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On total coloring the direct product of complete graphs☆

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Abstract

A *k-total coloring* of a graph G is an assignment of k colors to the elements (vertices and edges) of G so that adjacent or incident elements have different colors. The total chromatic number is the smallest integer k for which G has a k-total coloring. The well known Total Coloring Conjecture states that the total chromatic number of a graph is either $\Delta(G) + 1$ or $\Delta(G) + 2$, where $\Delta(G)$ is the maximum degree of G. We consider the direct product of complete graphs $K_m \times K_n$. It is known that if at least one of the numbers m or n is even, then $K_m \times K_n$ has total chromatic number equal to $\Delta(K_m \times K_n) + 1$, except when $m = n = 2$. We prove that the graph $K_m \times K_n$ has total chromatic number equal to $\Delta(K_m \times K_n) + 1$ when both m and n are odd numbers, ensuring in this way that all graphs $K_m \times K_n$ have total chromatic number equal to $\Delta(K_m \times K_n) + 1$, except when $m = n = 2$.

Keywords: total coloring; direct product; regular graph; complete graph.

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. A *k-total coloring* of a graph G is an assignment of k colors to the elements (vertices and edges) of G so that adjacent or incident elements have different colors. The *total chromatic number*, denoted by $\chi_T(G)$, is the smallest integer k for which G has a k-total coloring. Clearly, $\chi_T(G) \geq \Delta(G) + 1$ and the *Total Coloring Conjecture* (TCC), posed independently by Vizing [11] and Behzad et al. [2], states that $\chi_T(G) \leqslant \Delta(G) + 2$, where $\Delta(G)$ is the maximum degree of G. Graphs with $\chi_T(G) = \Delta(G) + 1$ are

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said to be *Type* 1 and graphs with $\chi_T(G) = \Delta(G) + 2$ are said to be *Type* 2. The TCC has been verified in restricted cases, such as cubic graphs [10] and graphs with large maximum degree [7], but has not been settled for all regular graphs for more than fifty years.

We denote an undirected edge $e \in E(G)$ whose ends are u and v by uv. The *direct product* (also called *tensor product* or *Kronecker product*) of two graphs G and H is a graph denoted by $G \times H$, whose vertex set is the Cartesian product $V(G) \times V(H)$, for which vertices (u, v) and (u', v') are adjacent if and only if $uu' \in E(G)$ and $vv' \in E(H)$, whose maximum degree $\Delta(G \times H) = \Delta(G) \cdot \Delta(H)$, and $G \times H$ is regular if and only if both G and H are regular graphs. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs on the same vertex set V and where $E_1 \cap E_2 = \emptyset$, and denote by $\bigoplus_{i=1}^{2} G_i$ the direct sum graph $G = (V, E_1 \cup E_2)$ of graphs G_1 and G_2. In this work, we use the well known property that the direct product is distributive over edge disjoint union of graphs, that is, if $G = \bigoplus_{i=1}^{t} G_i$, where G_i are edge-disjoint subgraphs of G and $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_t)$, then $H \times G = \bigoplus_{i=1}^{t}(H \times G_i)$.

The complete graph on n vertices is denoted by K_n. The direct product of complete graphs $K_m \times K_n$ is a regular graph of degree $\Delta(K_m \times K_n) = (m-1)(n-1)$ and can be described as an n-partite graph with m vertices in each part. The total chromatic number of $K_m \times K_n$ has been determined when m or n is an even number. When $m = n = 2$, we have the disconnected $2K_2$, which is Type 2, since each connected component K_2 is Type 2. When $m \geq 3$, $K_m \times K_2$ is the complete bipartite graph $K_{m,m}$ minus a perfect matching, and Yap [12] proved that this graph is Type 1. When $n \geq 4$ and n is an even number, Geetha and Somasundaram [6] proved that $K_n \times K_n$ is Type 1. Janssen and Mackeigan [8] recently proved that $K_m \times K_n$ is Type 1 when m or n is an even number, with $m, n \geq 3$. As far as we know, for the remaining case, when both m and n are odd numbers, it is not known whether $K_m \times K_n$ is Type 1 or Type 2. In this work, we establish the total chromatic number of $K_m \times K_n$, when m and n are odd numbers, by proving that this graph is Type 1. Thus, we can conclude that, except for $m = n = 2$, the graph $K_m \times K_n$ is Type 1.

In order to achieve the claimed total colorings for all graphs $K_m \times K_n$, when m and n are odd numbers, we prove two theorems according to whether m and n are both large enough or not. In Section 2, we recall the conformable necessary condition to be Type 1 and a known lower bound on the vertex degree for regular graphs which ensures the equivalence, and we prove Lemma 2.1 and Theorem 2.2 which together provide the required total colorings of the direct product of complete graphs $K_m \times K_n$, for odd numbers $m, n \geq 13$. In Section 3, we present preliminary concepts on Hamiltonian decompositions used to obtain a guiding color for the remaining target total colorings. In Section 4, we prove Theorem 4.1 which provides the required total colorings of $K_m \times K_n$, for odd numbers $m, n \geq 3$ and $m < 13$.

2. The conformable condition is enough for odd numbers $m, n \geq 13$

A regular graph G is *conformable* [3] if G admits a vertex coloring with $\Delta(G) + 1$ colors such that the number of vertices in each color class has the same parity as $|V(G)|$.

