Received: 29 November 2022 | Revised: 9 November 2023 | Accepted: 14 March 2024

DOI: 10.1002/net.22220

RESEARCH ARTICLE

WILEY

Parameterized algorithms for Steiner tree and (connected) dominating set on path graphs

Celina M. H. de Figueiredo[1] | **Raul Lopes**[2] | **Alexsander A. de Melo**[1] | **Ana Silva**[3]

[1]Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil
[2]Université Paris-Dauphine, PSL University, CNRS UMR7243, Paris, France
[3]Universidade Federal do Ceará, Fortaleza, Brazil

Correspondence
Ana Silva, Universidade Federal do Ceará, Fortaleza, Brazil.
Email: anasilva@mat.ufc.br

Funding information
Agence Nationale de la Recherche, Grant/Award Number: ANR-18-CE40-0025-01; Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro, Grant/Award Number: E-26/202.793/2017; Fundação Cearense de Apoio ao Desenvolvimento Científico e Tecnológico, Grant/Award Number: PNE-0112-00061.01.00/16; Conselho Nacional de Desenvolvimento Científico e Tecnológico, Grant/Award Numbers: 140399/2017-8, 407635/2018-1, 303803/2020-7; Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, Grant/Award Number: Finance Code 001; STIC-AMSUD, Grant/Award Number: 88881.197438/2018-01.

Abstract

Chordal graphs are the intersection graphs of subtrees of a tree, while interval graphs of subpaths of a path. Undirected path graphs, directed path graphs and rooted directed path graphs are intermediate graph classes, defined, respectively, as the intersection graphs of paths of a tree, of directed paths of an oriented tree, and of directed paths of an out branching. All of these path graphs have *vertex leafage 2*. Dominating Set, Connected Dominating Set, and Steiner tree problems are W[2]-hard parameterized by the size of the solution on chordal graphs, NP-complete on undirected path graphs, and polynomial-time solvable on rooted directed path graphs, and hence also on interval graphs. We further investigate the (parameterized) complexity of all these problems when constrained to chordal graphs, taking the vertex leafage and the aforementioned classes into consideration. We prove that Dominating Set, Connected Dominating Set, and Steiner tree are FPT on chordal graphs when parameterized by the size of the solution plus the vertex leafage, and that Weighted Connected Dominating Set is polynomial-time solvable on strongly chordal graphs. We also introduce a new subclass of undirected path graphs, which we call in–out rooted directed path graphs, as the intersection graphs of directed paths of an in–out branching. We prove that Dominating Set, Connected Dominating Set, and Steiner tree are solvable in polynomial time on this class, generalizing the polynomiality for rooted directed path graphs proved by Booth and Johnson (SIAM J. Comput. **11** (1982), 191-199.) and by White et al. (Networks **15** (1985), 109-124.).

KEYWORDS
chordal graphs, dominating set, FPT algorithms, Steiner tree, undirected path graphs

1 | INTRODUCTION

Given a graph G and a positive integer κ, the Dominating Set problem consists of deciding whether G has a dominating set of size at most κ. Dominating Set is the canonical problem in the class W[2]-hard when parameterized by κ, which explains the great interest in it (see e.g., [20, 23, 24, 36]). Given its hardness, an approach often taken is to constrain the problem to certain graph classes. In this article, given the hardness of Dominating Set parameterized by the size of the solution on chordal graphs, we investigate Dominating Set and the related problems, Connected Dominating Set and Steiner tree, constrained to the subclasses of chordal graphs known as *path graphs*. Two structural parameters have been studied for understanding intractable problems on chordal graphs that admit polynomial-time algorithms on the proper subclass of interval graphs: the leafage measures how close a chordal graph is to being an interval graph and the vertex leafage measures the closeness to undirected path graphs [3, 28, 31].

