Discrete Applied Mathematics 281 (2020) 162–171

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Complexity-separating graph classes for vertex, edge and total colouring☆

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ARTICLE INFO

Article history:
Received 24 April 2018
Received in revised form 30 January 2019
Accepted 22 February 2019
Available online 19 March 2019

Keywords:
Analysis of algorithms and problem complexity
Graph algorithms
Structural characterization of types of graphs

ABSTRACT

Given a class $\mathcal{A}$ of graphs and a decision problem π belonging to NP, we say that a full complexity dichotomy of $\mathcal{A}$ was obtained if one describes a partition of $\mathcal{A}$ into subclasses such that π is classified as polynomial or NP-complete when restricted to each subclass. The concept of full complexity dichotomy is particularly interesting for the investigation of NP-complete problems: as we partition a class $\mathcal{A}$ into NP-complete subclasses and polynomial subclasses, it becomes clearer why the problem is NP-complete in $\mathcal{A}$. The class $\mathcal{C}$ of graphs that do not contain a cycle with a unique chord was studied by Trotignon and Vušković who proved a structure theorem which led to solving the vertex-colouring problem in polynomial time. In the present survey, we apply the structure theorem to study the complexity of edge-colouring and total-colouring, and show that even for graph classes with strong structure and powerful decompositions, the edge-colouring problem may be difficult. We discuss several surprising complexity dichotomies found in subclasses of $\mathcal{C}$, and the concepts of separating problem proposed by David S. Johnson and the dual concept of separating class.

1. NP-completeness ongoing guide

In the sixteenth edition of his NP-Completeness Column: An Ongoing Guide [30], David S. Johnson focused on graph restrictions and their effect, with emphasis on the restrictions to graph classes and how they affect the complexity of various NP-hard problems. Graph classes were selected because of their broad algorithmic significance. The presentation consisted of a summary table with 30 rows containing the selected classes of graphs, and 11 columns the first devoted to the complexity of determining whether a given graph is in the specified class followed by ten of the more famous NP-complete graph problems. The entry for a class and a problem was the complexity of the problem restricted to that class of graphs — polynomial-time solvable or NP-complete, if known. The goal was to identify interesting problems and interesting graph classes establishing the concept of complexity separation. According to [30], a *complexity-separating problem* π separates two graph classes $\mathcal{A} \subset \mathcal{B}$ if π is polynomial-time solvable when restricted to $\mathcal{A}$-graph inputs but π is NP-complete when restricted to $\mathcal{B}$-graph inputs. A dual concept was proposed in [41], a *complexity-separating graph class* $\mathcal{C}$ separates two problems π and σ if π is NP-complete when restricted to $\mathcal{C}$-graph inputs but σ is polynomial-time solvable when restricted to $\mathcal{C}$-graph inputs.

The chosen ten famous graph problems were: INDEPENDENT SET, CLIQUE, PARTITION INTO CLIQUES, CHROMATIC NUMBER, CHROMATIC INDEX, HAMILTONIAN CIRCUIT, DOMINATING SET, SIMPLE (unweighted) MAX CUT, (unweighted)

☆ This paper is based on an invited talk given at LAGOS 2017, the Latin-American Algorithms, Graphs and Optimization Symposium. This work is partially supported by Brazilian agencies CNPq and FAPERJ.

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https://doi.org/10.1016/j.dam.2019.02.039

STEINER TREE IN GRAPHS, and GRAPH ISOMORPHISM. The first nine problems were at the time known to be NP-complete for general graphs; the complexity of GRAPH ISOMORPHISM for general graphs is still a long-standing open problem, one of twelve open problems highlighted at the end of the NP-completeness guide by Garey and Johnson [21]. The well-known VERTEX COVER problem was not included as its complexity will always be the same as that of INDEPENDENT SET, unless we are considering parameterized complexity.

The chosen 30 significant graph classes were organized into four groups corresponding to the four subsections of the paper: Trees and Near-trees, Planarity and its Relations, A Catalogue of Perfect graphs, and Intersection Graphs.

The table revealed among its 330 entries the existence of a substantial collection of 71 open problems classified from *entertaining puzzles* as *P*? or *O*? to *may well be hard* or *are famous* as *O* or *O*! problems. It is remarkable that only one entry in the entire table deserved a famous open problem *O*! entry, the recognition for perfect graphs, and just two entries deserved a may well be hard problem *O* entry, edge colouring of planar graphs and hamilton circuit of permutation graphs. The two problems with most open entries were edge colouring and maxcut, and there are few updates in the column for maxcut, for instance we know today that maxcut is polynomial for cographs [5], but NP-complete for split graphs [5] and for strongly chordal graphs [60]. At that time, the edge colouring problem had 19 of its 30 entries classified *O*?, which meant *apparently open but possibly easy to resolve*, and 14 of those *O*? entries remain open after more than thirty years.

The choice of the ten famous graph problems and of the 30 significant graph classes reflected the importance of the famous open problem, the recognition for perfect graphs, for which the special *O*! entry was given. A graph is *perfect* if for every induced subgraph the chromatic number equals the maximum clique size. At the first edition of his NP-Completeness Column: An Ongoing Guide [29], David S. Johnson discussed the progress that had been made on the twelve open problems presented at the end of the NP-completeness guide [21]. Six of those open problems had been resolved, and the split was even: three had been shown to be solvable in polynomial time and three had been proved NP-complete. It is remarkable that today ten of those twelve open problems are resolved, and that the split is still even. David S. Johnson concluded the first edition by presenting as problem of the month, the recognition for perfect graphs, and explained that just recently in 1981 the class of imperfect graphs was shown to be in NP, equivalently the class of perfect graphs was shown to be in coNP. A more recent proof was obtained in 2006, since membership in NP follows from the strong perfect graph theorem that characterizes perfect graphs as such that neither the graph nor its complement contains an odd hole, where a hole is a chordless cycle of length at least 4 [16].

Comparability graphs, also known as transitively orientable graphs, are undirected graphs that can, by an appropriate assignment to directions to the edges, be turned into transitive directed graphs. Chordal graphs, also known as triangulated graphs, are graphs in which every cycle of length at least 4 has a chord. Comparability graphs are a superset of bipartite graphs and are incomparable to chordal graphs. Permutation graphs are comparability graphs whose complements are also comparability graphs. In 1985, HAMILTONIAN CIRCUIT was known to be polynomial-time solvable when restricted to graphs that are both permutation graphs and bipartite, but remained open for permutation graphs in general.

The first four chosen problems of the table were the four problems solvable in polynomial time for perfect graphs by algorithms based on the ellipsoid method. The sections A Catalogue of Perfect Graphs and Intersection Graphs explained how simpler and purely combinatorial methods work for each of the chosen subclasses of perfect graphs, some of them defined as intersection graphs. The seminal book of Golumbic [22] appeared at that time and the book has as its chapters subclasses of perfect graphs and of intersection graphs.