Lemma 2.1. *For odd numbers $m, n \geq 3$, the graph $K_m \times K_n$ is conformable.*

Proof. Consider $m \leq n$. We construct a vertex coloring with $(m-1)(n-1)+1$ colors such that each color class is composed by 1 or 3 vertices. Let $t = \frac{m+n-2}{2}$. Since $t < n$, vertices $(0, i), (1, i), (2, i)$ in the direct product $K_m \times K_n$ define an independent set and can receive the same color c_i, for $i = \{0, \ldots, t-1\}$. Now color each of the $mn - 3t$ remaining uncolored vertices with a different additional color, to obtain the desired vertex coloring with $t + (mn - 3t) = mn - 2t = mn - m - n + 2 = (m-1)(n-1) + 1 = \Delta(K_m \times K_n) + 1$ colors. □

Hilton and Hind [7] established the TCC for graphs G having $\Delta(G) \geq \frac{3}{4}|V(G)|$. Chetwynd et al. [4] proved that letting G be a regular graph of odd order and with degree $\Delta(G) \geq \frac{\sqrt{7}}{3}|V(G)|$, then G is Type 1 if and only if G is conformable. Chew [5] improved this result by showing that it is enough to require that $\Delta(G) \geq \frac{(\sqrt{37}-1)}{6}|V(G)|$. In Theorem 2.2, we establish that when $m, n \geq 13$ are odd numbers, then $\Delta(K_m \times K_n)$ satisfies the lower bound required by Chew, which together with Lemma 2.1 implies the desired result.

Theorem 2.2. *For odd numbers $m, n \geq 13$, the graph $K_m \times K_n$ is Type* 1.

Proof. Let $m, n \geq 13$ be two odd numbers. Hence, $(7-\sqrt{37})n - 6 \geq (7-\sqrt{37}) \cdot 13 - 6 \geq 0$ and $n \geq 13 \geq \frac{72}{13(7-\sqrt{37})-6}$. So, $13(7-\sqrt{37})n \geq 72 + 6n$ and $13(7-\sqrt{37})n - 13 \cdot 6 \geq 72 + 6n - 13 \cdot 6$, which implies that $13 \geq \frac{6(n-1)}{(7-\sqrt{37})n-6}$. Now, as $m \geq 13$, we have that $m \geq \frac{6(n-1)}{(7-\sqrt{37})n-6}$. Therefore, $(7-\sqrt{37})mn - 6m \geq 6n - 6$, which is equivalent to $(1-\sqrt{37})mn + 6mn - 6m - 6n + 6 \geq 0$. So, $mn - m - n + 1 = (m-1)(n-1) \geq \frac{(\sqrt{37}-1)}{6}mn$. Since $\Delta(K_m \times K_n) = (m-1)(n-1)$, we have that $\Delta(K_m \times K_n) \geq \frac{(\sqrt{37}-1)}{6}|V(K_m \times K_n)|$. Therefore, by Chew's result [5] and by Lemma 2.1, we have that $K_m \times K_n$ is Type 1. □

3. Hamiltonian decompositions to get a guiding color for odd numbers $m, n \geq 3$ and $m < 13$

We consider 5 infinite families: $K_3 \times K_n$, $K_5 \times K_n$, $K_7 \times K_n$, $K_9 \times K_n$ and $K_{11} \times K_n$, with $n \geq 3$ an odd number. For $K_3 \times K_n$, $K_5 \times K_n$ and $K_7 \times K_n$, in Subsection 3.1, we use Waleski's Hamiltonian decomposition of K_n to define suitable Hamiltonian decompositions of $K_m \times K_n$, first when $gcd(m, n) = 1$ and second when $gcd(m, n) \neq 1$; in Subsection 3.2, we apply the constructed Hamiltonian decomposition to define a guiding color representing a color class from which the target $(\Delta(K_m \times K_n) + 1)$-total coloring is finally obtained in Subsection 4.1. For $K_9 \times K_n$ and $K_{11} \times K_n$, we use Chew's result [5] and Lemma 2.1 to obtain that the family $K_9 \times K_n$ is Type 1 for $n \geq 23$ and the family $K_{11} \times K_n$ is Type 1 for $n \geq 15$ in Subsection 4.2.

3.1. Hamiltonian decompositions

A k-regular graph G has a *Hamiltonian decomposition* (or is *Hamiltonian decomposable*) if its edge set can be partitioned into $\frac{k}{2}$ Hamiltonian cycles when k is an even number, or into $\frac{(k-1)}{2}$ Hamiltonian cycles plus a one factor (or perfect matching) when k is an odd number. Please refer to [1] for a survey on Hamiltonian decompositions of various product graphs.

Consider the well known Waleski's Hamiltonian decomposition of the complete graph K_n for $n \geq 3$. We shall focus on an odd number n. Let $n = 2w + 1$ and label the vertices of K_n as $0, 1, \ldots, 2w$. Following the notation used in [1], let C_n be the Hamiltonian cycle $\langle 0, 1, 2, 2w, 3, 2w-1, 4, 2w-2, 5, 2w-3, \ldots, w+3, w, w+2, w+1, 0\rangle$. If σ is the permutation $(0)(1, 2, 3, 4, \ldots, 2w-1, 2w)$, then $\sigma^0(C_n), \sigma^1(C_n), \sigma^2(C_n), \ldots, \sigma^{w-1}(C_n)$ is a Hamiltonian decomposition of K_n. Observe that $\sigma^0(C_n) = C_n$. We write $K_n = \bigoplus_{i=1}^{w} \sigma^{i-1}(C_n)$. Denote by $\sigma^t(C_n)_z$, with $z = 0, 1, \ldots, n-1$ the z^{th}-vertex in the cycle $\sigma^t(C_n)$, and note that the vertex 0 is always the 0^{th}-vertex. Note that for $t \geq w$, the cycle $\sigma^t(C_n)$ is the opposite cycle of $\sigma^{t \bmod w}(C_n)$, that is, $\sigma^t(C_n)_z = \sigma^{t \bmod w}(C_n)_{n-z}$ for all $z \geq 1$.