For basic definitions and notation on graph theory, we refer the reader to [34]. And for formal definitions of all attacked problems and subclasses, we refer the reader to Section 2. Figure 1 and Table 1 summarize the complexity results cited along the text. The present full paper extends the preliminary results published at WALCOM 2022 [12] by establishing a hardness relation between DOMINATING SET, CONNECTED DOMINATING SET and STEINER TREE, including the weighted versions of the problems and considering complexity separating subclasses of chordal graphs. An independent related work has recently been presented at IPEC 2022 [16], where they study the complexity of these problems for the class of chordal graphs when parameterized by the leafage, improving results presented in [15].

DOMINATING SET, as well as CONNECTED DOMINATING SET and STEINER TREE, are still W[2]-hard when parameterized by the solution size, hereon denoted by κ, and constrained to chordal graphs (and even split graphs, which form a strict subclass of chordal graphs) [32]. However, they become polynomial-time solvable on rooted directed path graphs [4, 35], a superclass of the more widely known interval graphs. A natural question therefore is whether they are also polynomial-time solvable on undirected path graphs. This unfortunately is not the case, unless P $=$ NP, as they are NP-complete on these graphs [4, 11]. Nevertheless, here we prove that they are FPT when parameterized by κ on undirected path graphs (Theorem 1 below). This classification closes all the parameterized complexity open entries for undirected path graphs in a revised version of Column 16 of Johnson's table [26] presented in [11].

Theorem 1. *DOMINATING SET, CONNECTED DOMINATING SET and STEINER TREE can be solved on an undirected path graph G in time $2^{2\kappa \log \kappa} \cdot n^{O(1)}$, where $n = |V(G)|$ and κ denotes the solution size.*

We mention that the complexity above can be improved to $2^{O(\kappa)} \cdot n^{O(1)}$ by applying our results together with an algorithm recently proposed in [16]. We nevertheless present our slightly slower algorithm because, being designed specifically to undirected path graphs, it is much simpler than the one in [16]. In Theorem 2 below, however, we apply the algorithm in [16] directly.

Undirected path graphs can also be defined as the chordal graphs with vertex leafage 2 [9]; the vertex leafage of a chordal graph G is denoted by $v\ell(G)$. Because the investigated problems are NP-complete on undirected path graphs [4, 11], we get that they are NP-complete on chordal graphs with vertex leafage k for every fixed $k \geq 2$. This fact prevents the existence

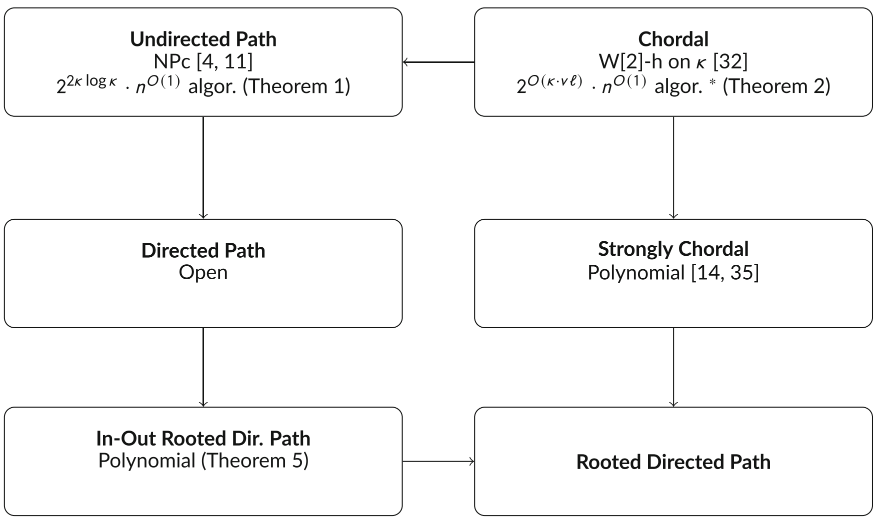

FIGURE 1 Complexity of DOMINATING SET, CONNECTED DOMINATING SET and STEINER TREE on the investigated classes. An arrow from class A to class B indicates that class A contains class B. * Theorem 2 holds provided a tree model with optimal vertex leafage.

TABLE 1 Complexities of the weighted problems.

