Theoretical Computer Science 410 (2009) 2072–2083

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

The complexity of clique graph recognition ☆

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ARTICLE INFO

Article history:
Received 19 March 2007
Received in revised form 9 December 2008
Accepted 17 January 2009
Communicated by O. Watanabe

Keywords:
Clique graphs
NP-complete problems
Helly property
Intersection graphs

ABSTRACT

A *complete set* of a graph G is a subset of vertices inducing a complete subgraph. A *clique* is a maximal complete set. Denote by $\mathcal{C}(G)$ the *clique family* of G. The *clique graph* of G, denoted by $K(G)$, is the intersection graph of $\mathcal{C}(G)$. Say that G is *a clique graph* if there exists a graph H such that $G = K(H)$. The clique graph recognition problem asks whether a given graph is a clique graph. A sufficient condition was given by Hamelink in 1968, and a characterization was proposed by Roberts and Spencer in 1971. However, the time complexity of the problem of recognizing clique graphs is a long-standing open question. We prove that the clique graph recognition problem is NP-complete.

1. Introduction

Consider finite, simple and undirected graphs. V and E denote the vertex set and the edge set of the graph G, respectively. A *complete set* of G is a subset of V inducing a complete subgraph. A *clique* is a maximal complete set. The *clique family* of G is denoted by $\mathcal{C}(G)$. The *clique graph* of G is the intersection graph of $\mathcal{C}(G)$.

The *clique operator*, K, assigns to each graph G its clique graph which is denoted by $K(G)$. On the other hand, say that G is *a clique graph* if G belongs to the image of the clique operator, i.e. if there exists a graph H such that $G = K(H)$.

Clique operator and its image were widely studied. First articles focused on recognizing clique graphs [20,36]. In [4,13], graphs for which the clique graph changes whenever a vertex is removed are considered. Graphs fixed under the operator K or fixed under the iterated clique operator, K^n, for some positive integer n; and the behavior under these operators of parameters such as the number of vertices or diameter were studied in [5,8,9,12,26,30] and more recently in [7,14,21–23,29]. For several classes of graphs, the image of the class under the clique operator was characterized [10,18,19,24,34,37]; and, in some cases, also the inverse image of the class [16,28,35]. Results of the previous bibliography can be found in the survey [39]. Clique graphs have been much studied as intersection graphs and are included in several books [11,25,33].

In this paper we are concerned with the time complexity of the problem of recognizing clique graphs, this is the time complexity of the following decision problem.

☆ An extended abstract published in: Proceedings of Graph-theoretic concepts in computer science, WG 2006, Lecture Notes in Comput. Sci., vol. 4271, Springer, 2006, pp. 269–277. This research was partially supported by FAPERJ–Cientistas do Nosso Estado project, by PROCIÊNCIA project, and by CNPq Project PROSUL–Proc. no. 490333/04-4.

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0304-3975/$ – see front matter
doi:10.1016/j.tcs.2009.01.018

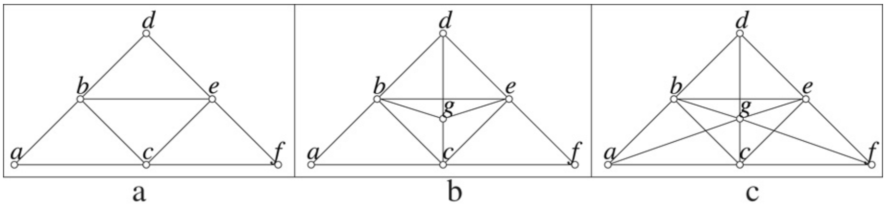

Fig. 1. (a) Non-clique graph thus non-clique-Helly graph; (b) non-clique-Helly graph but clique graph; (c) clique-Helly graph thus clique graph.

CLIQUE GRAPH
INSTANCE: A graph $G = (V, E)$.
QUESTION: Is there a graph H such that $G = K(H)$?

In spite of the characterizations of clique graphs given in [36] and more recently in [1], the time complexity of CLIQUE GRAPH is a long-standing open question [11,32,36,39].

Our main theorem proves that CLIQUE GRAPH is NP-complete by a reduction from a specially chosen version of the 3-satisfiability problem. In Section 2, definitions and basic concepts about clique graphs are presented. Besides, we detail a proof that CLIQUE GRAPH is in NP, and we state the selected 3SAT$_{\bar{3}}$ version of satisfiability problem.
In Section 3, we describe the construction of instance G_I of CLIQUE GRAPH from instance $I = (U, C)$ of 3SAT$_{\bar{3}}$; and analyze some of its properties. In Section 4, we state and prove the main theorem by showing that the constructed graph G_I is a clique graph if and only if C is satisfiable. In Section 5, we have our concluding remarks.

The extended abstract [3] recently published contains the description of the special graph G_I constructed from the satisfiability instance $I = (U, C)$ but omits most of the proof. The present paper presents the full required proof – a difficult and long case analysis – and highlights the properties of the constructed graph G_I for the full understanding of the complexity of the recognition problem and the subsequent study of the problem for special classes of graphs.

2. Definitions and basic concepts

Given a *set family* $\mathcal{F} = (F_i)_{i\in I}$, the sets F_i are called *members* of the family. $F \in \mathcal{F}$ means that F is a member of $\mathcal{F}$. The family is *pairwise intersecting* if the intersection of any two members is not the empty set. The *intersection* or *total intersection* of $\mathcal{F}$ is the set $\bigcap \mathcal{F} = \bigcap_{i\in I} F_i$. The family $\mathcal{F}$ has the *Helly property*, if any pairwise intersecting subfamily has nonempty total intersection.

The edge with end vertices u and v is represented by uv. We say that the complete set C *covers* the edge uv when u and v belong to C. A *complete set edge cover* of a graph G is a family of complete sets of G covering all edges of G.

The following theorem is a well-known characterization of Clique Graphs.

Theorem 1 (*Roberts and Spencer [36]*). *G is a clique graph if and only if there exists a complete set edge cover of G satisfying the Helly property.*

Notice that for any graph G the clique family $\mathcal{C}(G)$ is a complete set edge cover of G, but, in general, this family does not satisfy the Helly property. Graphs such that $\mathcal{C}(G)$ satisfies the Helly property are called *clique-Helly* graphs. It follows from Theorem 1 that every clique-Helly graph is a clique graph. The reciprocal implication is not true: there exist clique graphs which are not clique-Helly graphs. We have depicted in Fig. 1 three examples: (a) a non-clique graph (no complete set edge cover satisfies the Helly property [36]); (b) a clique graph that is not a clique-Helly graph (the clique family does not satisfy the Helly property, but the complete set edge cover $\{a, b, c\}$, $\{c, e, f\}$, $\{b, d, g\}$, $\{d, e, g\}$, $\{b, c, e, g\}$ does); and (c) a clique graph that is a clique-Helly graph (the clique family has the Helly property). Examples given in Fig. 1 also show that being a clique graph or being a clique-Helly graph are not hereditary properties.

In [38], clique-Helly graphs are characterized and a polynomial-time algorithm for their recognition is presented. Next lemma extends that result and leads to a polynomial-time algorithm to check if a given complete set edge cover of a graph satisfies the Helly property; this lemma is used to show that CLIQUE GRAPH is in NP and in the proof of Theorem 8.

A *triangle* is a complete set with exactly 3 vertices. The set of triangles of G is denoted by $T(G)$. Let $\mathcal{F}$ be a complete set edge cover of G and T a triangle, and denote by $\mathcal{F}_T$ the subfamily of $\mathcal{F}$ formed by all the members containing at least two vertices of T.

Lemma 2 (*Alcón and Gutierrez [2]*). *Let $\mathcal{F}$ be a complete set edge cover of G. The following conditions are equivalent:*
(i) *$\mathcal{F}$ has the Helly property.*
(ii) *For every $T \in T(G)$, the subfamily $\mathcal{F}_T$ has the Helly property.*
(iii) *For every $T \in T(G)$, the subfamily $\mathcal{F}_T$ has nonempty intersection, this means $\bigcap \mathcal{F}_T \neq \emptyset$.*

As noted by Roberts and Spencer [36], Theorem 1 yields a polynomial certificate of G being a clique graph. First, for the polynomial size of the edge cover certificate, note that if $\mathcal{F}$ has the Helly property, then every subfamily $\mathcal{F}'$ of $\mathcal{F}$ has the Helly property as well. In addition, we prove that if G admits a complete set edge cover $\mathcal{F}$, then G admits a complete set edge cover $\mathcal{F}'$ of size at most $|E|$ which is our considered certificate: just sequentially scan the edges of E, select for $\mathcal{F}'$

one complete set of $\mathcal{F}$ covering the first edge, and for each edge e not yet covered by $\mathcal{F}'$, select for $\mathcal{F}'$ one complete set of $\mathcal{F}$ covering e. Clearly this procedure labels each selected set with a corresponding scanned edge of E, yielding a subfamily $\mathcal{F}'$ of size at most $|E|$. Notice, in addition, that no member of $\mathcal{F}'$ is contained in another one. Second, for the polynomial verification of the certificate, a result of Berge [6] says that a family of sets has the Helly property, if and only if for any triple of elements, the subfamily of sets containing at least two out of these three elements has nonempty intersection. Actually, by Lemma 2, it is enough to consider the triples of vertices a, b, c of G defining a triangle T. We consider the members of $\mathcal{F}'_T$ and check for every vertex v of V if v belongs to $\bigcap \mathcal{F}'_T$. This produces an $O(|V|^4|E|)$ algorithm that checks if a complete set edge cover $\mathcal{F}'$ of size $O(|E|)$ is Helly. Thus CLIQUE GRAPH belongs to NP.