For instance consider $n = 5$, write $n = 2w + 1$ and thus $w = 2$, to get the Hamiltonian decomposition $K_5 = \bigoplus_{i=1}^{2} \sigma^{i-1}(C_5)$, where $\sigma^0(C_5) = \langle 0, 1, 2, 4, 3, 0\rangle$ and $\sigma^1(C_5) = \langle 0, 2, 3, 1, 4, 0\rangle$, as highlighted in Figure 1. Note that $\sigma^2(C_5) = \langle 0, 3, 4, 2, 1, 0\rangle$ is the opposite cycle of $\sigma^0(C_5)$, and $\sigma^3(C_5) = \langle 0, 4, 1, 3, 2, 0\rangle$ is the opposite cycle of $\sigma^1(C_5)$.

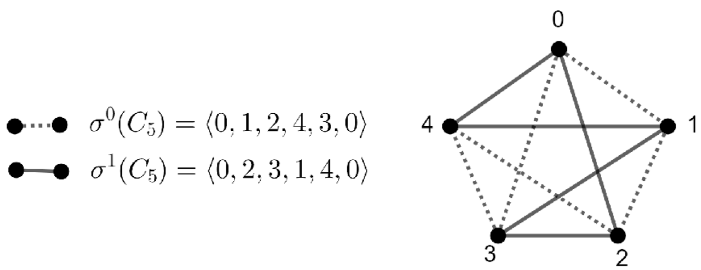

Fig. 1. Waleski's Hamiltonian decomposition of $K_5 = \sigma^0(C5) \bigoplus \sigma^1(C5)$.

It is well known and not hard to see that the direct product of cycle graphs is Hamiltonian decomposable if and only if at least one of them is an odd cycle [9]. In what follows, for both m and n odd numbers, we shall use Waleski's

Hamiltonian decomposition of the complete graph K_n and the well known distributive property of the direct product to define a Hamiltonian decomposition of $K_m \times K_n$, for $m = 3, 5, 7$ and odd number $n \geq 3$ suitable to our target total coloring.

Write odd numbers $m, n \geq 3$ as $m = 2q + 1$ and $n = 2w + 1$. Let $gcd(m, n) = d$. For $j = 1, \dots, 2q$, $i = 1, \dots, 2w$ and $k = 0, \dots, d-1$, denote by $C(j,i)^k$ the cycle on $\frac{mn}{d}$ vertices $\langle C(j,i)^k_z \rangle_{z=0,\dots,\frac{mn}{d}}$, where $C(j,i)^k_z = (\sigma^{j-1}(C_m)_{(z+k) \bmod m}, \sigma^{i-1}(C_n)_{z \bmod n})$, with $z = 0, \dots, \frac{mn}{d}$, is the z^{th}-vertex of the cycle $C(j,i)^k$. Observe that according to the notation for vertex $C(j,i)^k_z$, we have $C(j,i)^k_0 = C(j,i)^k_{\frac{mn}{d}}$, and the vertex $(0,0)$ is always the 0^{th}-vertex of $C(j,i)^0$.

We consider next the construction of a Hamiltonian decomposition of $K_m \times K_n$ according to whether $gcd(m, n) = 1$ or not. Case 1 considers $gcd(m, n) = 1$ which gives a single $k = 0$ and that each $C(j,i)^0$ is a Hamiltonian cycle which gives that $\{C(j,i) = C(j,i)^0 \mid j = 1, \dots q \text{ and } i = 1, \dots, 2w\}$ is a Hamiltonian decomposition of $K_m \times K_n$. Case 2 considers $gcd(m, n) \neq 1$ which implies that each cycle $C(j,i)^k$ is not a Hamiltonian cycle. We construct a Hamiltonian decomposition of $K_m \times K_n$ given by $\{C(j,i) \mid j = 1, \dots, 2q \text{ and } i = 1, \dots, w\}$ where each Hamiltonian cycle is composed by d paths obtained from the cycles $C(j,i)^k$, such that, for each $k = 0, \dots, d-1$, the cycle $C(j,i)^k$ becomes a path by removing one edge.

Case 1: $gcd(m, n) = 1$. Consider $\{C(j,i) \mid j = 1, \dots q \text{ and } i = 1, \dots, 2w\}$, a Hamiltonian decomposition of $K_m \times K_n$, where $C(j,i) = C(j,i)^0$, see an example in Figure 2. Indeed, consider $K_m = \bigoplus_{j=1}^{q} (\sigma^{j-1}(C_m))$ and $K_n = \bigoplus_{i=1}^{w} (\sigma^{i-1}(C_n))$ the Waleski's Hamiltonian decompositions of K_m and K_n, respectively. Thus we write $K_m \times K_n = \bigoplus_{j=1}^{q} \bigoplus_{i=1}^{w} (\sigma^{j-1}(C_m) \times \sigma^{i-1}(C_n))$. As the degree $\Delta(\sigma^{j-1}(C_m) \times \sigma^{i-1}(C_n)) = 4$, for any $j = 1, 2, \dots, q$ and for any $i = 1, 2, \dots, w$, each subgraph $\sigma^{j-1}(C_m) \times \sigma^{i-1}(C_n)$ of $K_m \times K_n$ has two Hamiltonian cycles: $C(j,i)$ and $C(j, i+w)$, and so, it is enough to consider $C(j,i)$ for $j = 1, \dots, q$ and $i = 1, \dots, 2w$.

For instance, consider $K_3 \times K_5$ in Figure 2. As $gcd(3, 5) = 1$ we use $K_3 \times K_5 = \bigoplus_{j=1}^{1} \bigoplus_{i=1}^{2} (\sigma^{j-1}(C_3) \times \sigma^{i-1}(C_5))$, the 2 Hamiltonian cycles of the subgraph $\sigma^0(C_3) \times \sigma^0(C_5)$ of $K_3 \times K_5$ are $C(1,1)$ and $C(1,3)$. Analogously, the 2 Hamiltonian cycles of the subgraph $\sigma^0(C_3) \times \sigma^1(C_5)$ of $K_3 \times K_5$ are $C(1,2)$ and $C(1,4)$.