	Weighted problem		
Graph class	STTREE	CONNDOMSET	DOMSET
CHORDAL	W[2]-h [32]	W[2]-h [32]	W[2]-h [32]
STRONGLY CHORDAL	NPc [35]	P (Theorem 4)	P [14]
UNDIRECTED PATH	NPc [11]	NPc [11]	NPc [4]
ROOTED DIRECTED PATH	Open	P (Theorem 4)	P [14]

Note: Nothing is known about the complexities on (in–out rooted) directed path graphs.

of FPT algorithms parameterized by the vertex leafage of chordal graphs unless P = NP. We prove that they are FPT when parameterized by $\kappa + v\ell(G)$, provided a tree model with optimal vertex leafage is given.

Theorem 2. *Let G be a chordal graph on n vertices. If a tree model $\mathcal{T}$ of G such that $v\ell(\mathcal{T}) = v\ell(G)$ is given, then* Dominating Set, Connected Dominating Set *and* Steiner tree *can be solved in time $2^{O(\kappa \cdot v\ell(G))} \cdot n^{O(1)}$.*

In fact, we solve explicitly a generalization of Dominating Set (see Theorem 6), and get the results on Connected Dominating Set and Steiner tree by providing parameterized reductions from Connected Dominating Set to Dominating Set, and from Steiner tree to Connected Dominating Set. We write $\Pi \preceq_\kappa \Pi'$ to denote that there exists a parameterized reduction from Π to Π', both having parameter κ. We improve on the reductions for Steiner tree and Connected Dominating Set provided in [35], because here we preserve leafage and vertex leafage, and because our reduction from Connected Dominating Set to Dominating Set also works on the weighted versions of the problems.

Theorem 3. *When constrained to the class of chordal graphs, and of chordal graphs with bounded (vertex) leafage, we have that*:

$$\text{Steiner Tree} \preceq_\kappa \text{Connected Dominating Set} \preceq_\kappa \text{Dominating Set}.$$

Now, because our reduction from Connected Dominating Set to Dominating Set also works on the weighted versions of the problems, and since Weighted Dominating Set is polynomial-time solvable on strongly chordal graphs [14], a superclass of rooted directed path graphs, it follows that Weighted Connected Dominating Set is also polynomial-time solvable on rooted directed path graphs. We prove that, in fact, our reduction works also on strongly chordal graphs.

Theorem 4. *Weighted Connected Dominating Set is polynomial-time solvable on strongly chordal graphs.*

The above theorem gives us that strongly chordal graphs separate these problems from Weighted Steiner tree since the latter is NP-complete on strongly chordal graphs [35]. However, it should be noted that there is a fundamental difference between the weighted versions of these problems, as Weighted Steiner tree considers weights on the edges, while the others consider weights on the vertices. It should also be mentioned that Steiner tree is polynomial-time solvable on strongly chordal graphs [35]. Table 1 summarizes what is known about the weighted versions of these problems.

Observe that the complexity of all the problems constrained to directed path graphs is widely open, except that we know their unweighted versions are FPT when parameterized by κ, as this is a subclass of undirected path graphs. As an attempt to understand the complexity on these graphs, we introduce the *in–out rooted directed path* graphs, which are the intersection graphs of directed paths in an in–out branching. We prove that the unweighted problems on in–out rooted directed path graphs are all polynomial-time solvable.

Theorem 5. *Dominating Set, Connected Dominating Set and Steiner tree are polynomial-time solvable on in–out rooted directed path graphs.*