A consequence of the above analysis is that a graph admits a complete set edge cover with the Helly property if and only if the graph admits a complete set edge cover with the Helly property such that no member is contained in another; such cover is called an *RS-family* of the graph. Thus Theorem 1 is equivalent to the following simpler statement:

G is a clique graph if and only if G admits an RS-family.

The following two facts are stated and proved by Roberts and Spencer [36] and as we explain after each statement, they are explicitly used in our NP-completeness proof.

Fact 3 (*Lemma* 1 *of [36]*). *Let $\mathcal{F}$ be an RS-family of a graph G. Then $\mathcal{F}$ contains a complete set of size* 2 *if and only if this complete set is a clique of G.*

Our constructed instance of CLIQUE GRAPH is a graph where every edge is in a triangle, which means that no complete set of size 2 is a clique. In our proof we apply the definition of RS-family and Fact 3 to our constructed graph to know that any possible RS-family does not contain complete sets of size 1 or 2.

Fact 4 (*Proof of Theorem* 3 *of [36]*). *If a triangle T is a clique of G, then T is a member of every complete set edge cover of G that satisfies the Helly property.*

Our constructed instance of CLIQUE GRAPH has many auxiliary vertices of degree 2. Each vertex of degree 2 is contained in exactly one corresponding auxiliary triangle T, which implies that this auxiliary triangle T is a clique. In our proof we use Fact 4 to know that every such auxiliary triangle T of our constructed graph is a member of any possible RS-family.

We show that CLIQUE GRAPH is NP-complete by a reduction from the following version of the 3-satisfiability problem with at most 3 occurrences per variable. Let $U = \{u_i, 1 \le i \le n\}$ be a set of boolean variables. A literal is either a variable u_i or its complement $\overline{u_i}$. A clause over U is a set of literals of L. Let $C = \{c_j, 1 \le j \le m\}$ be a collection of clauses over U. We say that variable u_i occurs in clause c_j (and then in C) if u_i or $\overline{u_i} \in c_j$. We say that variable u_i occurs in clause c_j as literal u_i (or that literal u_i occurs in c_j) if $u_i \in c_j$, and as literal $\overline{u_i}$ (or that literal $\overline{u_i}$ occurs in c_j) if $\overline{u_i} \in c_j$.

3SAT$_{\overline{3}}$

INSTANCE: $I = (U, C)$, where $U = \{u_i, 1 \le i \le n\}$ is a set of boolean variables, and $C = \{c_j, 1 \le j \le m\}$ a set of clauses over U such that each clause has two or three variables, each variable occurs at most three times in C.

QUESTION: Is there a truth assignment for U such that each clause in C has at least one true literal?

It is a known result that 3SAT$_{\overline{3}}$ is an NP-complete problem [15,27]. Note that the instances of the chosen version of 3SAT must satisfy a linear dependency between parameters n and m: every 3SAT$_{\overline{3}}$ instance satisfies $2m \le 3n$.

In order to reduce 3SAT$_{\overline{3}}$ to CLIQUE GRAPH we need to construct in polynomial time a particular instance G_I of CLIQUE GRAPH from a generic instance $I = (U, C)$ of 3SAT$_{\overline{3}}$, in such a way that the constructed graph G_I is a clique graph if and only if C is satisfiable.

3. Construction of G_I from $I = (U, C)$

Let $I = (U, C)$ be any instance of 3SAT$_{\overline{3}}$. We assume with no loss of generality that each variable occurs two or three times in C, and no variable occurs twice in the same clause. In addition, if variable u_i occurs twice in C, then we assume it is once as literal u_i and once as literal $\overline{u_i}$; and if variable u_i occurs three times in C, then we assume it is once as literal u_i and twice as literal $\overline{u_i}$.

For each variable u_i, let j_i be the subindex of the unique clause where variable u_i occurs as literal u_i; and $\bar{J}_i = \{j \mid$ literal $\overline{u_i}$ occurs in $c_j\}$.

For each clause c_j with $|c_j| = 3$, let $I_j = \{i \mid$ variable u_i occurs in $c_j\}$; and for each clause c_j with $|c_j| = 2$, let $I_j = \{i \mid$ variable u_i occurs in $c_j\} \cup \{n+1\}$. Notice that in any case $|I_j| = 3$. Given $I_j = \{i_1, i_2, i_3\}$, with $i_1 < i_2 < i_3$, let $i_1^* = i_2, i_2^* = i_3$ and $i_3^* = i_1$.

From instance $I = (U, C)$, we construct a graph $G_I = (V, E)$ as follows (we give immediately after an easier alternative modular description of the graph G_I):

The vertex set V is the union:

$$V = \bigcup_{1 \le i \le n} \{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}, e^i_{j_i}, f^i_{j_i}, g^i_{j_i}, h^i_{j_i}\} \cup \bigcup_{1 \le i \le n} \bigcup_{j \in \bar{J}_i} \{a^i_j, c^i_j, d^i_j, e^i_j, f^i_j, g^i_j, h^i_j, z^i_j, v^i_j, w^i_j\}$$
$$\cup \bigcup_{1 \le j \le m, |c_j|=2} \{a^{n+1}_j, c^{n+1}_j, d^{n+1}_j, e^{n+1}_j, f^{n+1}_j, g^{n+1}_j, h^{n+1}_j\}.$$

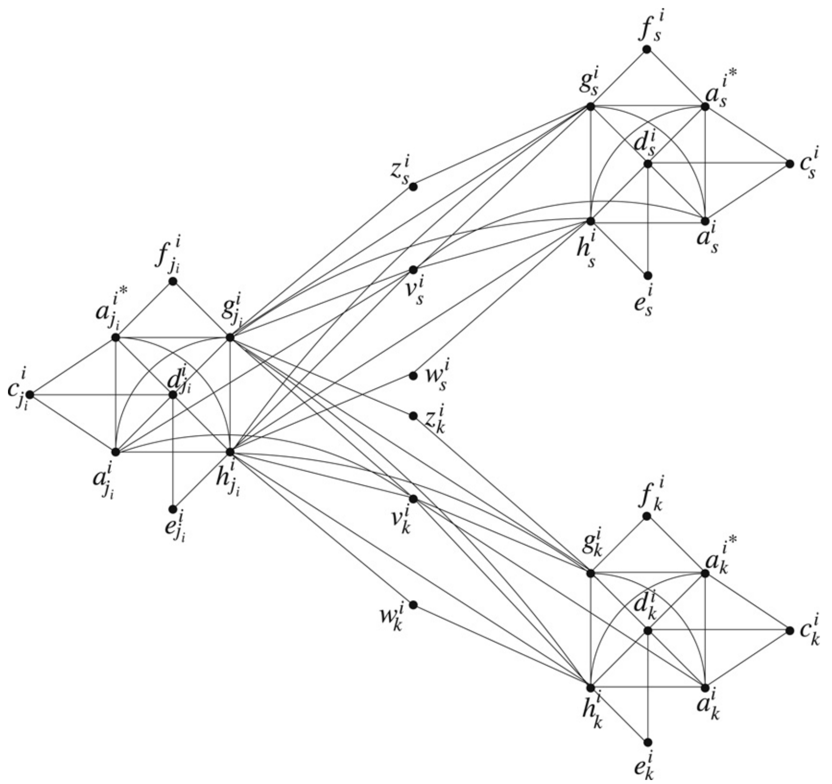

Fig. 2. Truth Setting component T_i for variable u_i with $\bar{J}_i = \{s, k\}$.

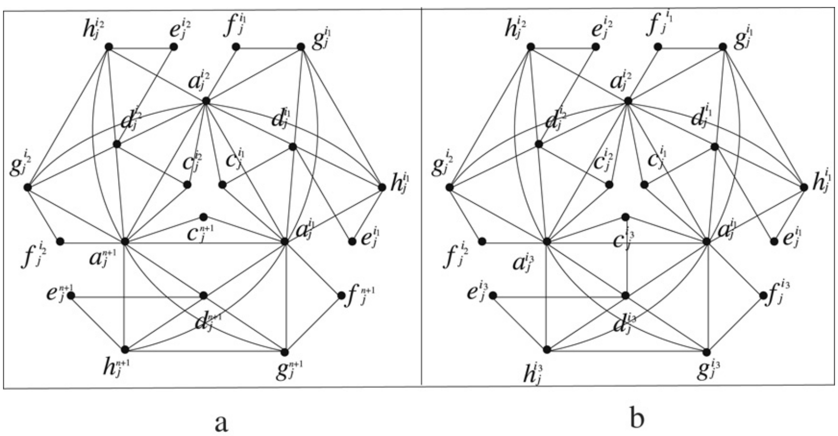

Fig. 3. Satisfaction Testing component S_j for clause c_j with: (a) 2 literals corresponding to variables u_{i_1} and u_{i_2}; and (b) 3 literals corresponding to variables u_{i_1}, u_{i_2} and u_{i_3}. Vertices $\{a_j^i, d_j^i, g_j^i, h_j^i \mid i \in I_j\}$ induce a complete graph on 12 vertices but for simplicity some edges are not drawn.