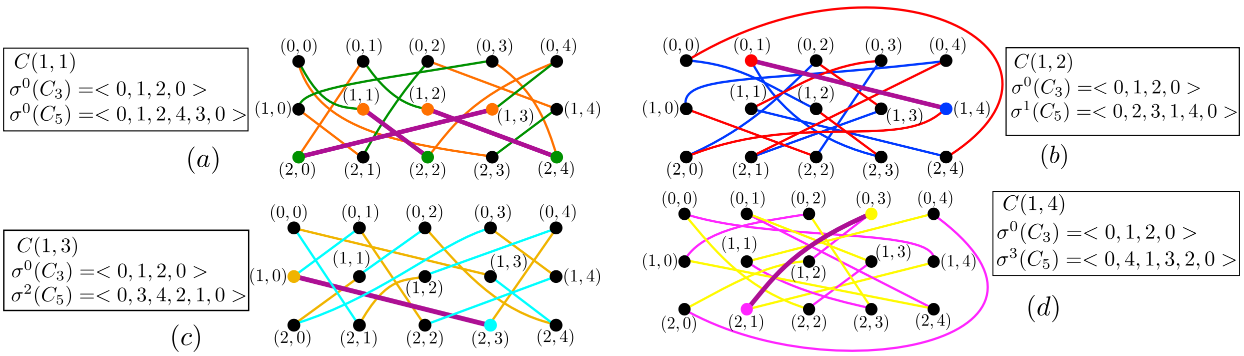

Fig. 2. A depiction of $K_3 \times K_5$ partitioned into 4 Hamiltonian cycles. In (*a*) we have the Hamiltonian cycle $C(1,1)$ with 3 colors: the edges $(1,1)(2,2)$, $(1,2)(2,4)$ and $(1,3)(2,0)$ are colored with the guiding purple color; the endvertices of the purple edges and the remaining edges of $C(1,1)$ are colored with colors orange and dark green. In (*b*) we have the Hamiltonian cycle $C(1,2)$ also colored with 3 colors: the edge $(0,1)(1,4)$ also colored with the guiding purple color; the endvertices of the purple edge and the remaining edges of $C(1,2)$ are colored with colors red and dark blue. In (*c*) we have $C(1,3)$ also with 3 colors: the edge $(1,0)(2,3)$ also colored with the guiding purple color; the endvertices of the purple edge and the remaining edges of $C(1,3)$ are colored with colors light blue and light green. Finally in (*d*) we have $C(1,4)$ also colored with 3 colors: the edge $(0,3)(2,1)$ also colored with the guiding purple color; the endvertices of the purple edge and the remaining edges of $C(1,4)$ are colored with colors pink and yellow.

Case 2: $gcd(m, n) = d > 1$. By definition, in this case, each $C(j,i)^k$ is not a Hamiltonian cycle. For $k = 0, \dots, d-1$, denote by $P(j,i)^k$ the path induced by the $\frac{mn}{d}$ vertices $C(j,i)^k_z$, with $z = 0, \dots, \frac{mn}{d} - 1$, obtained from $C(j,i)^k$ by removing one edge. Consider $\{C(j,i) \mid j = 1, \dots, 2q \text{ and } i = 1, \dots, w\}$ a Hamiltonian decomposition of $K_m \times K_n$, where the Hamiltonian cycles are defined as follows.

(*i*) For $m = 3$, $C(j, i) = \langle P(j, i)^0, P(j, i)^1, P(j, i)^2, (0, 0)\rangle$
For $i = 1, \ldots, w$, the cycles $C(1, i)$ and $C(2, i)$ form a Hamiltonian decomposition of $\sigma^0(C_3) \times \sigma^{i-1}(C_n)$.
For instance, consider $K_3 \times K_9$ in Figure 3. As $gcd(3, 9) = 3$ we write $K_3 \times K_9 = \bigoplus_{i=1}^{4}(\sigma^0(C_3) \times \sigma^{i-1}(C_9))$. The 2 Hamiltonian cycles of the subgraph $\sigma^0(C_3) \times \sigma^0(C_9)$ are $C(1, 1)$ and $C(2, 1)$; and analogously of the subgraph $\sigma^0(C_3) \times \sigma^1(C_9)$ are $C(1, 2)$ and $C(2, 2)$; of the subgraph $\sigma^0(C_3) \times \sigma^2(C_9)$ are $C(1, 3)$ and $C(2, 3)$; finally of the subgraph $\sigma^0(C_3) \times \sigma^3(C_9)$ are $C(1, 4)$ and $C(2, 4)$.