The polynomial-time algorithm that obtains an optimal vertex colouring of a perfect graph is not purely combinatorial and is usually considered impractical. Recently, a purely graph-theoretical algorithm that produces an optimal vertex colouring for every square-free perfect graph in polynomial time has been described [15], and theorists are drawing closer to perfect colouring [68]. The focus given in [30] on perfect graphs and subclasses of perfect graphs classified vertex colouring as polynomial, when restricted to most of the selected classes of graphs. It is natural to consider successful strategies towards the solution of the more studied vertex colouring problem when trying to better understand the challenging edge colouring problem, and the total colouring problem where we colour both vertices and edges with no conflicts.

In Section 2, we present several full dichotomies for edge and total colouring problems by considering first structure theorems and decomposition which have been successfully used for recognition and vertex colouring, and second the overfull conjecture which has been successfully used for edge colouring. In Section 3, we go beyond vertex, edge and total colourings, by considering clique and biclique colourings, two variants of the classical vertex colouring, and in order to determine the exact complexity, we need to define higher levels of the polynomial hierarchy. We conclude in Section 4 with the updated table, more than thirty years later, from 1985 to 2018.

2. Full dichotomies

Given a class $\mathcal{A}$ of graphs and a graph (decision) problem π belonging to NP, we say that a *full complexity dichotomy* of $\mathcal{A}$ is obtained if one describes a partition of $\mathcal{A}$ into subclasses $\mathcal{A}_1, \mathcal{A}_2, \ldots$ such that π is classified as polynomial or NP-complete when restricted to each subclass $\mathcal{A}_i$. The concept of *full complexity dichotomy* is particularly interesting for the investigation of NP-complete problems: as we partition a class $\mathcal{A}$ into NP-complete subclasses and polynomial subclasses, it becomes clearer why the problem π is NP-complete in $\mathcal{A}$. We describe next full complexity dichotomies for edge and total colouring problems. We begin by defining our target colouring problems: vertex, edge and total, with the corresponding parameters, which will be used to define the target graph classes and presented full complexity dichotomies.

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. A *vertex colouring* is a map $\lambda : V(G) \to \mathcal{L}$ with $\lambda(u) \neq \lambda(v)$ for any two adjacent vertices $u, v \in V(G)$. If $\mathcal{L} = \{1, \ldots, k\}$, then λ is a k-vertex colouring. The smallest integer k for which a *k-vertex colouring* exists is the *chromatic number* of G, denoted $\chi(G)$. Clearly, $\chi(G) \geq \omega(G)$, where $\omega(G)$ denotes the size of a maximum clique in G.

An *edge colouring* is a map $\lambda : E(G) \to \mathcal{L}$ with $\lambda(e) \neq \lambda(f)$ for any two adjacent edges $e, f \in E(G)$. If $\mathcal{L} = \{1, \ldots, k\}$, then λ is a k-edge colouring. The smallest integer k for which a *k-edge colouring* exists is the *chromatic index* of G, denoted $\chi'(G)$. Clearly, $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of a vertex in G. Vizing's theorem states that every simple graph G has an edge colouring with $\Delta(G) + 1$ colours. Therefore, either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. If a graph G has chromatic index $\Delta(G)$, then G is said to be *class 1*; otherwise, G is said to be *class 2*.

An *element* of G is one of its vertices or edges. A *total colouring* is a map $\lambda : E(G) \cup V(G) \to \mathcal{L}$ with $\lambda(x) \neq \lambda(y)$ for any two adjacent or incident elements $x, y \in E(G) \cup V(G)$. The smallest integer k for which a k-total colouring exists is the *total chromatic number* of G, denoted $\chi_T(G)$. Clearly, $\chi_T(G) \geq \Delta(G) + 1$. The *Total Colouring Conjecture (TCC)*, posed independently by Behzad in 1965 and by Vizing in 1964, states that every simple graph G has a total colouring with $\Delta(G) + 2$ colours. By the TCC, either $\chi_T(G) = \Delta(G) + 1$ or $\chi_T(G) = \Delta(G) + 2$. If a graph G has $\chi_T(G) = \Delta(G) + 1$, then G is said to be *type 1*; if G has $\chi_T(G) = \Delta(G) + 2$, then G is said to be *type 2*. The TCC has been verified in restricted cases, such as cubic graphs, and graphs with maximum degree $\Delta(G) \leq 5$, but the general problem has been open for more than fifty years, exposing how challenging the problem of total colouring is. The TCC has not been settled for regular graphs, for planar graphs, and not even for cographs.

The chromatic number problem is one of the 21 problems proved to be NP-complete in the seminal paper by Karp [31], and is problem [GT4] in the NP-completeness guide [21]. The chromatic index was one of the twelve open problems highlighted at the end of the NP-completeness guide [21], and reported to be NP-complete in the first edition of the NP-completeness ongoing guide [29]. It is NP-complete to determine whether the chromatic index of an arbitrary graph G is $\Delta(G)$ [28], and it is NP-complete to determine whether the total chromatic number of an arbitrary graph G is $\Delta(G)+1$ [55]. Remark that the original NP-completeness proof for the total colouring problem was a reduction from the edge colouring problem, suggesting that, for most graph classes, total colouring would be harder than edge colouring. The edge colouring problem remains NP-complete when restricted to k-regular graph inputs [28,36], and when restricted to k-regular comparability (hence perfect) inputs [8], for each fixed $k \geq 3$. The total colouring problem remains NP-complete when restricted to k-regular bipartite inputs [47], for each fixed $k \geq 3$. It is natural to investigate the complexity of total colouring restricted to classes for which the complexity of edge colouring is already established.

There are classes of graphs that satisfy the TCC, and yet it is NP-complete to determine whether the total chromatic number of a graph in the class is type 1, for instance the class of bipartite graphs, or the class of cubic graphs. On the other hand, there are classes of graphs for which the total colouring problem remains NP-complete when restricted to graphs in the class, and yet the TCC has not been settled for the class, for instance regular graphs, or unichord-free graphs [42].

In Section 2.1, we consider χ-bounded graph classes as a generalization of perfect graphs, and the class of unichord-free graphs where we are able to obtain full dichotomies with respect to edge and total colourings. In Section 2.2, we consider the overfull conjecture for edge colouring and Hilton's condition for total colouring to obtain full dichotomies with respect to subclasses of chordal graphs.

2.1. Structure theorems and decomposition

Trotignon and Vušković [62] studied the class $\mathcal{C}$ of graphs that do not contain a cycle with a unique chord, which are called unichord-free graphs. The main motivation to investigate this class was to find a structure theorem for it, a kind of result which is not very frequent in the literature. Basically, this structure result states that every graph in $\mathcal{C}$ can be built starting from a restricted set of basic graphs and applying a series of known "gluing" operations. Another interesting property of this class is that it belongs to the family of the χ-bounded graphs, introduced by Gyárfás [25] as a natural extension of perfect graphs. A family of graphs $\mathcal{G}$ is *χ-bounded* with *χ-binding function* f if, for every induced subgraph H of $G \in \mathcal{G}$, $\chi(H) \leq f(\omega(H))$, where $\chi(H)$ denotes the chromatic number of H and $\omega(H)$ denotes the size of a maximum clique in H. The research in this area is mainly devoted to understanding for what choices of forbidden induced subgraphs, the resulting family of graphs is χ-bounded. Note that perfect graphs are a χ-bounded family of graphs with χ-binding function $f(x) = x$, and perfect graphs are characterized by excluding odd holes and their complements [16]. Also, by Vizing's Theorem, the class of line graphs of simple graphs is a χ-bounded family with χ-binding function $f(x) = x + 1$ (this special upper bound is known as the *Vizing bound*) and line graphs are characterized by nine forbidden induced subgraphs [66]. The class $\mathcal{C}$ is also χ-bounded with the Vizing bound [62]. In [45], the complexity of determining the chromatic index of graphs in $\mathcal{C}$ is considered, together with the subclasses obtained from $\mathcal{C}$ by forbidding 4-holes and/or 6-holes. The goal was to investigate how structure results can be used to solve the edge-colouring problem.