Since $|\bar{J}_i| \leq 2$, $|V|$ is bounded by $n \times 7 + n \times 2 \times 10 + m \times 7 = 27n + 7m$. Actually, the linear dependency $2m \leq 3n$ between parameters n and m gives a linear upper bound on n: $|V|$ is bounded by $27n + 7m \leq 27n + \frac{21}{2}n = \frac{75}{2}n$.

The edge set E contains:

For each j, $1 \leq j \leq m$, the edges of the complete graph induced by the vertex set $K_{12}(j) = \{a_j^i, d_j^i, g_j^i, h_j^i \mid i \in I_j\}$; the edges of the sets $\{c_j^i d_j^i \mid i \in I_j, i \neq n+1\}$ and $\{c_j^i a_j^i, c_j^i a_j^{i^*}, e_j^i d_j^i, e_j^i h_j^i, f_j^i g_j^i, f_j^i a_j^{i^*} \mid i \in I_j\}$.

And for each i, $1 \leq i \leq n$, for each $j \in \bar{J}_i$, the edges of the complete graph induced by the vertex set $K_5(j, i) = \{h_{j_i}^i, g_{j_i}^i, v_j^i, h_j^i, g_j^i\}$; and the edges of the set $\{h_{j_i}^i w_j^i, w_j^i h_j^i, g_{j_i}^i z_j^i, z_j^i g_j^i, a_{j_i}^i v_j^i, v_j^i a_j^i\}$.

Notice that for each variable u_i, graph G_I contains an induced subgraph, Truth Setting component T_i, the graph depicted in Fig. 2; and for each clause c_j, graph G_I contains as induced subgraph, Satisfaction Testing component S_j, the graph depicted either in Fig. 3(a) or (b), depending on the number of variables in clause c_j. Notice that in that figure some edges have been omitted for simplicity. A subgraph T_i intersects a subgraph S_j if and only if variable u_i occurs in clause c_j; and, in that case, the

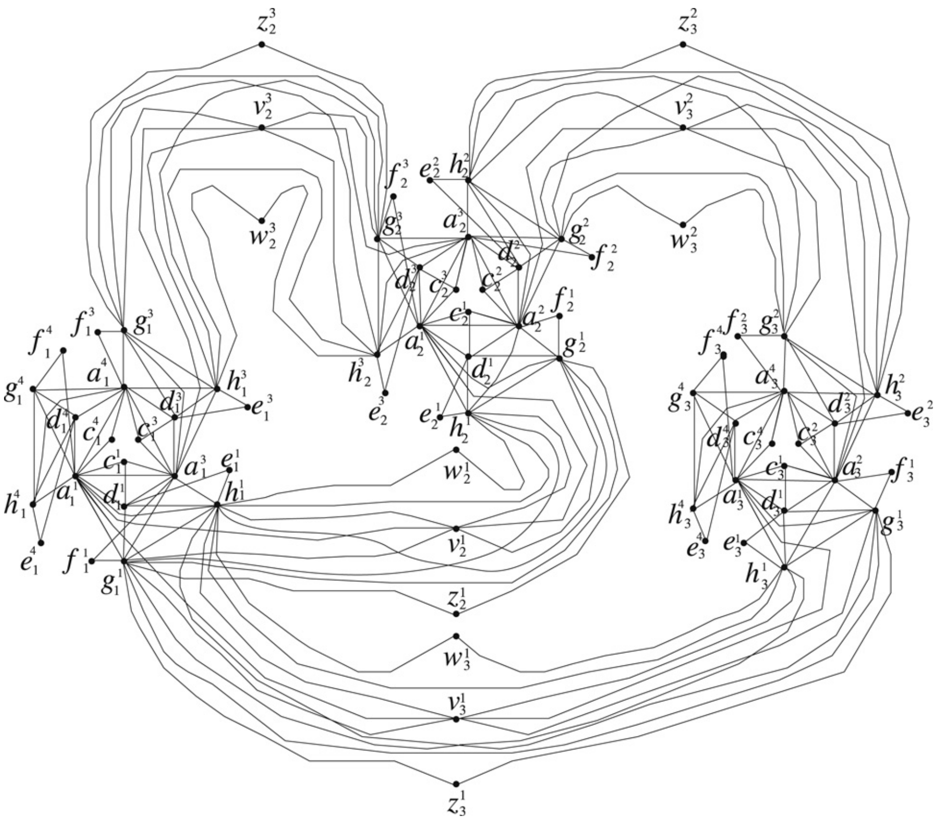

Fig. 4. Graph G_I obtained from the $3\text{SAT}_{\overline{3}}$ instance $I = (U, C)$, $U = \{u_1, u_2, u_3\}$, $C = \{\{u_1, u_3\}, \{\overline{u_1}, u_2, \overline{u_3}\}, \{\overline{u_1}, \overline{u_2}\}\}$.

intersection is the subgraph induced by vertices $a^i_j, a^{i^*}_j, c^i_j, d^i_j, e^i_j, f^i_j, g^i_j$, and h^i_j. We obtain the whole graph G_I by superposing the subgraphs T_i and S_j, $1 \le i \le n$, $1 \le j \le m$.

For the convenience of the reader we offer an example in Fig. 4 of graph G_I obtained from the instance $I = (U, C)$, $U = \{u_1, u_2, u_3\}$, $C = \{\{u_1, u_3\}, \{\overline{u_1}, u_2, \overline{u_3}\}, \{\overline{u_1}, \overline{u_2}\}\}$.

In Section 3.1, we shall prove some properties about the RS-families of the constructed graph G_I. Note that in the constructed graph G_I, every edge is in a triangle, so by Fact 3 we know that every complete set in an RS-family of G_I has size at least 3. Note the presence in the constructed graph G_I of auxiliary vertices of degree 2: $e^i_j, f^i_j, w^i_j, z^i_j$. The auxiliary triangles containing those auxiliary degree 2 vertices are cliques and by Fact 4 must be present in every RS-family of the constructed graph G_I. For the convenience of the reader, we list all cliques of the constructed graph G_I.

Fact 5. *The cliques of G_I can be listed as follows:*

- *For each j, $1 \le j \le m$, the complete set $K_{12}(j) = \{a^i_j, d^i_j, g^i_j, h^i_j \mid i \in I_j\}$;*
- *For each i, $1 \le i \le n$, for each $j \in \bar{J}_i$, the complete set $K_5(j, i) = \{h^i_{j_i}, g^i_{j_i}, v^i_j, h^i_j, g^i_j\}$;*
- *For each j, $1 \le j \le m$, $i \in I_j$, $i \ne n + 1$, one complete set K_4 containing c^i_j: $\{a^i_j, c^i_j, d^i_j, a^{i^*}_j\}$;*
- *For each i, $1 \le i \le n$, for each $j \in \bar{J}_i$, two complete sets K_4 containing v^i_j: $\{a^i_{j_i}, h^i_{j_i}, g^i_{j_i}, v^i_j\}$, $\{a^i_j, h^i_j, g^i_j, v^i_j\}$;*
- *All triangles with a vertex of degree 2. For each j, $1 \le j \le m$, for each $i \in I_j$, the triangles $\{f^i_j, a^{i^*}_j, g^i_j\}$, $\{e^i_j, d^i_j, h^i_j\}$; and the triangle $\{c^{n+1}_j, a^{n+1^*}_j, a^{n+1}_j\}$ when $n + 1 \in I_j$. For each i, $1 \le i \le n$, for each $j \in \bar{J}_i$, the triangles $\{g^i_{j_i}, z^i_j, g^i_j\}$, $\{h^i_{j_i}, w^i_j, h^i_j\}$.*

3.1. About RS-families of graph G_I

Our main theorem is proved in the next section by showing that the constructed graph G_I admits an RS-family if and only if there exists a truth assignment for U that satisfies C. The following lemmata are used in that proof when given an RS-family of G_I we exhibit a truth assignment for U that satisfies C. The truth value of each variable u_i will be assigned depending on the member of the RS-family of G_I covering the edge $a^i_{j_i} c^i_{j_i}$.

For the convenience of the reader, we recall our notation. The notation explicitly defines a correspondence between indices of variables and clauses of instance I, and the vertices of the constructed graph G_I. For each variable u_i, let j_i be the subindex of the unique clause where variable u_i occurs as literal u_i; and $\bar{J}_i = \{j \mid \text{ literal } \overline{u_i} \text{ occurs in } c_j\}$. For each clause c_j with

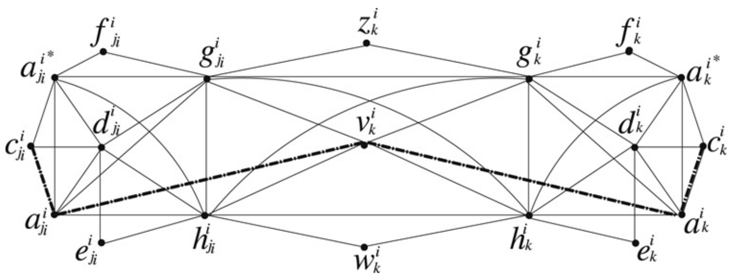

Fig. 5. Figure used in the proof of Lemma 6. Any RS-family of G_I must contain the highlighted triangles. In order to cover highlighted edge $a^i_{j_i}c^i_{j_i}$, exactly one of the triangles $\{a^i_{j_i}, c^i_{j_i}, a^{i^*}_{j_i}\}$ or $\{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}\}$ must belong to $\mathcal{F}$.