A parameter closely related to vertex leafage, and in fact an upper bound for it, is the *leafage of* G, denoted by $\ell(G)$ [28]. Surprisingly enough, a tree model with $\ell(G)$ leaves can be computed in polynomial time [22]. This unfortunately is not the case for the vertex leafage parameter, as it is known [9] that it is NP-complete to decide whether a chordal graph G has vertex leafage at most 3; they also give an algorithm to compute $v\ell(G)$ in time $n^{\ell(G)}$, which is XP when parameterized by $\ell(G)$. In [15] they provide an FPT algorithm for Dominating Set when parameterized by $\ell(G)$, which has been recently improved in [16]. Since $v\ell(G) \le \ell(G)$, we get that $v\ell(G)$ is a more restrictive parameter than $\ell(G)$, and hence these algorithms are not readily applicable to Dominating Set parameterized by κ and $v\ell(G)$. Nevertheless, we show that positive instances of Dominating Set and Connected Dominating Set must have bounded leafage, which brought us to the question about whether the same holds for generalizations of Dominating Set. Indeed, we have found that the broader class of problems, called Min-LC-VSP problems [8, 15], have the same property. Given a graph G on n vertices and subsets $\sigma, \rho \subseteq \{0, \dots, n-1\}$, a subset $S \subseteq V(G)$ is a (σ, ρ)*-set* if: $|N(v) \cap S| \in \sigma$ for every $v \in S$, and $|N(v) \cap S| \in \rho$ for every $v \in V(G) \setminus S$. Fixing σ, ρ, and given a graph G and an integer κ, the Min-LC-VSP$_{\sigma,\rho}$ problem consists in deciding whether there exists a (σ, ρ)-set S of size at most κ. Observe that if $0 \in \rho$, then the answer is always yes since taking the emptyset satisfies the constraints; this is why we suppose $0 \notin \rho$ in what follows. Min-LC-VSP problems generalize a number of optimization problems, as for example, Dominating Set, d-Dominating Set, Total Dominating Set, Induced d-regular Subgraph and so forth [8]. We state our result and its corollary obtained from $v\ell(G) \le \ell(G)$.

Theorem 6. *Let $\sigma, \rho \subseteq \{0, \dots, n-1\}$ be such that $0 \notin \rho$, G a chordal graph and κ a positive integer. If (G, κ) is a* Yes *instance of* Min-LC-vsp$_{\sigma,\rho}$*, then $\ell(G) \le \kappa \cdot v\ell(G)$.*

Corollary 1. *Let $\sigma, \rho \subseteq \{0, \dots, n-1\}$, G be a chordal graph and κ be a positive integer. If* Min-LC-vsp$_{\sigma,\rho}$ *is* FPT *when parameterized by $v\ell(G)$, then* Min-LC-vsp$_{\sigma,\rho}$ *is also* FPT *when parameterized by $\ell(G)$. And if* Min-LC-vsp$_{\sigma,\rho}$

is FPT *when parameterized by* $\ell(G)$ *and a tree model* $\mathcal{T}$ *with* $v\ell(\mathcal{T}) = v\ell(G)$ *is provided, then* $\text{MIN-LC-VSP}_{\sigma,\rho}$ *is also* FPT *when parameterized by* $\kappa + v\ell(G)$.

The article is organized as follows. In Section 2, we present formal definitions and some basic results. We also establish the containment relations among the various graph classes investigated in this article, as well as some other related classes. In Section 3, we prove an important lemma that directly implies Theorem 6, and the part of Theorem 2 that concerns DOMINATING SET. In Section 4, we prove the part of Theorem 1 that concerns DOMINATING SET. Section 5 contains the reduction from CONNECTED DOMINATING SET to DOMINATING SET, while Section 6 contains the reduction from STEINER TREE to CONNECTED DOMINATING SET. These reductions finish the proof of Theorems 1–4. The proof of Theorem 5 is presented in Section 7, and finally Section 8 presents some final remarks and open questions.

2 | PRELIMINARIES

In this section, we formally define all investigated problems and graph subclasses. We also give some basic definitions of parameterized complexity. For further definitions, we refer the reader to [10, 34].

2.1 | Dominating set and variations

For a graph G, a set $D \subseteq V(G)$ is *dominating* if, for every vertex $v \in V(G)\setminus D$, we have that v has a neighbor in D. Given a graph G and a positive integer κ, the DOMINATING SET problem consists of deciding whether G has a dominating set of size at most κ, while the CONNECTED DOMINATING SET asks the same but requires additionally that $G[D]$ is connected. Given also a subset $X \subseteq V(G)$, called the set of *terminals*, the STEINER TREE problem consists of deciding whether there exists a subset $S \subseteq V(G)\setminus X$, called a *Steiner set*, such that $|S| \leq \kappa$ and $G[S \cup X]$ is connected—and hence $G[S \cup X]$ has a spanning tree T, called a *Steiner tree* for (G, X). The natural parameter of all these problems is κ. We restrict our attention to the case in which G is connected, which ensures that connected dominating sets and Steiner trees do exist. Additionally, observe that DOMINATING SET can also be reduced to the case where G is connected, since it suffices to solve the problem on the connected components of G.