Trotignon and Vušković [62] give a decomposition result for graphs in $\mathcal{C}$ in the following form: every graph in $\mathcal{C}$ either belongs to a basic class or has a cutset. Three cutsets are used: 1-cutset composed by just one vertex, proper 2-cutset composed by two non adjacent vertices, and proper 1-join composed by a set of edges. A graph with neither of the three cutsets is basic if it is a complete graph, a hole with at least five vertices, a strongly 2-bipartite graph, or an induced

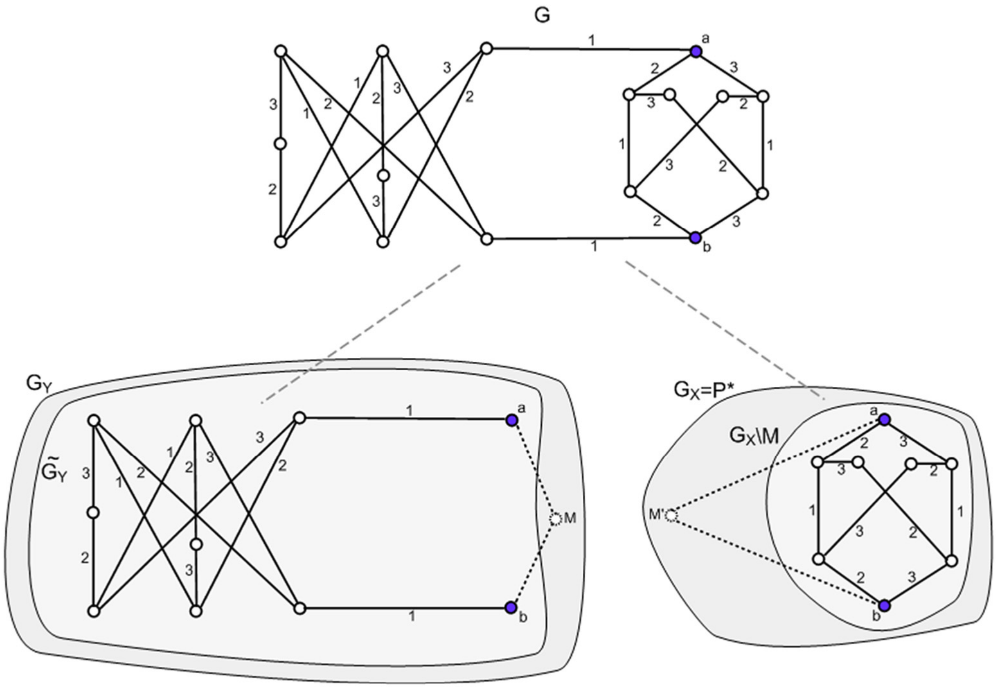

Fig. 1. Decomposition with respect to a proper 2-cutset $\{a, b\}$. G is class 1: Δ colours suffice, but $G_X = P^*$ is class 2: $\Delta + 1$ colours are needed.

subgraph of the Petersen graph or of the Heawood graph. In Fig. 1, we present a unichord-free graph G decomposed with respect to a proper 2-cutset into block graphs G_X and G_Y. The graph G_X is obtained from the Petersen graph by removal of one vertex and is called P^*, a basic graph for the class $\mathcal{C}$. Observe that the marker vertices and their incident edges – identified by dashed lines – do not belong to the original graph. It is not necessary that a block of decomposition of G is $\Delta(G)$-edge-colourable in order that G itself is $\Delta(G)$-edge-colourable.

In [45], edge-colouring is proved hard even for graphs of $\mathcal{C}$ that do not have a 4-hole, a subclass obtained by forbidding the 1-join operation. In [41], total-colouring is proved hard even for graphs of $\mathcal{C}$ that are bipartite. The structure theorem of [62] is applied to edge colouring and total colouring graphs of $\mathcal{C}$ that do not have a 4-hole to show [42,44,45]: every non-complete {4-hole, unichord}-free graph with maximum degree at least 4 is class 1, every non-complete {4-hole, unichord}-free graph with maximum degree at least 3 is type 1. Note that a natural subclass of unichord-free graphs is the class of chordless graphs, which are the graphs in which all cycles are chordless. Every chordless graph G with $\Delta(G) \geq 3$ is class 1 and type 1 [43]. See the obtained full dichotomy in Fig. 2, where for each graph class, we have depicted the complexities of edge colouring and total colouring problems. The outer circle refers to edge colouring and the inner circle refers to total colouring, while light grey means polynomial-time solvable and dark grey means NP-complete. The dichotomies are more than just polynomial versus NP-complete, but actually constant time versus NP-complete, these problems can be either trivial or very hard.

2.2. The overfull conjecture

A graph $G = (V, E)$ is *overfull* if $|E| > \Delta(G)\lfloor |V|/2 \rfloor$. An overfull graph G has more edges than any $\Delta(G)$-edge-colouring can colour, and so G is necessarily class 2. For some graph classes, being overfull is equivalent to being class 2, for instance the class of complete graphs, since the complete graphs with an odd number of vertices are overfull, whereas every complete graph with an even number of vertices cannot be overfull and admits a $\Delta(G)$-edge-colouring [66]. Note that a graph G with universal vertices and an even number of vertices is a spanning subgraph of a $\Delta(G)$-edge-colourable graph. Plantholt [52] proved the following characterization:

Theorem 1. *Let G be a simple graph with an odd number of vertices. If G has a universal vertex, then G is class 2 if and only if $|E| > \Delta(G)\lfloor |V|/2 \rfloor$.*

A graph G is *subgraph overfull* if it has a subgraph H with $\Delta(G) = \Delta(H)$ such that H is overfull. Subgraph-overfull graphs are class 2, and it can be verified in polynomial time whether a graph is subgraph overfull [49]. For some graph classes, being subgraph overfull is equivalent to being class 2. Examples of such classes are graphs with a universal vertex [52], complete multipartite graphs [27], and split graphs with odd maximum degree [11]. The *overfull conjecture* by Chetwynd