$|c_j| = 3$, let $I_j = \{i \mid$ variable u_i occurs in $c_j\}$; and for each clause c_j with $|c_j| = 2$, let $I_j = \{i \mid$ variable u_i occurs in $c_j\} \cup \{n+1\}$. Notice that in any case $|I_j| = 3$. Given $I_j = \{i_1, i_2, i_3\}$, with $i_1 < i_2 < i_3$, let $i^*_1 = i_2$, $i^*_2 = i_3$ and $i^*_3 = i_1$.

Notice that, in case $i = n+1$, vertex c^{n+1}_j is of degree 2 contained in precisely one triangle $\{a^{n+1}_j, c^{n+1}_j, a^{(n+1)^*}_j\}$ which is the only clique covering edge $a^{n+1}_j c^{n+1}_j$. From Fact 4, triangle $\{a^{n+1}_j, c^{n+1}_j, a^{(n+1)^*}_j\}$ must belong to any RS-family. Lemma 6 considers the possible ways an RS-family may cover edge $a^i_j c^i_j$, when $i \neq n+1$.

Lemma 6 (*Two Cover Lemma*). *Let $\mathcal{F}$ be an RS-family of the graph G_I. For each j, $1 \le j \le m$, and for each $i \in I_j$, $i \neq n+1$, exactly one of the triangles $\{a^i_j, c^i_j, a^{i^*}_j\}$, $\{a^i_j, c^i_j, d^i_j\}$ belongs to $\mathcal{F}$.*

Proof. Consider any j, $1 \le j \le m$, and $i \in I_j$, $i \neq n+1$. Assume with no loss of generality, $j = j_i$, and refer to the left side of Fig. 5. Indeed, to prove the case $j \in \bar{J}_i$, refer to the right side of Fig. 5 accordingly.

We have to prove that exactly one of the triangles $\{a^i_{j_i}, c^i_{j_i}, a^{i^*}_{j_i}\}$, $\{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}\}$ of Fig. 5 belongs to $\mathcal{F}$.

Notice that, by Fact 4, triangles $\{d^i_{j_i}, e^i_{j_i}, h^i_{j_i}\}$ and $\{a^{i^*}_{j_i}, f^i_{j_i}, g^i_{j_i}\}$ highlighted in Fig. 5 belong to $\mathcal{F}$, as $e^i_{j_i}$ and $f^i_{j_i}$ are vertices of degree 2 in G_I.

From Facts 3 and 5, in order to cover the highlighted edge $\mathbf{a}^i_{j_i}\mathbf{c}^i_{j_i}$, at least one of the three complete sets $\{\mathbf{a}^i_{j_i}, \mathbf{c}^i_{j_i}, a^{i^*}_{j_i}, d^i_{j_i}\}$, $\{\mathbf{a}^i_{j_i}, \mathbf{c}^i_{j_i}, a^{i^*}_{j_i}\}$ or $\{\mathbf{a}^i_{j_i}, \mathbf{c}^i_{j_i}, d^i_{j_i}\}$ must belong to $\mathcal{F}$.

We claim that the complete set $\{\mathbf{a}^i_{j_i}, \mathbf{c}^i_{j_i}, a^{i^*}_{j_i}, d^i_{j_i}\} \notin \mathcal{F}$.

Indeed, from Facts 3 and 5, in order to cover the highlighted edge $\mathbf{a}^i_{j_i}\mathbf{v}^i_k$, at least one of the three complete sets $\{\mathbf{a}^i_{j_i}, g^i_{j_i}, \mathbf{v}^i_k, h^i_{j_i}\}$, $\{\mathbf{a}^i_{j_i}, g^i_{j_i}, \mathbf{v}^i_k\}$, or $\{\mathbf{a}^i_{j_i}, h^i_{j_i}, \mathbf{v}^i_k\}$ must belong to $\mathcal{F}$. Now, assuming $\{\mathbf{a}^i_{j_i}, \mathbf{c}^i_{j_i}, a^{i^*}_{j_i}, d^i_{j_i}\} \in \mathcal{F}$, implies that no one of these three complete sets is a member of $\mathcal{F}$, as each of the following three subfamilies:
$\{\{\mathbf{a}^i_{j_i}, g^i_{j_i}, \mathbf{v}^i_k, h^i_{j_i}\}, \{\mathbf{a}^i_{j_i}, \mathbf{c}^i_{j_i}, a^{i^*}_{j_i}, d^i_{j_i}\}, \{a^{i^*}_{j_i}, f^i_{j_i}, g^i_{j_i}\}\}$,
$\{\{\mathbf{a}^i_{j_i}, g^i_{j_i}, \mathbf{v}^i_k\}, \{\mathbf{a}^i_{j_i}, \mathbf{c}^i_{j_i}, a^{i^*}_{j_i}, d^i_{j_i}\}, \{a^{i^*}_{j_i}, f^i_{j_i}, g^i_{j_i}\}\}$, and
$\{\{\mathbf{a}^i_{j_i}, h^i_{j_i}, \mathbf{v}^i_k\}, \{\mathbf{a}^i_{j_i}, \mathbf{c}^i_{j_i}, a^{i^*}_{j_i}, d^i_{j_i}\}, \{d^i_{j_i}, e^i_{j_i}, h^i_{j_i}\}\}$ violate the Helly property.

It follows that $\{a^i_{j_i}, c^i_{j_i}, a^{i^*}_{j_i}, d^i_{j_i}\} \notin \mathcal{F}$, and that it remains to prove that both triangles $\{a^i_{j_i}, c^i_{j_i}, a^{i^*}_{j_i}\}$ and $\{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}\}$ are not together in $\mathcal{F}$.

Indeed, again by considering the highlighted edge $\mathbf{a}^i_{j_i}\mathbf{v}^i_k$, the presence of triangles: $\{a^{i^*}_{j_i}, f^i_{j_i}, g^i_{j_i}\}$, $\{d^i_{j_i}, e^i_{j_i}, h^i_{j_i}\}$, $\{a^i_{j_i}, c^i_{j_i}, a^{i^*}_{j_i}\}$, and $\{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}\}$ together in $\mathcal{F}$ implies that none of the three complete sets that may possibly cover $\mathbf{a}^i_{j_i}\mathbf{v}^i_k$ is a member of $\mathcal{F}$, as each of the following three subfamilies:
$\{\{\mathbf{a}^i_{j_i}, g^i_{j_i}, \mathbf{v}^i_k, h^i_{j_i}\}, \{a^i_{j_i}, c^i_{j_i}, a^{i^*}_{j_i}\}, \{a^{i^*}_{j_i}, f^i_{j_i}, g^i_{j_i}\}\}$,
$\{\{\mathbf{a}^i_{j_i}, g^i_{j_i}, \mathbf{v}^i_k\}, \{a^i_{j_i}, c^i_{j_i}, a^{i^*}_{j_i}\}, \{a^{i^*}, f^i_{j_i}, g^i_{j_i}\}\}$, and
$\{\{\mathbf{a}^i_{j_i}, h^i_{j_i}, \mathbf{v}^i_k\}, \{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}\}, \{d^i_{j_i}, e^i_{j_i}, h^i_{j_i}\}$ violate the Helly property.

It follows that exactly one of the triangles $\{a^i_{j_i}, c^i_{j_i}, a^{i^*}_{j_i}\}$ or $\{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}\}$ belongs to $\mathcal{F}$; and the proof is completed. □

Lemma 7 (*Literal Communication Lemma*). *Let $\mathcal{F}$ be an RS-family of the graph G_I. For each i, $1 \le i \le n$, and for each $j \in \bar{J}_i$, if $\{a^i_j, c^i_j, d^i_j\} \in \mathcal{F}$ then $\{a^i_{j_i}, c^i_{j_i}, a^{i^*}_{j_i}\} \in \mathcal{F}$, (Fig. 6(a)), and if $\{a^i_j, c^i_j, a^{i^*}_j\} \in \mathcal{F}$ then $\{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}\} \in \mathcal{F}$, (Fig. 6(b)).*

Proof. Please refer to Fig. 6. Consider any i, $1 \le i \le n$, and $j \in \bar{J}_i$. First assume $\{a^i_j, c^i_j, d^i_j\} \in \mathcal{F}$ and refer to the right side of Fig. 6(a), where the assumed triangle $\{a^i_j, c^i_j, d^i_j\}$ is a filled triangle.