In the WEIGHTED DOMINATING SET, we are given a graph G, a positive number κ and a weight function ω on $V(G)$, and the problem consists of deciding whether there exists a dominating set $S \subseteq V(G)$ such that $\sum_{u\in S}\omega(u) \leq \kappa$. The WEIGHTED CONNECTED DOMINATING SET is analogous, except that we want additionally that $G[S]$ is connected. Finally, in the WEIGHTED STEINER TREE problem, we are given a graph G, a subset $X \subseteq V(G)$, a positive number κ and a weight function ω on $E(G)$, and are asked whether there is a Steiner tree T for (G, X) such that $\sum_{e\in E(T)}\omega(e) \leq \kappa$.

As said in the introduction, we solve STEINER TREE and CONNECTED DOMINATING SET by presenting the reductions that prove Theorem 3, then solving DOMINATING SET. But in order to solve the latter, we actually solve a more general problem, defined next. Given a graph $G = (V, E)$ and a subset $B \subseteq V$, we say that $S \subseteq V(G)$ is a *B-dominating set of* (G, B) if $N[u] \cap S \neq \emptyset$ for every $u \in B$. The B-DOMINATING SET is the problem of, given a graph G, a subset $B \subseteq V(G)$ and a positive integer κ, deciding whether (G, B) has a B-dominating set of size at most κ.

2.2 | Chordal graphs and subclasses

Given a graph G and a family of subsets $\mathcal{S} = \{S_u\}_{u\in V(G)}$ of a set U, we say that G is the *intersection graph of* $\mathcal{S}$ if $uv \in E(G)$ if and only if $S_u \cap S_v \neq \emptyset$, and that $(U, \mathcal{S})$ is a intersection *model* (or simply, *model*) of G. In particular, a *tree model* of G is an intersection model $\mathcal{T} = (T, \{T_u\}_{u\in V(G)})$ such that T is a tree, called *characteristic tree*, T_u is a subtree of T for every $u \in V(G)$, and two vertices are adjacent in the graph G if and only if their two corresponding subtrees have a nonempty vertex intersection. A cycle of a graph G is *chordless* if any two vertices of the cycle are adjacent in G if and only if they are consecutive in the cycle. *Chordal* graphs are defined as graphs having no chordless cycle of size bigger than three, but it is known that they are precisely the intersection graphs admitting a tree model, that is, they are the vertex intersection graphs of subtrees of a characteristic tree [17]. Nested subclasses of chordal graphs are defined by putting constraints in either the characteristic tree, or the subtrees. *Interval* graphs are the intersection graphs of subpaths of a path [5]; *rooted directed path* graphs are the intersection graphs of directed paths of an out-branching [18] (an oriented rooted tree with all vertices being reachable from the root); *directed path* graphs are the intersection graphs of directed paths of an oriented tree [29]; and *undirected path* graphs are the intersection graphs of paths of a tree [19]. The cited papers give polynomial-time recognition algorithms that also provide tree models for these classes. Chordal graphs have a linear number of maximal cliques, which can be listed in polynomial time, and a characteristic tree having the maximal cliques as nodes can be obtained in polynomial time [17].

In Section 7, we give a polynomial time algorithm to solve Dominating Set on a subclass of directed path graphs that contains the rooted directed path graphs. Let T be an oriented tree. A *leaf* of T is a vertex of degree 1. We say that T is an *in–out branching rooted at* r if every path between r and a leaf ℓ is oriented either towards r or towards ℓ. A graph G is an *in–out rooted directed path graph* if G is the intersection graph of directed paths of a rooted in–out branching.