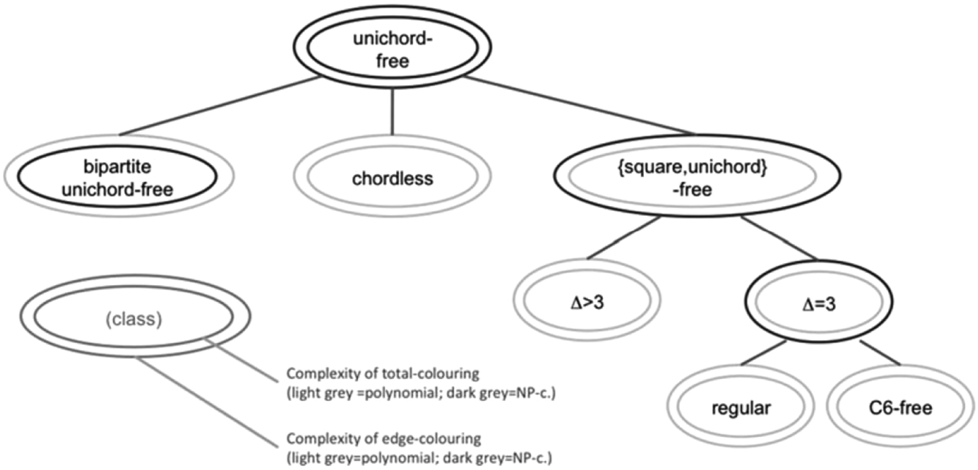

Fig. 2. Complexity of edge colouring and total colouring in subclasses of unichord-free graphs.

and Hilton [12] states that a graph $G = (V, E)$ with $\Delta(G) > |V|/3$ is class 2 if and only if it is subgraph overfull. In fact, for most graph classes for which the edge-colouring problem can be solved in polynomial time, the equivalence "class 2 $=$ subgraph overfull" holds. It is remarkable that the majority of these classes are composed of graphs whose maximum degree is large — always larger than one third of the number of vertices. So, for these graph classes, the equivalence "class 2 $=$ subgraph overfull" – and the consequent polynomial-time algorithm for the edge-colouring problem – would be a direct consequence of the overfull conjecture, in case of its validity.

We say that a general graph satisfies *Plantholt's condition* if the subgraph induced by the closed neighbourhood of a maximum degree vertex is class 2. It is a natural step to investigate the complexity of total colouring restricted to classes for which the complexity of edge colouring is already established. The total chromatic number has been determined for the class of complete graphs: complete graphs with an odd number of vertices are type 1, whereas complete graphs with an even number of vertices are type 2 [69]. Note that a graph G with universal vertices and an odd number of vertices is a spanning subgraph of a $\Delta(G) + 1$-total-colourable graph. An analogue of Plantholt's theorem for edge colouring graphs with a universal vertex [52] was presented by Hilton for total colouring [26]:

Theorem 2. *Let G be a simple graph with an even number of vertices. If G has a universal vertex, then G is type 2 if and only if $|E(\overline{G})| + \alpha'(\overline{G}) < |V|/2$ where $\alpha'(\overline{G})$ is the cardinality of a maximum independent set of edges of the complement graph $\overline{G}$.*

Note that graphs with universal vertices and an even number of vertices satisfy the TCC because they are spanning subgraphs of a type 2 graph. Theorem 2 in fact characterizes the cases when graphs with an even number of vertices and universal vertices are type 1 or type 2. Actually, Theorem 2 can be applied to a closed neighbourhood of a maximum degree vertex to determine cases for which a general graph G cannot be type 1. Therefore, we say that a general graph satisfies *Hilton's condition* if the subgraph induced by the closed neighbourhood of a maximum degree vertex is type 2.

Recall that a *clique* is a set of pairwise adjacent vertices and an *independent set* is a set of pairwise non-adjacent vertices in the graph. A graph is a *split* graph if its vertex set can be partitioned into a clique and an independent set. A *proper interval* or *indifference* graph is the intersection graph of a set of unit intervals of a straight line. An *indifference* order of a graph is a linear order on its vertex set such that the vertices of each maximal clique are consecutive with respect to the linear order. Roberts in 1971 gave the characterization: a graph is an indifference graph if and only if it admits an indifference order. Split graphs and indifference graphs are two classes of graphs for which the TCC has been settled, and split graphs and indifference graphs, with $\Delta(G)$ even, have $\chi_T(G) = \Delta(G) + 1$ [11,20]. The motivation to investigate the total chromatic number of split-indifference graphs is twofold. On the one hand, it is the intersection of two graph classes for which the total colouring problem is still open. On the other hand, it is a graph class for which the edge colouring problem was solved. Ortiz et al. [50] presented a characterization of split-indifference graphs and applied it to show that all class 2 split-indifference graphs satisfy Plantholt's condition. Campos et al. [9] applied the same characterization to show that all type 2 split-indifference graphs satisfy Hilton's condition.

A connected simple graph G is a *k-clique* graph if, and only if, G has exactly k distinct (not necessarily disjoint) maximal cliques. If G is a 3-clique graph and has no universal vertex, then G is an indifference graph [19]. Hence, 3-clique graphs do satisfy the TCC. It remains to determine which 3-clique graphs G without universal vertices and with odd maximum degree are type 1 and which are type 2.

For all graph classes listed in Table 1, every odd maximum degree graph is class 1 [11,19], and every even maximum degree graph is type 1 [11,20]. A general question, which we leave open, is to determine the largest graph class for which all its odd maximum degree graphs are class 1 and for which all its even maximum degree graphs are type 1. A related

Table 1
Classes considered with respect to Plantholt's condition on edge colouring and Hilton's condition on total colouring. Each entry considers the edge colouring followed by the total colouring of the graph class.

Graph class	Even Δ	Odd Δ
Complete	Class 2 (overfull), type 1	Class 1, type 2 (Hilton's condition)
Univ. vertex	Plantholt's condition [52], type 1	Class 1, Hilton's condition [26]
Split	Open, type 1 [11]	Class 1 [11], open
Indifference	Open, type 1 [20]	Class 1 [19], open
Split-indifference	Plantholt's condition [50], type 1 [20]	Class 1 [50], Hilton's condition [9]
3-clique	Plantholt's condition [19], type 1 [20]	Class 1 [19], open

question is to determine the largest graph class for which all its class 2 graphs satisfy Plantholt's condition and all its type 2 graphs satisfy Hilton's condition. A necessary condition for such a class is that its odd maximum degree graphs are class 1 and its even maximum degree graphs are type 1. All graph classes listed in Table 1 satisfy the TCC, but the total chromatic number has not been determined for split graphs, for indifference graphs, and nor for 3-clique graphs.