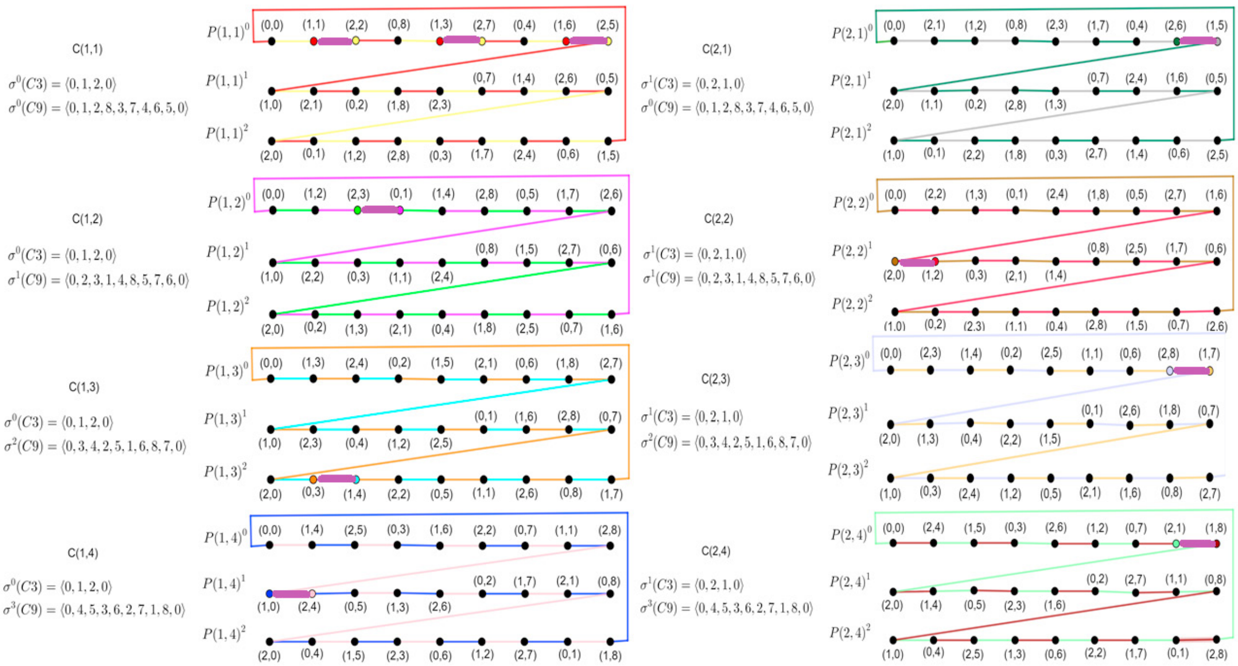

Fig. 3. A depiction of $K_3 \times K_9$ partitioned into 8 Hamiltonian cycles. We have the Hamiltonian cycle $C(1, 1)$ with 3 colors: the edges $(1, 1)(2, 2)$, $(1, 3)(2, 7)$ and $(1, 6)(2, 5)$ are colored with the guiding purple color; the endvertices of the purple edges and the remaining edges of $C(1, 1)$ are colored with colors red and yellow. In the remaining 7 Hamiltonian cycles, each of them has one edge with the guiding purple color whose endvertices and the remaining edges of the cycle are colored with additional new two colors. The vertices $(0, 0), (0, 2), (0, 4), (0, 5), (0, 6), (0, 7)$ and $(0, 8)$ are an independent set and can be colored with the guiding purple color obtaining a 17-total coloring of $K_3 \times K_9$.

(*ii*) For $m = 5$, $C(j, i) = \begin{cases} \langle P(j, i)^0, P(j, i)^1, P(j, i)^2, P(j, i)^3, P(j, i)^4, (0, 0)\rangle, \text{if } j = 1, 3 \\ \langle P(j, i)^0, P(j, i)^2, P(j, i)^4, P(j, i)^1, P(j, i)^3, (0, 0)\rangle, \text{if } j = 2, 4 \end{cases}$
For $i = 1, \ldots, w$, the set of cycles $\{C(j, i) \mid j = 1, \ldots, 4\}$ is a Hamiltonian decomposition of $K_5 \times \sigma^{i-1}(C_n)$.

(*iii*) For $m = 7$, $C(j, i) = \begin{cases} \langle P(j, i)^0, P(j, i)^3, P(j, i)^4, P(j, i)^5, P(j, i)^1, P(j, i)^2, P(j, i)^6, (0, 0)\rangle, \text{if } j = 1, 3, 5 \\ \langle P(j, i)^0, P(j, i)^4, P(j, i)^1, P(j, i)^3, P(j, i)^6, P(j, i)^2, P(j, i)^5, (0, 0)\rangle, \text{if } j = 2, 4, 6 \end{cases}$
For $i = 1, \ldots, w$, the set of cycles $\{C(j, i) \mid j = 1, \ldots, 6\}$ is a Hamiltonian decomposition of $K_7 \times \sigma^{i-1}(C_n)$.

3.2. $\Delta((K_m \times K_n) + 1)$-*total coloring from elements of a guiding color*

We are ready to explain how a $(\Delta(K_m \times K_n) + 1)$-total coloring of $K_m \times K_n$ is obtained by considering the Hamiltonian decomposition of $K_m \times K_n$ into Hamiltonian cycles $C(i, j)$ defined in Subsection 3.1. In a $(\Delta(K_m \times K_n) + 1)$-total coloring, each color class is such that each vertex is either inside the color class or is incident to an edge of the color class. We shall choose a guiding color with the additional property that its color class contains one or three edges of each Hamiltonian cycle. Note that each Hamiltonian cycle is an odd cycle and, by Vizing's theorem [11], admits a 3-edge coloring. Thus, for each cycle, we assign two additional colors to the remaining edges of the Hamiltonian cycle and to the endvertices of the edges with the guiding color, as Figures 2 and 3. With suitable choices for the edges of the matching colored by the guiding color, the so far uncolored vertices define an independent set which can be also colored with the guiding color as Figure 4.

In order to obtain a $(\Delta(K_m \times K_n) + 1)$-total coloring, we give a table composed by the elements of the guiding color class. We identify the edges of the guiding color on the corresponding Hamiltonian cycle where they belong. If the Hamiltonian cycle contains a unique edge of the guiding color, then its endvertices and the remaining edges of the cycle are easily colored using two additional colors. If the Hamiltonian cycle contains three edges of the guiding color, then we can easily see that their endvertices define two independent sets that can be colored with two colors as also the remaining edges of the cycle.

