths (one for all short intervals, another for all the long intervals);
- for each $e_j = v_i v_{i'} \in E(G)$, with $i < i'$, both intervals $I^1_{i,j}$ and $I^2_{i,j}$ may be coincident in any model, and therefore may have the same length. The same holds for both intervals $I^1_{i',j}$ and $I^2_{i',j}$.

Therefore, $\mathsf{ic}(G') \leq 2m + 2 = 3n + 2 = \Theta(\sqrt[4]{n'})$. Therefore, the NP-completeness result derived from the original reduction in [1] can be strengthened to state that MaxCut is NP-complete for interval graphs G having interval count $O(\sqrt[4]{|V(G)|})$.

Second, we show that there is a resulting model $\mathcal{M}'$ produced in the reduction, defined in terms of particular orderings π_V, π_E for which $\mathsf{ic}(\mathcal{M}') = \Omega(\sqrt[4]{n'})$. Consider the cubic graph G depicted in Figure 6(a) which consists of an even cycle $(v_1, v_2, \ldots, v_n)$ with the addition of the edges $(v_i, v_{i+\frac{n}{2}})$ for all $1 \leq i \leq n/2$. For the ordering $\pi_V = (v_n, v_{n-1}, \ldots, v_1)$ and any ordering π_E in which the first n edges are the edges of the cycle $(v_1, v_2, \ldots, v_n)$, in this order, the reduction yields a model $\mathcal{M}'$ for which there is a chain $I^1_{1,1} \subset I^1_{2,2} \subset \ldots \subset I^1_{n,n}$ of nested intervals (see Figure 6(b)), which shows that $\mathsf{ic}(\mathcal{M}') \geq n$, and thus $\mathsf{ic}(\mathcal{M}') = \Omega(\sqrt[4]{n'})$.

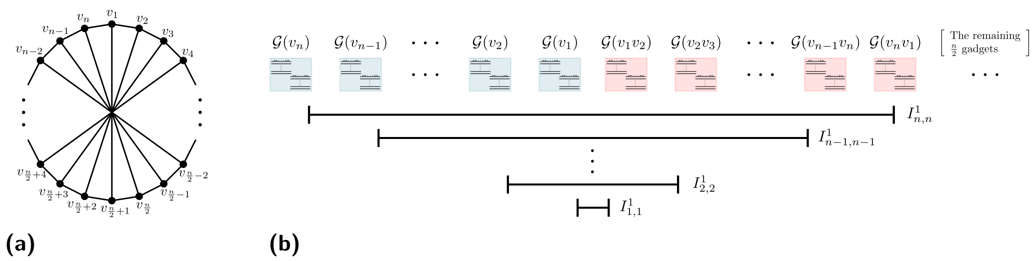


Figure 6 (a) A cubic graph G, and (b) a chain of nested intervals in the model $\mathcal{M}'$.

It can be argued from the proof of NP-completeness for MaxCut when restricted to cubic graphs [2] that the constructed cubic graph may be assumed to have no bridges. This fact was not used in the original reduction of [1]. In an attempt to obtain a model $\mathcal{M}$ having fewer lengths for bridgeless cubic graphs, we have derived Lemma 8. Although the number of lengths in this new upper bound has decreased by the constant factor of 4/9, it is still $\Theta(n) = \Theta(\sqrt[4]{n'})$.

▶ **Lemma 8.** *Let G be a cubic bridgeless graph with $n = |V(G)|$. There exist particular orderings π_V of $V(G)$ and π_E of $E(G)$ such that:*

1. *there is a resulting model $\mathcal{M}$ produced in the original reduction of* MaxCut *such that* $\mathsf{ic}(\mathcal{M}) \leq \frac{4n}{3} + 3$.
2. *for all such resulting models $\mathcal{M}$, we have that* $\mathsf{ic}(\mathcal{M}) \geq 5$ *if G is not a Hamiltonian graph.*

As a concluding remark, we note that the interval count of the interval model $\mathcal{M}$ produced in the original reduction is highly dependent on the assumed orderings of $V(G)$ and $E(G)$, and may achieve $\mathsf{ic}(\mathcal{M}) = \Omega(\sqrt[4]{n'})$. The model $\mathcal{M}'$ produced in our reduction enforces that $\mathsf{ic}(\mathcal{M}') = 4$ which is invariant for any such orderings. On the perspective of the problem of interval count 2 and beyond, for which very little is known, our NP-completeness result on a class of bounded interval count graphs is also of interest.

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