3. Beyond vertex, edge, and total colourings

Unichord-free graphs proved to have a rich structure that can be used to obtain interesting results with respect to the study of the complexity of colouring problems. In the context of clique-colouring and biclique-colouring problems, a *clique* of a graph G is a maximal set of vertices that induces a complete subgraph of G with at least one edge. A *biclique* of G is a maximal set of vertices that induces a complete bipartite subgraph of G with at least one edge. A *clique-colouring* of G is a colouring of the vertices such that no clique is monochromatic. If the colouring uses at most k colours, then we say that it is a *k-clique-colouring*. A *biclique-colouring* of G is a colouring of the vertices such that no biclique is monochromatic. If the colouring uses at most k colours, then we say that it is a *k-biclique-colouring*. The *clique-chromatic number* of G, denoted by $\kappa(G)$, is the least k for which G has a k-clique-colouring. The *biclique-chromatic number* of G, denoted by $\kappa_B(G)$, is the least k for which G has a k-biclique-colouring. A *star* is a maximal set of vertices that induces a complete bipartite graph with a universal vertex and at least one edge. A *star-colouring* is a colouring of the vertices such that no star is monochromatic. If the colouring uses at most k colours, then we say that it is a *k-star-colouring*. The *star-chromatic number* of G, denoted by $\kappa_S(G)$, is the least k for which G has a k-star-colouring.

In [39], clique-colouring and biclique-colouring problems restricted to unichord-free graphs are investigated. It is shown that the clique-chromatic number of a unichord-free graph is at most 3, and that the 2-clique-colourable unichord-free graphs are precisely those that are perfect. Moreover, an $O(|V \parallel E|)$-time algorithm that returns an optimal clique-colouring of a unichord-free graph input $G = (V, E)$ is described. The biclique-chromatic number of a unichord-free graph is proved to be either the increment of or exactly the size of a largest twin set, and an $O(|V|^2|E|)$-time algorithm that returns an optimal biclique-colouring of a unichord-free graph input is described. The clique-chromatic and the biclique-chromatic numbers are not monotone with respect to induced subgraphs. The biclique-chromatic number presents an extra unexpected difficulty, as it is not the maximum over the biconnected components, which is overcome by considering additionally the star-biclique-chromatic number.

Clique-colouring and biclique-colouring have some similarities with usual vertex-colouring; in particular, any vertex-colouring is also a clique-colouring and a biclique-colouring. In other words, both the clique-chromatic number κ and the biclique-chromatic number κ_B are bounded above by the vertex-chromatic number χ. Optimal vertex-colourings and clique-colourings coincide in the case of K_3-free graphs, while optimal vertex-colourings and biclique-colourings coincide in the (much more restricted) case of $K_{1,2}$-free graphs.

Clique-colouring and biclique-colouring share essential differences with respect to usual vertex-colouring. A clique-colouring (resp. biclique-colouring) may not be a clique-colouring (resp. biclique-colouring) when restricted to a subgraph. Subgraphs may even have a larger clique-chromatic number (resp. larger biclique-chromatic number). Indeed, consider an odd hole with five vertices C_5 and a wheel graph with six vertices W_6 (resp. a triangle K_3 and a diamond $K_4 \setminus e$). Most remarkably, although $\kappa(G)$ is the maximum of the clique-chromatic numbers of the biconnected components, the parameter $\kappa_B(G)$ may not behave well under 1-cutset composition. Another difference is that a large clique is not an obstruction for clique colourability: even 2-clique-colourable graphs can contain arbitrarily large cliques. In [10], perfect graphs of arbitrarily large clique-chromatic number were constructed, answering a conjecture of [18].

Unichord-free and diamond-free graphs have a number of cliques linear in the number of vertices and edges, respectively. Unichord-free perfect graphs are a natural subclass of diamond-free perfect graphs, a class that attracted much attention in the context of clique-colouring — clique-colouring diamond-free perfect graphs is notably recognized as a difficult problem [1]. The results that every unichord-free graph is 3-clique-colourable, and that the 2-clique-colourable unichord-free graphs are precisely those that are perfect [39] are related to the result that every diamond-free perfect graph is 3-clique-colourable [14].

Clique colouring is harder than vertex colouring. It is coNP-complete to check whether a 2-clique-colouring is valid [1]. It is NP-hard to decide whether a perfect graph is 2-clique-colourable [35]. In order to determine the exact complexity of 2-clique-colourability, we consider higher levels of the polynomial hierarchy.

Table 2
Computational complexity of colouring problems restricted to subclasses of unichord-free graphs.

Problem	Class			
	General	Unichord-free	{*square*, *unichord*}-free	{*triangle*, *unichord*}-free
Vertex-col.	NPc [30]	P [62]	P [62]	P [62]
Edge-col.	NPc [30]	NPc [45]	NPc [45]	NPc [45]
Total-col.	NPc [47]	NPc [42]	P [42]	NPc [42]
Clique-col.	Σ_2^pc [46]	P [39]	P [39]	P ($\kappa = \chi$)
Biclique-col.	Σ_2^pc [23]	P [39]	P [39]	P ($\kappa_B = 2$) [39]
Star-col.	Σ_2^pc [23]	P [39]	P [39]	P ($\kappa_S = 2$) [39]

Similarly to NP, the class Σ_2^p has a characterization using certificates. A problem is in NP if there is a polynomial-size certificate for each yes-instance, and verifying this certificate is a problem in P. The characterization of the class Σ_2^p is similar, but here we require only that verifying the certificate is in coNP. The problem 2-clique-colouring is in Σ_2^p, since it is coNP to check that a colouring of the vertices is a clique-colouring: a monochromatic maximal clique is a polynomial-time verifiable certificate that the colouring is not proper. In [46], the exact complexity of 2-clique-colourability is determined as Σ_2^p-complete. The completeness result gives more information than knowing that the problem is NP-hard, because it rules out the possibility that the problem is in NP ∩ coNP, unless the polynomial hierarchy collapses.

Notice that the perfect graph subclasses for which the 2-clique-colouring problem is in NP mentioned in Table 2 satisfy that the number of cliques is polynomial. The complement of a matching has an exponential number of cliques and yet the 2-clique-colouring problem is in NP, since every such graph is 2-clique-colourable. It would be interesting to find subclasses of perfect graphs where not all graphs are 2-clique-colourable and yet the 2-clique-colouring problem is in P when restricted to the class.

A biclique-colouring which is also a star-colouring is the key to provide an $O(|V|^2|E|)$-time algorithm that returns an optimal biclique-colouring by returning an optimal star-biclique-colouring of unichord-free graphs. We call triangle the complete graph K_3, and square the 4-hole C_4. We remark that by definition, clique-colouring and vertex-colouring coincide for triangle-free graphs, while biclique-colouring and star-biclique-colouring coincide for square-free graphs. In [23], a polynomial-time 2-star-colouring algorithm for triangle-free graphs is given, and the biclique-colouring complexity for triangle-free graphs is left as an open problem.

Table 2 highlights the computational complexity of colouring problems restricted to classes related to unichord-free graphs. It is interesting to note that the class of {square, unichord}-free graphs provides: for total-colouring, the surprising example of a class for which total-colouring is polynomial although edge-colouring is NP-complete; while for biclique-colouring, since biclique-colouring and star-colouring coincide, the challenge is to colour the stars. For edge-colouring, the difficult decomposition is the proper 1-join, while for biclique-colouring, surprisingly, it is the 1-cutset.

4. Every graph is easy or hard

In 2014, Dániel Marx gave a plenary talk at the 9th International Colloquium on Graph Theory and Combinatorics, entitled Every Graph is Easy or Hard: Dichotomy Theorems for Graph Problems. He highlighted three features of dichotomy theorems: dichotomy theorems give good research programmes, easy to formulate, but can be hard to complete; the search for dichotomy theorems may uncover algorithmic results that no one has thought of; proving dichotomy theorems may require good command of both algorithmic and hardness proof techniques.