Notice that, by Fact 4, triangles $\{e^i_j, h^i_j, d^i_j\}$, $\{g^i_{j_i}, g^i_j, z^i_j\}$, $\{e^i_{j_i}, h^i_{j_i}, d^i_{j_i}\}$ highlighted in Fig. 6 belong to $\mathcal{F}$, as e^i_j, z^i_j and $e^i_{j_i}$ are vertices of degree 2 in G_I.

As in the proof of the previous Lemma, in order to cover edge $\mathbf{a}^i_j\mathbf{v}^i_j$, the complete set $\{\mathbf{a}^i_j, g^i_j, \mathbf{v}^i_j\}$ must be in $\mathcal{F}$, as each of the following two subfamilies:
$\{\{\mathbf{a}^i_j, h^i_j, g^i_j, \mathbf{v}^i_j\}, \{a^i_j, c^i_j, d^i_j\}, \{e^i_j, h^i_j, d^i_j\}\}$, and
$\{\{\mathbf{a}^i_j, h^i_j, \mathbf{v}^i_j\}, \{a^i_j, c^i_j, d^i_j\}, \{e^i_j, h^i_j, d^i_j\}\}$ violate the Helly property.

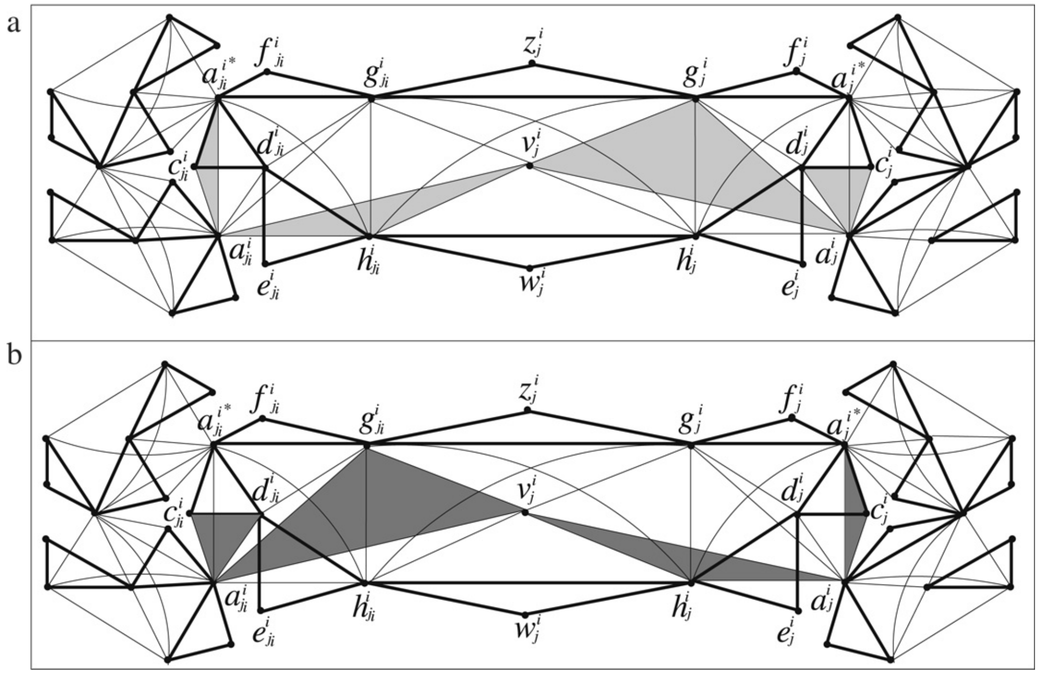

Fig. 6. Figure used in the proof of Lemma 7. For any variable u_i and $j \in \bar{J}_i$, any RS-family of G_I must contain either the filled triangles in (a) or the filled triangles in (b). In any case the bold triangles must belong to the RS-family.

Now, again, in order to cover the edge $\mathbf{a}^i_{j_i}\mathbf{v}^i_j$, the complete set $\{\mathbf{a}^i_{j_i}, h^i_{j_i}, \mathbf{v}^i_j\}$ must be in $\mathcal{F}$, as each of the following two subfamilies:
$\{\{\mathbf{a}^i_{j_i}, g^i_{j_i}, \mathbf{v}^i_j, h^i_{j_i}\}, \{a^i_j, g^i_j, v^i_j\}, \{g^i_{j_i}, g^i_j, z^i_j\}\}$, and
$\{\{\mathbf{a}^i_{j_i}, g^i_{j_i}, \mathbf{v}^i_j\}, \{a^i_j, g^i_j, v^i_j\}, \{g^i_{j_i}, g^i_j, z^i_j\}\}$ violate de Helly property.

So far, the assumption of triangle $\{a^i_j, c^i_j, d^i_j\} \in \mathcal{F}$, forced that the triangles $\{a^i_j, g^i_j, v^i_j\}$ and $\{a^i_{j_i}, h^i_{j_i}, v^i_j\}$ are both in $\mathcal{F}$. Note these three triangles are filled triangles in Fig. 6(a).

Please refer to the left side of Fig. 6(a). By the previous lemma, we know that precisely one of the two triangles $\{a^i_{j_i}, c^i_{j_i}, a^{i^*}_{j_i}\}$ or $\{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}\}$ belongs to $\mathcal{F}$.

Since the assumption of $\{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}\} \in \mathcal{F}$ implies a subfamily $\{\{a^i_{j_i}, h^i_{j_i}, v^i_j\}, \{e^i_{j_i}, d^i_{j_i}, h^i_{j_i}\}, \{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}\}\}$ that violates de Helly property, we conclude that $\{a^i_{j_i}, c^i_{j_i}, a^{i^*}_{j_i}\} \in \mathcal{F}$. Note, this is the fourth filled triangle in Fig. 6(a).

To get the second implication, an analogous argument refers to Fig. 6(b), starts with the assumption of $\{a^i_j, c^i_j, a^{i^*}_j\} \in \mathcal{F}$, a filled triangle on the right side of Fig. 6(b), and obtains as a consequence and in turn the other three filled triangles of Fig. 6(b): $\{a^i_j, h^i_j, v^i_j\}$, $\{a^i_{j_i}, g^i_{j_i}, v^i_j\}$, and finally $\{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}\}$. □

These two lemmata are the basis of the proof of the main theorem. The first implies that given any variable u_i and any clause c_j where u_i occurs, any RS-family of G_I is forced to choose exactly one of the triangles $\{a^i_j, a^{i^*}_j, c^i_j\}$, $\{a^i_j, c^i_j, d^i_j\}$ to cover the edge $a^i_jc^i_j$. The second lemma implies that if c_k and c_{j_i} are clauses where variable u_i occurs as different literals, then any RS-family of G_I is forced to choose different types of triangles to cover respectively the edges $a^i_kc^i_k$ and $a^i_{j_i}c^i_{j_i}$. It follows from the structure of the Truth Setting component T_i, that if c_s and c_k are clauses where variable u_i occurs as the same literal, then any RS-family of G_I is forced to choose the same type of triangle to cover the edges $a^i_sc^i_s$ and $a^i_kc^i_k$, respectively.
The correspondence between the two possible truth assignments of variable u_i and the two possible triangles used to cover the edge $a^i_{j_i}c^i_{j_i}$ is clear.

4. Main theorem

Theorem 8. CLIQUE GRAPH *is NP-complete.*

Proof. As shown in Section 2, CLIQUE GRAPH belongs to NP.

Given any instance $I = (U, C)$ of $3\text{SAT}_{\overline{3}}$, let G_I be the graph obtained by Section 3 process. We show that G_I is a clique graph if and only if C is satisfiable.

First, suppose G_I is a clique graph and let $\mathcal{F}$ be an RS-family for G_I. We exhibit a truth assignment for U that satisfies C:
For each variable $u_i \in U$, set

$$u_i \text{ equal to true if and only if } \{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}\} \in \mathcal{F}. \tag{1}$$

To see that this truth assignment satisfies C consider a clause c_j.

Please refer to Fig. 3. The Helly property of $\mathcal{F}$ implies there exists $i \in I_j$ such that the triangle $\{a^i_j, c^i_j, a^{i^*}_j\} \notin \mathcal{F}$; notice that $i \neq n+1$, because c^{n+1}_j is a vertex of degree 2 in G_I. It follows, by the Two Cover Lemma, that

$$\{a^i_j, c^i_j, d^i_j\} \in \mathcal{F}. \quad (2)$$

Observe that, since $i \neq n+1$, subindex i corresponds to a variable u_i which occurs in clause c_j. There are two possibilities:

$\mathbf{u_i}$ **occurs as literal** $\mathbf{u_i}$ **in** $\mathbf{c_j}$: then $j = j_i$ and condition (2) says $\{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}\} \in \mathcal{F}$. Then, by (1), variable u_i is true; and clause c_j is satisfied.

$\mathbf{u_i}$ **occurs as literal** $\overline{\mathbf{u_i}}$ **in** $\mathbf{c_j}$: then $j \in \bar{J}_i$.
By condition (2) and Literal Communication Lemma, $\{a^i_{j_i}, a^{i^*}_{j_i}, c^i_{j_i}\} \in \mathcal{F}$; then, by Two Cover Lemma, $\{a^i_{j_i}, c^i_{j_i}, d^i_{j_i}\} \notin \mathcal{F}$; thus, by (1), u_i is false; and clause c_j is satisfied.