Given a tree model $\mathcal{T} = (T, \{T_u\}_{u\in V(G)})$ of a chordal graph G, the *vertex leafage of* $\mathcal{T}$ is the maximum number $v\ell(\mathcal{T})$ of leaves in a subtree T_u, while the *vertex leafage of* G is the minimum vertex leafage over all of its tree models [9]. Similarly, the *leafage of* G, denoted by $\ell(G)$, is the minimum number of leaves $\ell(\mathcal{T})$ in the tree of a tree model $\mathcal{T}$ of G [28].

Still considering a tree model $\mathcal{T} = (T, \{T_u\}_{u\in V(G)})$ of G, given a node $t \in V(T)$, we denote by V_t the set $\{u \in V(G) : t \in V(T_u)\}$. We say that $u \in V(G)$ is a *leafy vertex* of G (with respect to $\mathcal{T}$) if $V(T_u) = \{\ell_u\}$ and ℓ_u is a leaf in T; denote by $\mathcal{L}(G, \mathcal{T})$ the set of leafy vertices of G with respect to $\mathcal{T}$, and for each $u \in \mathcal{L}(G, \mathcal{T})$, denote by ℓ_u the unique node in T_u.

A tree model $(T, \{T_u\}_{u\in V(G)})$ of G is said to be *minimal* if there are no two adjacent nodes $t, t' \in V(T)$ such that $V_t \subseteq V_{t'}$. Even though obtaining a minimal tree model, given a tree model of G, is a standard operation, we prove it explicitly in order to show that also the vertex leafage does not increase. The arguments used in the proposition are also presented along the text in [9] without an explicit statement.

Proposition 1. *Let G be a chordal graph, and $\mathcal{T} = (T, \{T_u\}_{u\in V(G)})$ be a tree model of G. Then, a minimal tree model $\mathcal{T}' = (T', \{T'_u\}_{u\in V(G)})$ of G with $v\ell(\mathcal{T}') \leq v\ell(\mathcal{T})$ and $\ell(T') \leq \ell(T)$ can be computed in polynomial time.*

Proof. We iterate on the nodes of T, starting from the leaves. Recall that V_t denotes the set of vertices of G whose tree contains t. If a leaf ℓ with neighbor t is such that $V_\ell \subseteq V_t$, simply delete ℓ from T and from every subtree T_u containing ℓ. Observe that this operation cannot increase the number of leaves of T or of any subtree T_u. Now, let $t, t' \in V(T)$ be such that $V_{t'} \subseteq V_t$ and $d_T(t') > 1$. Also, let $t_1, \dots, t_q$ be the neighbors of t' different from t. Remove t' from T, and add the edges $\{tt_1, \dots, tt_q\}$, obtaining T'; this is equivalent to doing a contraction of the edge tt' in T. Observe that, because $V_{t'} \subseteq V_t$, we get that $t \in V(T_u)$ whenever $t' \in V(T_u)$. Thus we can do the same with the subtrees containing t', that is, if $t' \in V(T_u)$, then contract the edge tt' in T_u to obtain T'_u. Denote by $\mathcal{T}'$ the obtained model. Clearly edge contractions cannot increase the number of leaves in a tree, so $v\ell(\mathcal{T}') \leq v\ell(\mathcal{T})$ and $\ell(T') \leq \ell(T)$. Now we need to argue that this is still a tree model for G. This can be seen to hold because T_u changes if and only if $t' \in V(T_u)$, if and only if $tt' \in E(T_u)$, and because $t \in V(T'_u)$ for every u that had its subtree modified. ■

The following lemma will also be useful.