The updated Table 3 shows the progress after thirty years. The ten famous graph problems INDEPENDENT SET, CLIQUE, PARTITION INTO CLIQUES, CHROMATIC NUMBER, CHROMATIC INDEX, HAMILTONIAN CIRCUIT, DOMINATING SET, SIMPLE (unweighted) MAX CUT, (unweighted) STEINER TREE IN GRAPHS, and GRAPH ISOMORPHISM follow the corresponding abbreviations used by David S. Johnson: IndSet, Clique, CliPar, ChrNum, ChrInd, HamCirc, DomSet, MaxCut, StTree, and GraphIso. Two main open problems have been resolved: perfect graph recognition and hamilton circuit of permutation graphs; while the main open problem edge colouring of planar graphs remains open. Edge colouring still is the problem with most entries open, while maxcut had more progress particularly with respect to perfect graphs and intersection graphs.

In 1994, HAMILTONIAN CIRCUIT was shown to be polynomial-time solvable for permutation graphs by Deogun and Steiner [17], by settling the complexity in the more general class of cocomparability graphs. A polynomial-time algorithm for constructing a Hamiltonian path and a Hamiltonian cycle is presented, based on the relationship between the Hamiltonian problem in a cocomparability graph and the bump number problem in a partial order corresponding to the transitive orientation of its complementary graph.

In 2005, perfect graph recognition was shown to be polynomial-time solvable by Chudnovsky, Cornuéjols, Liu, Seymour, and Vušković [13]. The proof exploits the strong perfect graph theorem [16], which reduces the problem to that of testing for the existence of an odd hole. When odd holes exist in both the graph and its complement, there is no guarantee as to which type of hole the algorithm will find. The perfect graph recognition algorithm is a major breakthrough, but the problem of determining whether a given graph contains an odd hole remains open. See excellent surveys on the strong perfect graph theorem by Trotignon [61] and on Truemper configurations by Vušković [65].

Table 3
The updated table from 1985 to 2018, there are 23 **new references**, classifying 33 former open problems. We keep the abbreviations used by [30]: P = Polynomial-time solvable, P? = Appears to be polynomial-time by standard techniques, N = NP-complete, I = Open, but equivalent in complexity to general GRAPH ISOMORPHISM, O? = Apparently open, but possibly easy to resolve, O = Open, and may well be hard, T (as a reference) = restriction trivializes the problem, and GJ (as a reference) = the Guide [21]. We use abbreviation OG (as a reference) = the Ongoing guide [30], please refer to this reference for the entry. There are 33 new entries, all in bold and with numbers as references.

Graph Class	Member	IndSet	Clique	CliPar	ChrNum	ChrInd	HamCir	DomSet	MaxCut	StTree	GraphIso
Trees/Forests	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [GJ]	P [T]	P [GJ]
Almost trees (k)	P	P [OG]	P [T]	P?	P?	P?	P?	P [OG]	P?	P?	P?
Partial k-trees	P [OG]	P [OG]	P [T]	**P** [3]	P [OG]	**P** [4]	P [OG]	P [OG]	**P** [5]	**P** [34]	**P** [38]
Bandwidth-k	P [OG]	P [OG]	P [T]	P?	P [OG]	P?	P?	P [OG]	P [OG]	P?	P [OG]
Degree-k	P [T]	N [GJ]	P [T]	N [GJ]	N [GJ]	N [OG]	N [GJ]	N [GJ]	N [GJ]	N [GJ]	P [OG]
Planar	P [GJ]	N [GJ]	P [T]	N [OG]	N [GJ]	O	N [GJ]	N [GJ]	P [GJ]	N [OG]	P [GJ]
Series parallel	P [OG]	P [OG]	P [T]	**P** [3]	P [OG]	P [OG]	P [OG]	P [OG]	P [GJ]	P [OG]	P [GJ]
Outerplanar	P	P [OG]	P [T]	P [OG]	P [OG]	P [OG]	P [T]	P [OG]	P [GJ]	P [OG]	P [GJ]
Halin	P	P [OG]	P [T]	P [OG]	P [OG]	P [OG]	P [T]	P [OG]	P [GJ]	**P** [67]	P [GJ]
k-outerplanar	P	P [OG]	P [T]	P [OG]	P [OG]	O?	P [OG]	P [OG]	P [GJ]	P?	P [GJ]
Grid	P	P [GJ]	P [T]	P [T]	P [T]	P [GJ]	N [OG]	N [OG]	P [T]	N [OG]	P [GJ]
$K_{3,3}$-free	P [OG]	N [GJ]	P [T]	N [GJ]	N [GJ]	O?	N [GJ]	N [GJ]	P [OG]	N [GJ]	**I** [6]
Thickness-k	N [OG]	P [GJ]	P [T]	N [GJ]	N [GJ]	N [OG]	N [GJ]	N [GJ]	N [OG]	N [GJ]	O?
Genus-k	P [OG]	P [GJ]	P [T]	N [GJ]	N [GJ]	O?	N [GJ]	N [GJ]	O?	N [GJ]	P [OG]
Perfect	**P** [13]	P [OG]	P [OG]	P [OG]	P [OG]	**N** [8]	N [OG]	N [OG]	**N** [5]	N [GJ]	I [GJ]
Chordal	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [OG]	N [OG]	**N** [5]	N [OG]	I [GJ]
Split	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [OG]	N [OG]	**N** [5]	N [OG]	I [OG]
Strongly chordal	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	**N** [48]	P [OG]	**N** [60]	P [OG]	**I** [63]
Comparability	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	**N** [8]	N [OG]	N [OG]	**N** [53]	N [GJ]	I [GJ]
Bipartite	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [OG]	N [OG]	P [T]	N [GJ]	I [GJ]
Permutation	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	**P** [17]	P [OG]	O?	P [OG]	P [OG]
Cographs	P [T]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	**P** [5]	P [OG]	P [OG]
Undirected path	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	**N** [2]	N [OG]	**N** [5]	O?	I [GJ]
Directed path	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	**N** [51]	P [OG]	O?	P [OG]	O?
Interval	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	O?	P [OG]	P [OG]
Circular Arc	P [OG]	P [OG]	P [OG]	P [OG]	N [OG]	O?	**P** [57]	P [OG]	O?	P [OG]	O?
Circle	P [OG]	P [GJ]	P [OG]	**N** [33]	N [OG]	O?	P [OG]	**N** [32]	**N** [7]	P [OG]	O?
Proper Circ. Arc	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	O?	P [OG]	**P** [37]
Edge (or Line)	P [OG]	P [GJ]	P [T]	N [GJ]	N [OG]	**N** [8]	N [OG]	N [GJ]	**P** [24]	N [OG]	I [OG]
Claw-free	P [T]	P [OG]	**N** [54]	N [GJ]	N [OG]	**N** [8]	N [OG]	N [GJ]	**N** [5]	N [OG]	I [OG]

Split graphs constitute a very structured subclass of chordal graphs, being graphs such that both the graph and its complement are required to be chordal. This strong requirement forces the vertex set of a split graph to admit a partition into a stable set and a clique. In 1985, Johnson [30] stated "I know of no problem that separates the two classes in complexity". In 2003, Spinrad in his book [59] gave a survey of results on graph classes, an update of the complexity status regarding complexity-separating problems. There are a few complexity-separating problems (for instance, triangle packing and pathwidth) for which the problem is NP-complete for chordal but polynomial for split graphs. Spinrad [59] states: "split graphs often are at the core of algorithms and hardness results for chordal graphs".