Conversely, given a truth assignment of U that satisfies C, we exhibit a complete set edge cover $\mathcal{F}$ of G_I.

For each j, $1 \leq j \leq m$, complete set $K_{12}(j) = \{a^i_j, d^i_j, g^i_j, h^i_j \mid i \in I_j\}$.

For each j, $1 \leq j \leq m$, for each $i \in I_j$, the triangles $\{f^i_j, a^{i^*}_j, g^i_j\}$, $\{e^i_j, d^i_j, h^i_j\}$.

For each j, $1 \leq j \leq m$, for each $i \in I_j$, $i \neq n+1$, $\{c^i_j, a^{i^*}_j, d^i_j\}$; and for $i = n+1$, $\{c^{n+1}_j, a^{(n+1)^*}_j, a^{n+1}_j\}$.

For each i, $1 \leq i \leq n$, for each $j \in \bar{J}_i$, the complete set $K_5(j, i) = \{h^i_{j_i}, g^i_{j_i}, v^i_j, h^i_j, g^i_j\}$.

For each i, $1 \leq i \leq n$, for each $j \in \bar{J}_i$, $\{g^i_{j_i}, z^i_j, g^i_j\}$, $\{h^i_{j_i}, w^i_j, h^i_j\}$.

For each i, $1 \leq i \leq n$, such that variable u_i is true, $\{c^i_{j_i}, d^i_{j_i}, a^i_{j_i}\}$; and for each $j \in \bar{J}_i$, $\{a^i_{j_i}, g^i_{j_i}, v^i_j\}$, $\{v^i_j, h^i_j, a^i_j\}$, $\{a^i_j, a^{i^*}_j, c^i_j\}$.

For each i, $1 \leq i \leq n$, such that variable u_i is false, $\{c^i_{j_i}, a^{i^*}_{j_i}, a^i_{j_i}\}$; and for each $j \in \bar{J}_i$, $\{a^i_{j_i}, h^i_{j_i}, v^i_j\}$, $\{v^i_j, g^i_j, a^i_j\}$, $\{a^i_j, d^i_j, c^i_j\}$.

Please refer to Fig. 6. Notice that the triangles depicted in bold are present in the above defined complete set edge cover regardless of the truth assignment. On the other hand, according to the truth assignment of variable u_i, in the above defined complete set edge cover, there are the filled triangles depicted in either Fig. 6(a) or (b). Observe that, when $i \neq n+1$, according to the truth assignment of variable u_i, precisely one of the triangles $\{c^i_{j_i}, d^i_{j_i}, a^i_{j_i}\}$ or $\{c^i_{j_i}, a^{i^*}_{j_i}, a^i_{j_i}\}$ is selected in order to cover the edge $a^i_{j_i}c^i_{j_i}$. On the other hand, when $i = n+1$, the triangle $\{c^{n+1}_j, a^{(n+1)^*}_j, a^{n+1}_j\}$ is always selected to cover the edge $a^{n+1}_j c^{n+1}_j$, regardless of the truth assignment.

Observe that in particular given a 2-sized clause $c_j = \{u_{i_1}, u_{i_2}\}$, $i_1 < i_2$, a satisfiable truth assignment must set u_{i_1} or u_{i_2} to true, hence the complete set edge cover $\mathcal{F}$ defines the forced triangles $\{c^{n+1}_j, a^{n+1}_j, a^{i_1}_j\}$, $\{a^{i_2}_j, d^{i_1}_j, c^{i_1}_j\}$, $\{a^{n+1}_j, d^{i_2}_j, c^{i_2}_j\}$, and according to u_{i_1} or u_{i_2} be set to true respectively triangles $\{a^{i_1}_j, d^{i_1}_j, c^{i_1}_j\}$ or $\{a^{i_2}_j, d^{i_2}_j, c^{i_2}_j\}$ in the corresponding Satisfaction Testing component S_j.

The proof is concluded by showing that the complete set edge cover $\mathcal{F}$ of G_I has the Helly property. By Lemma 2, it is enough to show that for each triangle $T \in T(G_I)$, $\bigcap \mathcal{F}_T \neq \emptyset$.

If a triangle T contains an edge e for which any complete set of $\mathcal{F}$ covering e contains also T, then $\bigcap \mathcal{F}_T \neq \emptyset$. We call such a triangle an *easy triangle*. Note, in particular, that if T is a triangle of G_I with a vertex of degree 2, and e is any of the two edges of T incident to the vertex of degree 2, then the only member of the defined family $\mathcal{F}$ covering e is the triangle T itself — which clearly contains T. It follows that the triangles of G_I containing a vertex of degree 2 are easy triangles.

For the analysis below, please refer to Fig. 6. We classify the triangles of G_I into types according to either they are, or they are not contained in a $K_{12}(j)$ or in a $K_5(j, i)$.

(1) First we consider the triangles of G_I which are not contained in a $K_{12}(j)$ nor in a $K_5(j, i)$. These can be classified as follows.

(a) Triangles containing c^i_j, $1 \leq j \leq m$, $i \in I_j$, $i \neq n+1$. Note, in this case, vertex c^i_j is of degree 4, and is contained in precisely 3 triangles of G_I:

$\{c^i_j, a^i_j, a^{i^*}_j\}$, $\{c^i_j, a^i_j, d^i_j\}$, $\{c^i_j, a^{i^*}_j, d^i_j\}$.

For any of these three types of triangles T the members of $\mathcal{F}_T$ are:

If either u_i is true and occurs as literal u_i in c_j, or u_i is false and occurs as literal $\overline{u_i}$ in c_j, the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $\{c^i_j, a^{i^*}_j, d^i_j\}$, $\{a^i_j, c^i_j, d^i_j\}$. Thus $d^i_j \in \bigcap \mathcal{F}_T$.

If either u_i is true and occurs as literal $\overline{u_i}$ in c_j, or u_i is false and occurs as literal u_i in c_j, the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $\{c^i_j, a^{i^*}_j, d^i_j\}$, $\{a^i_j, c^i_j, a^{i^*}_j\}$. Thus $a^{i^*}_j \in \bigcap \mathcal{F}_T$.

(b) Triangles containing c^{n+1}_j, $1 \leq j \leq m$, $n+1 \in I_j$:

$\{c^{n+1}_j, a^{n+1}_j, a^{(n+1)^*}_j\}$. All these triangles are easy.

(c) Triangles containing e^i_j or f^i_j or w^i_j or z^i_j, $1 \leq j \leq m$, $i \in I_j$:

$\{e^i_j, d^i_j, h^i_j\}$, $\{f^i_j, a^{i^*}_j, g^i_j\}$, $\{w^i_j, h^i_{j_i}, h^i_j\}$, $\{z^i_j, g^i_{j_i}, g^i_j\}$. All these triangles are easy.

(d) Triangles containing v^i_j, $1 \leq i \leq n$, $j \in \bar{J}_i$,

$\{v^i_j, g^i_{j_i}, a^i_{j_i}\}$, $\{v^i_j, h^i_{j_i}, a^i_{j_i}\}$, $\{v^i_j, g^i_j, a^i_j\}$, $\{v^i_j, h^i_j, a^i_j\}$:

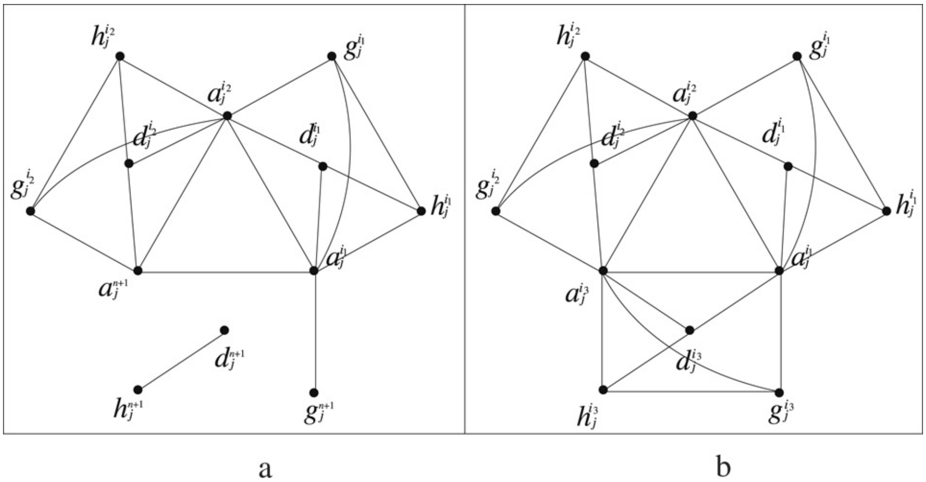

Fig. 7. Triangles of $K_{12}(j)$ which may be not easy, they contain edges covered not only by the $K_{12}(j)$ itself but also by other members of $\mathcal{F}$.

(i) $T = \{v_j^i, g_{j_i}^i, a_{j_i}^i\}$.