For instance, consider $K_3 \times K_5$ in Figure 4. We represent a table and a subgraph highlighting all elements (edges and vertices) colored by the guiding color and the colored vertices of Figure 2. We can identify which of the four Hamiltonian cycles contains which highlighted edges by observing the colors of their endvertices. In Fig 2(a), the six endvertices of the three edges colored with the guiding color (purple) in $C(1, 1)$ are the three vertices $(1, 1), (1, 2)$ and $(1, 3)$ defining an independent set that can be assigned with one color (orange), and the three vertices $(2, 0), (2, 2)$ and $(2, 4)$ defining another independent set that can be assigned with one color (green). The remaining edges of $C(1, 1)$ can be assigned with the colors orange and green. Analogously for the Hamiltonian cycles $C(1, 2), C(1, 3)$ and $C(1, 4)$, as in Figure 2. The remaining uncolored vertices $(0, 0), (0, 2)$ and $(0, 4)$ of Figure 2 represent an independent set that can be colored with the guiding color. Thus we can easily obtain a 9-total coloring of $K_3 \times K_5$ from the elements colored with the guiding color.

Elements of $K_3 \times K_5$ of the guiding color	
$C(1,1)$	$(1,1)(2,2), (1,3)(2,0), (1,2)(2,4)$
$C(1,2)$	$(0,1)(1,4)$
$C(1,3)$	$(1,0)(2,3)$
$C(1,4)$	$(2,1)(0,3)$
Vertices: $(0,0), (0,2), (0,4)$	

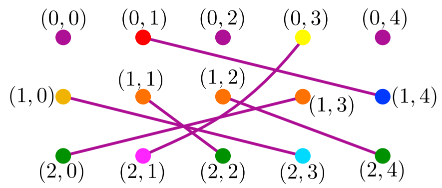

Fig. 4. A table composed by the elements of the guiding purple color in $K_3 \times K_5$, and its depiction using colors of the endvertices to identify the Hamiltonian cycles containing them.

4. Proof of Theorem 4.1

In this section, we consider only the direct product of odd complete graphs $K_m \times K_n$ with $m, n \geq 3$ and $m < 13$. Along the proof, we may sometimes omit the fact that m, n are odd numbers and $m, n \geq 3$, since it is clear that we work only with odd complete graphs greater than 2. Theorem 4.1 includes the five Type 1 infinite families of the direct product of complete graphs: $K_3 \times K_n$, $K_5 \times K_n$, $K_7 \times K_n$, $K_9 \times K_n$ and $K_{11} \times K_n$. Theorem 4.1 completes the result that $K_m \times K_n$ is Type 1, except when $m = n = 2$.

Theorem 4.1. *For odd numbers $m, n \geq 3$ with $m < 13$, the graph $K_m \times K_n$ is Type 1.*

We now present two subsections. In each subsection, for each considered family, we omit a finite number of particular graphs that are too small to obey the described pattern. For each particular graph, we were able to describe a particular Type 1 total coloring using the general strategy of first obtaining a particular Hamiltonian decomposition and then choosing a suitable guiding color. Their particular Hamiltonian decompositions and their tables containing the elements of the guiding color are omitted in the extended abstract.

4.1. Families $K_3 \times K_n$, $K_5 \times K_n$, $K_7 \times K_n$

In this subsection, we consider three Type 1 infinite families $K_m \times K_n$, for $m = 3, 5, 7$ and $n > m$ an odd number, dividing into two steps: when $gcd(m, n) = 1$ in Lemma 4.2 and when $gcd(m, n) = m$ in Lemma 4.3.

Lemma 4.2. *For $m = 3, 5, 7$ and an odd number $n > m$ with $gcd(m, n) = 1$, the graph $K_m \times K_n$ is Type 1.*

Proof. To obtain a $(\Delta(K_m \times K_n) + 1)$-total coloring for the three infinite families $K_m \times K_n$ for $m = 3, 5, 7$ and $n > m$ an odd number with $gcd(m, n) = 1$, first we use the Hamiltonian decomposition of $K_m \times K_n$ defined in Subsection 3.1 Case 1 to construct the three tables respectively with the elements of the guiding color.

- For $m = 3$. The general case for $K_3 \times K_n$, with $n \geq 11$ and $gcd(3, n) = 1$, is presented in Table 1. This case $m = 3$ has 2 omitted particular graphs: $K_3 \times K_5$ (solved in Subsection 3.2, see Figure 4) and $K_3 \times K_7$.

Table 1. Elements of $K_3 \times K_n$ of the guiding color, for $n = 2w + 1,\ n \geq 11$ and $gcd(3, n) = 1$.

Cycle	Edges	Cycle	Edges	Cycle	Edges
$C(1,1)$	$(1,1)(2,2),(1,3)(2,2w-1),(1,2w-2)(2,5)$	$C(1,3)$	$(0,3)(1,4)$	$C(1,2w-2)$	$(2,0)(0,2w-2)$
$C(1,i)$	$(1,i)(2,i+1),\ i = 2,5,6,\dots,2w-3,2w-1$	$C(1,4)$	$(1,0)(2,4)$	$C(1,2w)$	$(1,2w)(2,1)$
Vertices: $(0,i),\ i = 0,\dots,2w, i \neq 3, 2w-2$					

- For $m = 5$. The general case for $K_5 \times K_n$, with $n \geq 17$, $n \neq 21$ and $gcd(5, n) = 1$, is presented in Table 2. This case $m = 5$ has 5 omitted particular graphs: for $n = 7, 9, 11, 13, 21$.