The only remaining main open problem in the updated Table 3 is CHROMATIC INDEX for planar graphs. In 1964, Vizing [64] showed for each $k \in \{2, 3, 4, 5\}$ that there is a planar class 2 graph G with $\Delta(G) = k$. He proved that every planar graph with maximum vertex-degree at least 8 is a class 1 graph, and conjectured that every planar graph H with $\Delta(H) \in \{6, 7\}$ is a class 1 graph. Vizing's conjecture has been proved for planar graph with maximum vertex-degree 7 by Sanders and Zhao [56], and Zhang [70] independently. The problem of determining whether a given planar with maximum vertex-degree 6 is a class 1 graph remains open.

For most graph classes for which edge colouring can be solved in polynomial time, there is a bound on "number of vertices over maximum degree", in order to fit the overfull conjecture. We have presented the class of {square, unichord}-free graphs with maximum degree not 3 as a class of graphs with no such bound and yet class 2 = subgraph overfull for those graphs. The class of chordal graphs is another graph class with no such bound for which we conjecture class 2 = subgraph overfull.

The complexity of edge-colouring is unknown for well-studied strong structured graph classes, for which only partial results have been reported, such as cobipartite graphs [40], and join graphs [58]. By studying separating graph classes with respect to the problems vertex, edge, and total colourings, we may better understand the complexity of challenging colouring problems, and resolve edge colouring for the classes of chordal, split, cographs, and proper interval graphs.

Acknowledgements

I am grateful to Guillermo Durán, Mario Valencia-Pabon and the Programme Chairs of LAGOS 2017, for the opportunity of giving this invited talk. Special thanks are due to my co-authors: I have benefited enormously from our collaborations; I have freely adapted material from our joint work. Alexsander A. de Melo gave me invaluable help to update Table 3.

References

[1] G. Bacsó, S. Gravier, A. Gyárfás, M. Preissmann, A. Sebö, Coloring the maximal cliques of graphs, SIAM J. Discrete Math. 17 (2004) 361–376.
[2] A.A. Bertossi, M.A. Bonuccelli, Hamiltonian circuits in interval graph generalizations, Inform. Process. Lett. 23 (1986) 195–200.
[3] M. Blanchette, E. Kim, A. Vetta, Clique cover on sparse networks, in: Fourteenth Workshop on Algorithm Engineering and Experiments, SIAM, 2012, pp. 93–102.
[4] H.L. Bodlaender, Polynomial algorithms for graph isomorphism and chromatic index on partial *k*k-trees, J. Algorithms 11 (1990) 631–643.
[5] H.L. Bodlaender, K. Jansen, On the complexity of the maximum cut problem, in: Annual Symposium on Theoretical Aspects of Computer Science, Springer, 1994, pp. 769–780.
[6] K.S. Booth, C.J. Colbourn, Problems polynomially equivalent to graph isomorphism, Technical report, Report No. CS-77-04, Computer Science Department, University of Waterloo, 1979.
[7] C. Buchheim, L. Zheng, Fixed linear crossing minimization by reduction to the maximum cut problem, in: International Computing and Combinatorics Conference, Springer, 2006, pp. 507–516.
[8] L. Cai, J.A. Ellis, NP-completeness of edge-colouring some restricted graphs, Discrete Appl. Math. 30 (1991) 15–27.
[9] C.N. Campos, C.H. de Figueiredo, R. Machado, C.P. Mello, The total chromatic number of split-indifference graphs, Discrete Math. 312 (2012) 2690–2693.
[10] P. Charbit, I. Penev, S. Thomassé, N. Trotignon, Perfect graphs of arbitrarily large clique-chromatic number, J. Combin. Theory Ser. B 116 (2016) 456–464.
[11] B.L. Chen, H.-L. Fu, M.T. Ko, Total chromatic number and chromatic index of split graphs, J. Combin. Math. Combin. Comput. 17 (1995) 137–146.
[12] A.G. Chetwynd, A.J.W. Hilton, Star multigraphs with three vertices of maximum degree, Math. Proc. Camb. Phil. Soc. 100 (1986) 303–317.
[13] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, K. Vušković, Recognizing Berge graphs, Combinatorica 25 (2005) 143–186.
[14] M. Chudnovsky, I. Lo, Decomposing and clique-coloring (diamond, odd-Hole)-free graphs, J. Graph Theory 86 (2017) 5–41.
[15] M. Chudnovsky, I. Lo, F. Maffray, N. Trotignon, K. Vušković, Coloring square-free Berge graphs, J. Combin. Theory Ser. B 135 (2019) 96–128.
[16] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, The strong perfect graph theorem, Ann. Math. 164 (2006) 51–229.
[17] J.S. Deogun, G. Steiner, Polynomial algorithms for hamiltonian cycle in cocomparability graphs, SIAM J. Comput. 23 (1994) 520–552.
[18] D. Duffus, B. Sands, N. Sauer, R.E. Woodrow, Two-colouring all two-element maximal antichains, J. Comb. Theory A 57 (1991) 109–116.
[19] C.H. de Figueiredo, J. Meidanis, C.P. de Mello, On edge-colouring indifference graphs, Theoret. Comput. Sci. 181 (1997) 91–106.
[20] C.H. de Figueiredo, J. Meidanis, C.P. de Mello, Total-chromatic number and chromatic index of dually chordal graphs, Inform. Process. Lett. 70 (1999) 147–152.
[21] M.R. Garey, D.S. Johnson, Computers and Intractability, A Guide to the Theory of NP-completeness, WH Freeman, New York, 1979.
[22] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, Academic Press, 2004, (first edition) 1980, and Elsevier (second edition).
[23] M. Groshaus, F. Soulignac, P. Terlisky, Star and biclique coloring and choosability, J. Graph Algorithms Appl. 18 (2014) 347–383.
[24] V. Guruswami, Maximum cut on line and total graphs, Discrete Appl. Math. 92 (1999) 217–221.
[25] A. Gyárfás, Problems from the world surrounding perfect graphs, Zastos. Mat. 19 (1987) 413–441.
[26] A.J.W. Hilton, A total-chromatic number analogue of Plantholt's theorem, Discrete Math. 79 (1990) 169–175.
[27] D.G. Hoffman, C.A. Rodger, The chromatic index of complete multipartite graphs, J. Graph Theory 16 (1992) 159–163.
[28] I. Holyer, The np-completeness of edge-coloring, SIAM J. Comput. 10 (1981) 718–720.
[29] D.S. Johnson, NP-Completeness Columns, available at http://www.cos.ufrj.br/celina/npcc/.
[30] D.S. Johnson, Graph restrictions and their effect, J. Algorithms 6 (1985) 434–451.
[31] R.M. Karp, Reducibility among combinatorial problems, in: R.E. Miller, J.W. Thatcher (Eds.), Complexity of Computer Computations, Plenum Press, New York, 1972, pp. 85–103.
[32] J.M. Keil, The complexity of domination problems in circle graphs, Discrete Appl. Math. 42 (1993) 51–63.
[33] J.M. Keil, L. Stewart, Approximating the minimum clique cover and other hard problems in subtree filament graphs, Discrete Appl. Math. 154 (2006) 1983–1985.
[34] E. Korach, N. Solel, Linear time algorithm for minimum weight steiner tree in graphs with bounded treewidth, Technical report, Report No. 632, Israel Institute of Technology, 1990.
[35] J. Kratochvil, Z. Tuza, On the complexity of bicoloring clique hypergraphs of graphs, J. Algorithms 45 (2002) 40–54.
[36] D. Leven, Z. Galil, NP-completeness of finding the chromatic index of regular graphs, J. Algorithms 4 (1983) 35–44.
[37] M.C. Lin, F.J. Soulignac, J.L. Szwarcfiter, A simple linear time algorithm for the isomorphism problem on proper circular-arc graphs, in: Scandinavian Workshop on Algorithm Theory, Springer, 2008, pp. 355–366.
[38] D. Lokshtanov, M. Pilipczuk, M. Pilipczuk, S. Saurabh, Fixed-parameter tractable canonization and isomorphism test for graphs of bounded treewidth, SIAM J. Comput. 46 (2017) 161–189.
[39] H. Macedo, R.C.S. Machado, C.M.H. de Figueiredo, Efficient algorithms for clique-colouring and biclique-colouring unichord-free graphs, Algorithmica 77 (2017) 786–814.
[40] R.C.S. Machado, C.M.H. de Figueiredo, Decompositions for edge-coloring join graphs and cobipartite graphs, Discrete Appl. Math. 158 (2010) 1336–1342.
[41] R.C.S. Machado, C.M.H. de Figueiredo, Complexity separating classes for edge-colouring and total-colouring, J. Braz. Comp. Soc. 17 (2011) 281–285.
[42] R.C.S. Machado, C.M.H. de Figueiredo, Total chromatic number of unichord-free graphs, Discrete Appl. Math. 159 (2011) 1851–1864.
[43] R.C.S. Machado, C.M.H. de Figueiredo, N. Trotignon, Edge-colouring and total-colouring chordless graphs, Discrete Math. 313 (2013) 1547–1552.
[44] R.C.S. Machado, C.M.H. de Figueiredo, N. Trotignon, Complexity of colouring problems restricted to unichord-free and {square, unichord}-free graphs, Discrete Appl. Math. 164 (2014) 191–199.
[45] R.C.S. Machado, C.M.H. de Figueiredo, K. Vušković, Chromatic index of graphs with no cycle with unique chord, Theoret. Comput. Sci. 411 (2010) 1221–1234.
[46] D. Marx, Complexity of clique coloring and related problems, Theoret. Comput. Sci. 412 (2011) 3487–3500.
[47] C.J.H. McDiarmid, A. Sánchez-Arroyo, Total colouring regular bipartite graphs is NP-hard, Discrete Math. 124 (1994) 155–162.
[48] H. Müller, Hamiltonian circuits in chordal bipartite graphs, Discrete Math. 156 (1996) 291–298.
[49] T. Niessen, How to find overfull subgraphs in graphs with large maximum degree, Electron. J. Combin. 8 (2001) #R7.
[50] C. Ortiz, N. Maculan, J.L. Szwarcfiter, Characterizing and edge-colouring split-indifference graphs, Discrete Appl. Math. 82 (1998) 209–217.
[51] B. Panda, D. Pradhan, NP-completeness of hamiltonian cycle problem on rooted directed path graphs, arXiv preprint arXiv:0809.2443, 2008.
[52] M. Plantholt, The chromatic index of graphs with a spanning star, J. Graph Theory 5 (1981) 45–53.
[53] R.V. Pocai, The complexity of simple max-cut on comparability graphs, Electron. Notes Discrete Math. 55 (2016) 161–164, 14th Cologne-Twente Workshop on Graphs and Combinatorial Optimization (CTW16).
[54] S. Poljak, A note on stable sets and colorings of graphs, Comment. Math. Univ. Carolin. 15 (1974) 307–309.
[55] A. Sánchez-Arroyo, Determining the total colouring number is NP-hard, Discrete Math. 78 (1989) 315–319.