If u_i is true, then the members of $\mathcal{F}_T$ are: $K_{12}(j_i)$, $K_5(j, i)$, $\{a_{j_i}^i, g_{j_i}^i, v_k^i\}$, where $k \in \bar{J}_i$. Thus $g_{j_i}^i \in \bigcap \mathcal{F}_T$. If u_i is false, then the members of $\mathcal{F}_T$ are: $K_{12}(j_i)$, $K_5(j, i)$, $\{a_{j_i}^i, h_{j_i}^i, v_j^i\}$. Thus $h_{j_i}^i \in \bigcap \mathcal{F}_T$.

(ii) $T = \{v_j^i, h_{j_i}^i, a_{j_i}^i\}$:

If u_i is true, then the members of $\mathcal{F}_T$ are: $K_{12}(j_i)$, $K_5(j, i)$, $\{a_{j_i}^i, g_{j_i}^i, v_j^i\}$. Thus $g_{j_i}^i \in \bigcap \mathcal{F}_T$. If u_i is false, then the members of $\mathcal{F}_T$ are: $K_{12}(j_i)$, $K_5(j, i)$, $\{a_{j_i}^i, h_{j_i}^i, v_k^i\}$, where $k \in \bar{J}_i$. Thus $h_{j_i}^i \in \bigcap \mathcal{F}_T$.

(iii) $T = \{v_j^i, g_j^i, a_j^i\}$:

If u_i is true, then the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $K_5(j, i)$, $\{v_j^i, h_j^i, a_j^i\}$. Thus $h_j^i \in \bigcap \mathcal{F}_T$. If u_i is false, then the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $K_5(j, i)$, T. Thus $g_j^i \in \bigcap \mathcal{F}_T$.

(iv) $T = \{v_j^i, h_j^i, a_j^i\}$:

If u_i is true, then the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $K_5(j, i)$, T. Thus $h_j^i \in \bigcap \mathcal{F}_T$. If u_i is false, then the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $K_5(j, i)$, $\{v_j^i, g_j^i, a_j^i\}$. Thus $g_j^i \in \bigcap \mathcal{F}_T$.

(2) Consider the triangles of G_I contained in a $K_{12}(j)$ or in a $K_5(j, i)$.

(a) Now we study the triangles contained in $K_{12}(j) = \{a_j^i, d_j^i, g_j^i, h_j^i \mid i \in I_j\}$, $1 \le j \le m$. Among these triangles the ones with at least one edge covered only by $K_{12}(j)$ are easy triangles. The remaining triangles are the triangles whose three edges are edges of $K_{12}(j)$ covered by a complete set of $\mathcal{F}$ besides $K_{12}(j)$ itself.

Notice, by looking at the list of sets of the complete set edge cover $\mathcal{F}$ or by looking at Fig. 6, that the only edges of $K_{12}(j)$ that may be covered by a complete set of $\mathcal{F}$ besides $K_{12}(j)$ are:

$a_j^i a_j^{i^*}$, $a_j^i d_j^i$, $a_j^i g_j^i$, $a_j^i h_j^i$, $a_j^{i^*} g_j^i$, $a_j^{i^*} d_j^i$, $h_j^i d_j^i$, $h_j^i g_j^i$, for $i \neq n+1$; and

$a_j^i a_j^{i^*}$, $a_j^{i^*} g_j^i$, $h_j^i d_j^i$, for $i = n+1$, which are depicted in Fig. 7.

Thus the triangles of $K_{12}(j)$ which are not easy triangles can be classified into only five types, as follows (see Fig. 7):

(i) $T = \{a_j^i, i \in I_j\}$.

Since c_j is satisfied there exists $i \in I_j$ such that either u_i occurs as literal u_i in c_j and variable u_i is true or u_i occurs as literal $\overline{u_i}$ in c_j and variable u_i is false. In any case, by the construction of $\mathcal{F}$, $a_j^i a_j^{i^*}$ is covered only by $K_{12}(j)$. Since $K_{12}(j)$ also contains the third vertex of T, it follows that T is an easy triangle.

(ii) $T = \{a_j^i, a_j^{i^*}, d_j^i\}$, $i \in I_j$, $i \neq n+1$.

If either u_i is true and occurs as literal u_i in c_j, or u_i is false and occurs as literal $\overline{u_i}$ in c_j, the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $\{c_j^i, a_j^{i^*}, d_j^i\}$, $\{a_j^i, c_j^i, d_j^i\}$. Thus $d_j^i \in \bigcap \mathcal{F}_T$.

If either u_i is true and occurs as literal $\overline{u_i}$ in c_j, or u_i is false and occurs as literal u_i in c_j, the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $\{c_j^i, a_j^{i^*}, d_j^i\}$, $\{a_j^i, c_j^i, a_j^{i^*}\}$. Thus $a_j^{i^*} \in \bigcap \mathcal{F}_T$.

(iii) $T = \{a_j^i, d_j^i, h_j^i\}$, $i \in I_j$, $i \neq n+1$.

If either u_i is true and occurs as literal u_i in c_j, or u_i is false and occurs as literal $\overline{u_i}$ in c_j, the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $\{e_j^i, h_j^i, d_j^i\}$, $\{a_j^i, c_j^i, d_j^i\}$. Thus $d_j^i \in \bigcap \mathcal{F}_T$.

If u_i is true and occurs as literal $\overline{u_i}$ in c_j, the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $\{e_j^i, h_j^i, d_j^i\}$, $\{a_j^i, h_j^i, v_j^i\}$. Thus $h_j^i \in \bigcap \mathcal{F}_T$. If u_i is false and occurs as literal u_i in c_j, the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $\{e_j^i, h_j^i, d_j^i\}$, $\{a_j^i, h_j^i, v_k^i\}$, $k \in \bar{J}_i$. Please

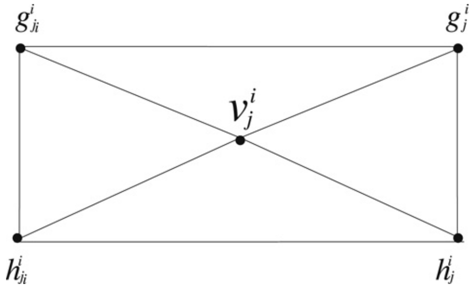

Fig. 8. Triangles of $K_5(j, i)$ which may be not easy, they contain edges covered not only by the $K_5(j, i)$ itself but also by other members of $\mathcal{F}$.

refer to the proposed complete set edge cover $\mathcal{F}$, where $\{a^i_j, h^i_j, v^i_k\}$ is listed in the 7th item as $\{a^i_{j_i}, h^i_{j_i}, v^i_j\}$. Thus $h^i_j \in \bigcap \mathcal{F}_T$.

(iv) $T = \{a^i_j, a^{i^*}_j, g^i_j\}, i \in I_j, i \neq n + 1$.

If u_i is true and occurs as literal u_i in c_j, the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $\{a^{i^*}_j, f^i_j, g^i_j\}$, $\{a^i_j, g^i_j, v^i_k\}$, $k \in \bar{J}_i$. Thus $g^i_j \in \bigcap \mathcal{F}_T$.

If either u_i is true and occurs as literal $\overline{u_i}$ in c_j, or u_i is false and occurs as literal u_i in c_j, the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $\{a^{i^*}_j, f^i_j, g^i_j\}$, $\{a^i_j, a^{i^*}_j, c^i_j\}$. Thus $a^{i^*}_j \in \bigcap \mathcal{F}_T$.

If u_i is false and occurs as literal $\overline{u_i}$ in c_j, the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $\{a^{i^*}_j, f^i_j, g^i_j\}$, $\{a^i_j, g^i_j, v^i_j\}$. Thus $g^i_j \in \bigcap \mathcal{F}_T$.

(v) $T = \{a^i_j, h^i_j, g^i_j\}, i \in I_j, i \neq n + 1$.

If u_i is true and occurs as literal u_i in c_j, the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $K_5(k, i)$, $\{a^i_j, g^i_j, v^i_k\}$, $k \in \bar{J}_i$. Thus $g^i_j \in \bigcap \mathcal{F}_T$.

If u_i is false and occurs as literal u_i in c_j, the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $K_5(k, i)$, $\{a^i_j, h^i_j, v^i_k\}$, $k \in \bar{J}_i$. Thus $h^i_j \in \bigcap \mathcal{F}_T$.

If u_i is true and occurs as literal $\overline{u_i}$ in c_j, the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $K_5(j, i)$, $\{a^i_j, h^i_j, v^i_j\}$. Thus $h^i_j \in \bigcap \mathcal{F}_T$.

If u_i is false and occurs as literal $\overline{u_i}$ in c_j, the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $K_5(j, i)$, $\{a^i_j, g^i_j, v^i_j\}$. Thus $g^i_j \in \bigcap \mathcal{F}_T$.

(b) Now we consider the triangles of G_I which are contained in a $K_5(j, i) = \{h^i_{j_i}, g^i_{j_i}, v^i_j, h^i_j, g^i_j\}$, for each $1 \leq i \leq n$ and $j \in \bar{J}_i$. Among these triangles the ones with at least one edge covered only by $K_5(j, i)$ are easy triangles. Notice the edges $g^i_{j_i}h^i_j$ and $h^i_{j_i}g^i_j$ are covered only by $K_5(j, i)$, which implies that six triangles formed by one of these two edges and an additional vertex of $K_5(j, i)$ are all easy. Every edge of the remaining four triangles may be covered by a complete set besides $K_5(j, i)$. Notice that such edges are: $g^i_{j_i}g^i_j, g^i_{j_i}v^i_j, g^i_{j_i}h^i_{j_i}, h^i_jv^i_j, h^i_jg^i_j, h^i_jh^i_{j_i}, v^i_jh^i_{j_i}, v^i_jg^i_j$, which are depicted in Fig. 8.