Table 2. Elements of $K_5 \times K_n$ of the guiding color, for $n = 2w + 1,\ n \geq 17, n \neq 21$ and $gcd(5, n) = 1$.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(2,2)(4,2w),(2,2w-2)(4,5),(2,7)(4,2w-5)$	$C(2,1)$	$(3,2)(1,2w)$
$C(1,i)$	$(2,i+1)(4,i-1),\ i = 2,\dots,2w-1, i \neq 6, 2w-4, 2w-3$	$C(2,i)$	$(3,i+1)(1,i-1),\ i = 2,\dots,2w-1,\ i \neq 5, 2w-5, 2w-4$
$C(1,6)$	$(0,w+6)(1,0)$	$C(2,5)$	$(4,0)(0,5)$
$C(1,2w-4)$	$(2,0)(4,2w-4)$	$C(2,2w-5)$	$(3,0)(1,2w-5)$
$C(1,2w-3)$	$(3,2w-4)(0,2w-1)$	$C(2,2w-4)$	$(0,2w-4)(2,2w-3)$
$C(1,2w)$	$(2,1)(4,2w-1)$	$C(2,2w)$	$(3,1)(1,2w-1),(3,2w-3)(1,4),(3,6)(1,2w-6)$
Vertices: $(0,i)$, for $i = 0,\dots,2w, i \neq 5, w+6, 2w-4, 2w-1$			

- For $m = 7$. The general case for $K_7 \times K_n$, with $n \geq 23$, $n \neq 25, 33$ and $gcd(7, n) = 1$, is presented in Table 3. This case $m = 7$ has 8 omitted particular graphs: for $n = 9, 11, 13, 15, 17, 19, 25, 33$.

Thus, the family $K_m \times K_n$, with odd numbers $m = 3, 5, 7,\ n > m$ and $gcd(m, n) = 1$, is Type 1. □

Table 3. Elements of $K_7 \times K_n$ of the guiding color, for $n = 2w + 1,\ n \geq 23,\ n \neq 25, 33$ and $gcd(7, n) = 1$.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(6,2w)(3,3),(6,6)(3,2w-4),(6,2w-7)(3,10)$	$C(2,2w-8)$	$(5,0)(0,2w-8)$
$C(1,i)$	$(6,i-1)(3,i+2),\ i = 2,\dots,2w-2,\ i \neq 7, 8, 2w-6$	$C(2,2w-1)$	$(1,2w-2)(4,1),(1,4)(4,2w-6),(1,2w-9)(4,8)$
$C(1,7)$	$(2,0)(6,7)$	$C(2,2w)$	$(1,2w-1)(4,2)$
$C(1,8)$	$(4,7)(0,10)$	$C(3,1)$	$(2,2w)(5,3)$
$C(1,2w-6)$	$(0,w-6)(1,0)$	$C(3,i)$	$(2,i-1)(5,i+2),\ i = 2,\dots,2w-2,\ i \neq 6, 7, 2w-7$
$C(1,2w-1)$	$(6,2w-2)(3,1)$	$C(3,6)$	$(4,0)(2,6)$
$C(1,2w)$	$(6,2w-1)(3,2)$	$C(3,7)$	$(0,6)(3,9)$
$C(2,1)$	$(1,2w)(4,3)$	$C(3,2w-7)$	$(6,0)(0,2w-7)$
$C(2,i)$	$(1,i-1)(4,i+2),\ i = 2,\dots,2w-2,\ i \neq 5, 6, 2w-8$	$C(3,2w-1)$	$(2,2w-2)(5,1)$
$C(2,5)$	$(3,0)(1,5)$	$C(3,2w)$	$(2,2w-1)(5,2),(2,5)(5,2w-5),(2,2w-8)(5,9)$
$C(2,6)$	$(5,8)(0,4)$		
Vertices: $(0,i),\ i = 0,\dots,2w, i \neq 4, 6, 10, w-6, 2w-8, 2w-7$			

Lemma 4.3. *For $m = 3, 5, 7$ and an odd number $n > m$ with $gcd(m, n) = m$, the graph $K_m \times K_n$ is Type 1.*

Proof. Analogous to the proof of Lemma 4.2, to obtain a $(\Delta(K_m \times K_n) + 1)$-total coloring for the families $K_m \times K_n$, when $m = 3, 5, 7,\ n \geq m$ are odd numbers and $gcd(m, n) = m$, first we use the Hamiltonian decomposition of $K_m \times K_n$ as Subsection 3.1 Case 2 to construct the three tables respectively with the elements of the guiding color.

- For $m = 3$. First, we construct a Hamilton decomposition of $K_3 \times K_n$ as Subsection 3.1 Case 2(i). The general case for $K_3 \times K_n$, with $n \geq 9$ and $gcd(3, n) = 3$, is presented in Table 4. This case $m = 3$ has one omitted particular graph $K_3 \times K_3$.

Table 4. Elements of $K_3 \times K_n$ of the guiding color, for $n = 2w + 1,\ n \geq 9$ and $gcd(3, n) = 3$.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(1,1)(2,2),(1,3)(2,2w-1),(1,2w-2)(2,5)$	$C(1,w-2)$	$(2,w-1)(0,w-3)$
$C(1,i)$	$(1,i)(2,i+1),\ i = 2,5,6,\dots,w-3,w-1,w$	$C(2,i)$	$(2,w+i+1)(1,w+i),\ i = 1,\dots,w-3,w-1$
$C(1,3)$	$(0,3)(1,4)$	$C(2,w-2)$	$(2,0)(1,w-2)$
$C(1,4)$	$(1,0)(2,4)$	$C(2,w)$	$(2,1)(1,2w)$
Vertices: $(0,i),\ i = 0,\dots,2w, i \neq 3, w-3$			