[56] D.P. Sanders, Y. Zhao, Planar graphs of maximum degree seven are class 1, J. Combin. Theory Ser. B 83 (2001) 201–212.
[57] W. Shih, T. Chern, W.-L. Hsu, An O($n^2 \log n$) algorithm for the hamiltonian cycle problem on circular-arc graphs, SIAM J. Comput. 21 (1992) 1026–1046.
[58] C. Simone, C.P. de Mello, Edge-colouring of join graphs, Theoret. Comput. Sci. 355 (2006) 364–370.
[59] J.P. Spinrad, Efficient Graph Representations, in: Fields Institute Monographs, vol. 19, AMS, 2003.
[60] R. Sucupira, L. Faria, S. Klein, A complexidade do problema do Corte Máximo para grafos fortemente cordais, in: XLV Simpósio Brasileiro de Pesquisa Operacional v. 01, 2013, pp. 2979–2988.
[61] N. Trotignon, Perfect graphs, in: L.W. Beineke, R.J. Wilson (Eds.), Topics in Chromatic Graph Theory, Cambridge University Press, 2015.
[62] N. Trotignon, K. Vušković, A structure theorem for graphs with no cycle with a unique chord and its consequences, J. Graph Theory 63 (2010) 31–67.
[63] R. Uehara, S. Toda, T. Nagoya, Graph isomorphism completeness for chordal bipartite graphs and strongly chordal graphs, Discrete Appl. Math. 145 (2005) 479–482.
[64] V.G. Vizing, On an estimate of the chromatic index of a p-graph, Metody Diskret. Analiz 3 (1964) 25–30.
[65] K. Vušković, The world of hereditary graph classes viewed through truemper configurations, in: Surveys in Combinatorics 2013, in: London Mathematical Society Lecture Note Series, vol. 409, Cambridge University Press, 2013, pp. 265–326.
[66] D.B. West, Introduction to Graph Theory, second ed., Prentice Hall, 2001.
[67] P. Winter, Steiner problem in halin networks, Discrete Appl. Math. 17 (1987) 281–294.
[68] N. Wolchover, Theorists Draw Closer to Perfect Coloring, Quanta Magazine, 2015.
[69] H.P. Yap, Total Colourings of Graphs, in: Lecture Notes in Mathematics, Springer-Verlag, Germany, 1996.
[70] L. Zhang, Every graph with maximum degree 7 is of class 1, Graphs Combin. 16 (2000) 467–495.