Thus we study four types of triangles contained in $K_5(j, i)$ (see Fig. 8): $\{g^i_{j_i}, g^i_j, v^i_j\}$, $\{h^i_{j_i}, h^i_j, v^i_j\}$, $\{g^i_{j_i}, h^i_{j_i}, v^i_j\}$, $\{g^i_j, h^i_j, v^i_j\}$.

(i) $T = \{g^i_{j_i}, g^i_j, v^i_j\}$.

If u_i is true, then the members of $\mathcal{F}_T$ are: $K_5(j, i)$, $\{g^i_{j_i}, g^i_j, z^i_j\}$, $\{a^i_{j_i}, g^i_{j_i}, v^i_j\}$. Thus $g^i_{j_i} \in \bigcap \mathcal{F}_T$. If u_i is false, then the members of $\mathcal{F}_T$ are: $K_5(j, i)$, $\{g^i_{j_i}, g^i_j, z^i_j\}$, $\{v^i_j, g^i_j, a^i_j\}$. Thus $g^i_j \in \bigcap \mathcal{F}_T$.

(ii) $T = \{h^i_{j_i}, h^i_j, v^i_j\}$.

If u_i is true, then the members of $\mathcal{F}_T$ are: $K_5(j, i)$, $\{h^i_{j_i}, h^i_j, w^i_j\}$, $\{v^i_j, h^i_j, a^i_j\}$. Thus $h^i_j \in \bigcap \mathcal{F}_T$. If u_i is false, then the members of $\mathcal{F}_T$ are: $K_5(j, i)$, $\{h^i_{j_i}, h^i_j, w^i_j\}$, $\{a^i_{j_i}, v^i_j, h^i_{j_i}\}$. Thus $h^i_{j_i} \in \bigcap \mathcal{F}_T$.

(iii) $T = \{g^i_{j_i}, h^i_{j_i}, v^i_j\}$.

If u_i is true, then the members of $\mathcal{F}_T$ are: $K_{12}(j_i)$, $\{a^i_{j_i}, g^i_{j_i}, v^i_j\}$, $K_5(k, i)$, where $k \in \bar{J}_i$. Thus $g^i_{j_i} \in \bigcap \mathcal{F}_T$. If u_i is false, then the members of $\mathcal{F}_T$ are $K_{12}(j_i)$, $\{a^i_{j_i}, h^i_{j_i}, v^i_j\}$, $K_5(k, i)$, where $k \in \bar{J}_i$. Thus $h^i_{j_i} \in \bigcap \mathcal{F}_T$.

(iv) $T = \{g^i_j, h^i_j, v^i_j\}$.

If u_i is true, then the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $K_5(j, i)$, $\{v^i_j, h^i_j, a^i_j\}$. Thus $h^i_j \in \bigcap \mathcal{F}_T$. If u_i is false, then the members of $\mathcal{F}_T$ are: $K_{12}(j)$, $K_5(j, i)$, $\{v^i_j, g^i_j, a^i_j\}$. Thus $g^i_j \in \bigcap \mathcal{F}_T$. □

In Fig. 9, we give an example of an RS-cover defined by a satisfying truth assignment, according to the proof of Theorem 8.

5. Concluding remarks

We have proved that deciding whether a given graph is a clique graph is an NP-complete problem. From the same proof, it follows that the problem remains NP-complete even for graphs with bounded clique size ω, and for bounded maximum degree Δ graphs. We say that the complete bipartite graph $K_{1,t}$ is a *t-claw*. A graph G is *t-claw free* if G does not contain a

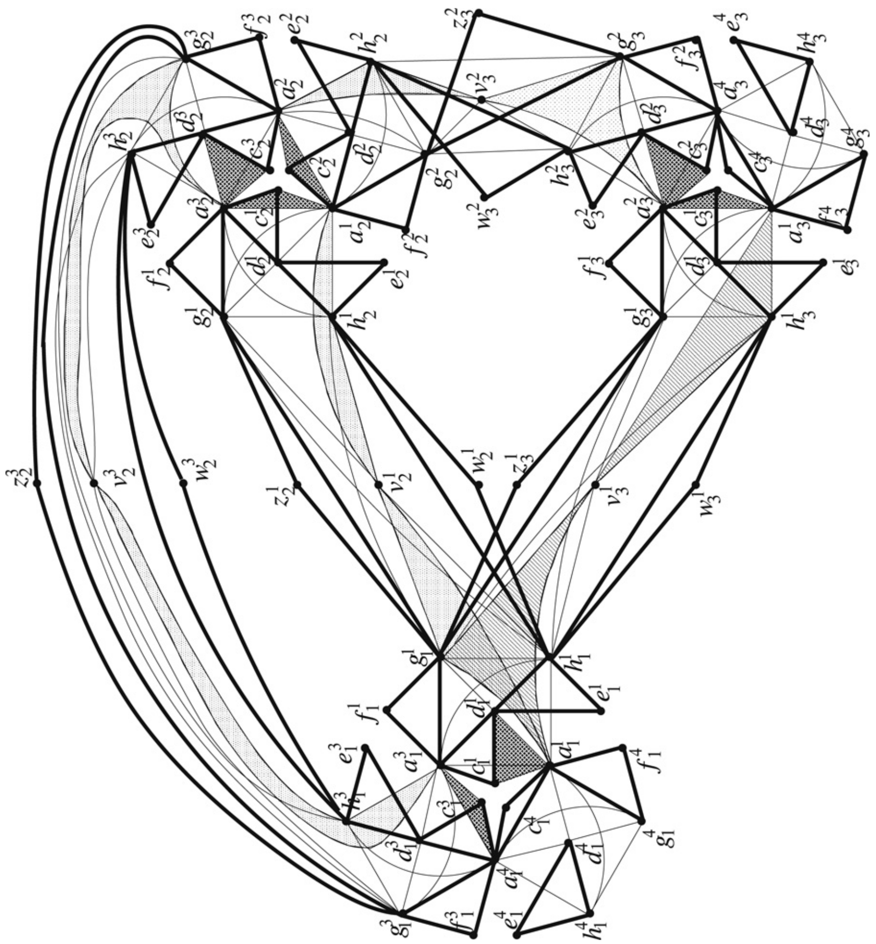

Fig. 9. RS-cover $\mathcal{F}$ for graph G_I of Fig. 4. The RS-cover is defined by the satisfying truth assignment where u_1 is true, and u_2 and u_3 are false. Bold edges highlight forced triangles present in every RS-cover of G_I. Filled connected regions depict triangles of $\mathcal{F}$ which depend on the truth assignment for $I = (U, C)$. Complete sets $K_5(j, i)$ and $K_{12}(j)$ belong to the RS-cover $\mathcal{F}$ but are not depicted in order to make simpler the drawing.

t-claw as an induced subgraph. The instance of CLIQUE GRAPH used to prove NP-completeness has clique size $\omega = 12$, has maximum degree $\Delta = 14$, and it is 7-claw free. However the problem is polynomial when restricted to graphs with clique size $\omega < 4$, and also when restricted to graphs with maximum degree $\Delta < 5$. Note that Theorem 3 of the fundamental paper by Roberts and Spencer [36] says: a K_4-free graph is a clique graph if and only if it is clique-Helly. A graph with $\Delta < 5$ is a clique graph if and only if it is hereditary clique-Helly [17,31]. G is a 1-claw free graph if and only if each connected component of G is K_1, and G is a 2-claw free graph if and only if each connected component of G is a complete graph. This suggests the search of the maximum values for the clique size $3 \leq \omega \leq 11$ and for the maximum degree $4 \leq \Delta \leq 13$ for which the problem is polynomial, and additionally the maximum value of $2 \leq t \leq 6$ such that the problem is polynomial for t-claw free graphs.

The only reference for the problem of recognizing clique graphs restricted to greater bounded maximum clique size graphs is the class of Planar graphs [2]. In that paper, a non-bounded degree subclass of planar clique graphs, larger than clique-Helly planar graphs, and admitting cliques of size 4, is characterized. In addition, a polynomial-time algorithm for the recognition of that subclass of planar clique graphs is given.

Several subclasses of clique graphs have been studied for which polynomial-time recognition is known. In particular, for several classes of graphs the corresponding class of clique graphs is known [39]. Note that it is known that the clique graph of a Chordal graph is a Dually chordal graph, but the complexity of deciding whether a Chordal graph is a clique graph is not known.

The NP-completeness of CLIQUE GRAPH suggests the study of the problem restricted to classes of graphs not properly contained in the class of clique graphs. We leave as open problems the recognition of Planar clique graphs, and of Chordal clique graphs.

Acknowledgments

We are grateful to the referees for their careful reading and the many suggestions which helped to improve the paper.

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