Table 5. Elements of $K_5 \times K_n$ of the guiding color, for $n = 2w+1,\ n \geq 15$ and $gcd(5,n) = 5$.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(2,2)(4,2w),(2,2w-2)(4,5),(2,7)(4,2w-5)$	$C(3,w-4)$	$(3,2w-4)(2,0)$
$C(1,i)$	$(2,i+1)(4,i-1),\ i=2,\ldots,w,\ i\neq 6$	$C(3,w-3)$	$(1,0)(0,w-3)$
$C(1,6)$	$(3,0)(0,6)$	$C(3,w)$	$(4,2w-1)(2,1)$
$C(2,1)$	$(3,2)(1,2w)$	$C(4,i)$	$(1,w+i-1)(3,w+i+1),\ i=1,\ldots,w-1, i\neq w-5,w-4$
$C(2,i)$	$(3,i+1)(1,i-1),\ i=2,\ldots,w, i\neq 5$	$C(4,w-5)$	$(4,2w-4)(1,2w-5)$
$C(2,5)$	$(4,0)(0,5)$	$C(4,w-4)$	$(2,2w-3)(0,2w-4)$
$C(3,i)$	$(4,w+i-1)(2,w+i+1),\ i=1,\ldots,w-1, i\neq w-4,w-3$	$C(4,w)$	$(1,2w-6)(3,6),(1,4)(3,2w-3),(1,2w-1)(3,1)$
Vertices: $(0,i),\ i=0,\ldots,2w,\ i\neq 5,6,w-3,2w-4$			

- For $m = 5$. First, we construct a Hamilton decomposition of $K_5 \times K_n$ as Subsection 3.1 Case 2(ii). The general case for $K_5 \times K_n$, with $n \geq 15$ and $gcd(5,n) = 5$, is presented in Table 5. This case $m = 5$ has one omitted particular graph $K_5 \times K_5$.
- For $m = 7$. First we construct a Hamilton decomposition of $K_7 \times K_n$ as Subsection 3.1 Case 2(iii). The general case for $K_7 \times K_n$, with $n \geq 35$ and $gcd(7,n) = 7$, is presented in Table 6. This case $m = 7$ has 2 omitted particular graphs $K_7 \times K_7$ and $K_7 \times K_{21}$.

Thus, the family $K_m \times K_n$, with odd numbers $m = 3,5,7,\ n > m$ and $gcd(m,n) = m$, is Type 1. □

Table 6. Elements of $K_7 \times K_n$ of the guiding color, for $n = 2w+1,\ n \geq 35$ and $gcd(7,n) = 7$.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(6,2w)(3,3),(6,6)(3,2w-4),(6,2w-7)(3,10)$	$C(4,i)$	$(3,w+i+2)(6,w+i-1),\ i=1,\ldots,w-2, i\neq w-6$
$C(1,i)$	$(6,i-1)(3,i+2),\ i=2,\ldots,w,\ i\neq 7,8$	$C(4,w-6)$	$(1,0)(0,w-6)$
$C(1,7)$	$(2,0)(6,7)$	$C(4,w-1)$	$(3,1)(6,2w-2)$
$C(1,8)$	$(4,7)(0,10)$	$C(4,w)$	$(3,2)(6,2w-1)$
$C(2,1)$	$(1,2w)(4,3)$	$C(5,i)$	$(4,w+i+2)(1,w+i-1),\ i=1,\ldots,w-2, i\neq w-8$
$C(2,i)$	$(1,i-1)(4,i+2),\ i=2,\ldots,w, i\neq 5,6$	$C(5,w-8)$	$(0,2w-8)(6,0)$
$C(2,5)$	$(3,0)(1,5)$	$C(5,w-1)$	$(4,8)(1,2w-9),(4,2w-6)(1,4),(4,1)(1,2w-2)$
$C(2,6)$	$(5,8)(0,4)$	$C(5,w)$	$(4,2)(1,2w-1)$
$C(3,1)$	$(2,2w)(5,3)$	$C(6,i)$	$(5,w+i+2)(2,w+i-1),\ i=1,\ldots,w-2, i\neq w-7$
$C(3,i)$	$(2,i-1)(5,i+2),\ i=2,\ldots,w, i\neq 6,7$	$C(6,w-7)$	$(0,2w-7)(5,0)$
$C(3,6)$	$(4,0)(2,6)$	$C(6,w-1)$	$(5,1)(2,2w-2)$
$C(3,7)$	$(0,6)(3,9)$	$C(6,w)$	$(5,9)(2,2w-8),(5,2w-5)(2,5),(5,2)(2,2w-1)$
Vertices: $(0,i),\ i=0,\ldots,2w,\ i\neq 4,6,10,w-6,2w-8,2w-7$			

4.2. *Families $K_9 \times K_n$ and $K_{11} \times K_n$*

Lemma 4.4. *For $m = 9, 11$ and an odd number $n \geq m$, the graph $K_m \times K_n$ is Type* 1.

Proof. In Section 2, we have actually proved that for odd numbers m,n the graph $K_m \times K_n$ is Type 1, provided that $\Delta(G) \geq \frac{(\sqrt{37}-1)}{6}|V(G)|$.

We show next that $K_9 \times K_n$ with $n \geq 23$ and $K_{11} \times K_n$ with $n \geq 15$ satisfy the required bound. Indeed, for $K_9 \times K_n$, when $n \geq 23$, we have that $n \geq 16/(16 - 3(\sqrt{37}-1))$. Therefore, $8(n-1) \geq \frac{(\sqrt{37}-1)}{6} \cdot 9n$, that is $\Delta(K_9 \times K_n) \geq \frac{(\sqrt{37}-1)}{6} \cdot \mid V(K_9 \times K_n) \mid$. For $K_{11} \times K_n$, when $n \geq 15$, we have that $n \geq 60/(60 - 11(\sqrt{37}-1))$. Therefore, $10(n-1) \geq \frac{(\sqrt{37}-1)}{6} \cdot 11n$, that is $\Delta(K_{11} \times K_n) \geq \frac{(\sqrt{37}-1)}{6} \cdot \mid V(K_{11} \times K_n) \mid$. Thus, we have that for $n \geq 23$, the graph $K_9 \times K_n$ is Type 1 and for $n \geq 15$, the graph $K_{11} \times K_n$ is Type 1.

The omitted particular graphs are $K_9 \times K_n$, for $n = 9, 11, 13, 15, 17, 19, 21$, and $K_{11} \times K_n$, for $n = 11, 13$. □

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