Lemma 1. *Let G be a chordal graph and $\mathcal{T} = (T, \{T_u\}_{u\in V(G)})$ be a tree model of G such that $V_t \neq \emptyset$ for every $t \in V(T)$. Then G is connected if and only if, for every $tt' \in E(T)$, there exists $u \in V(G)$ such that $tt' \in E(T_u)$.*

Proof. For the necessity part, suppose by contradiction that $tt' \in E(T)$ is such that $tt' \notin E(T_u)$ for any $u \in V(G)$. Let T_t be the component of $T - \{tt'\}$ containing t and $T_{t'}$ be the one containing t'. Also, let $V_t^* \subseteq V(G)$ be such that $T_u \subseteq T_t$ for every $u \in V_t^*$, and define $V_{t'}^* \subseteq V(G)$ similarly with relation to $T_{t'}$. Since $V_{t''} \neq \emptyset$ for every $t'' \in V(T)$, we get that V_t^* and $V_{t'}^*$ are non-empty and form a partition of $V(G)$, and by definition of tree model we know that there are no edges between V_t^* and $V_{t'}^*$. It follows that G is disconnected.

Now, suppose that, for every $tt' \in E(T)$, there exists $u \in V(G)$ such that $tt' \in E(T_u)$. If we prove that for every partition V_1, V_2 of $V(G)$, there exists an edge uv with $u \in V_1$ and $v \in V_2$, then it follows that G is connected. So consider an arbitrary partition V_1, V_2 of $V(G)$. Let T_1 be the subgraph of T induced by $\bigcup_{u\in V_1} V(T_u)$; define T_2 similarly with relation to V_2. If $V(T_1) \cap V(T_2) \neq \emptyset$, then we are done. Otherwise, since $V_t \neq \emptyset$ for every $t \in V(T)$, note that $V(T_1)$ and $V(T_2)$ define a partition of $V(T)$. By taking any tt' such that $t \in V(T_1)$ and $t' \in V(T_2)$, which must exist because V_1 and V_2 are non-empty, we get a contradiction as tt' is not contained in T_u for any $u \in V(G)$. ■

Since there is much confusion in the literature about the relations involving path graphs[1], we present the next proposition. See Figure 2. Given $k \geq 3$, the *k-sun* is obtained from a k-clique by appending k vertices in a way that the $2k$ vertices form a cycle alternating vertices in the clique and vertices not in the clique. Figure 3A depicts the 4-sun. We call a k-sun an *odd sun* if k is odd, and an *even sun*, otherwise. A graph is *strongly chordal* if it is a sun-free chordal graph [13]. Finally, a graph is *dually chordal* if it is the clique graph (intersection graph of maximal cliques) of a chordal graph.

[1]For instance, on the platform "Information System on Graph Classes and their Inclusions" (www.graphclasses.org/classes.cgi), Dominating Set is wrongly reported as linear-time solvable on directed path graphs as a consequence of these being a subclass of strongly chordal and of dually chordal.

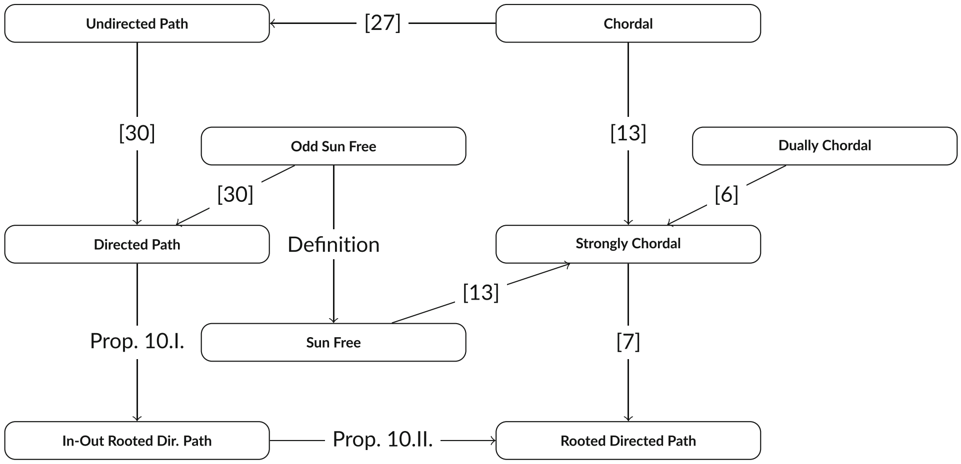

FIGURE 2 Relations between mentioned classes. An arrow from class A to class B means that A is a strict superclass of B.

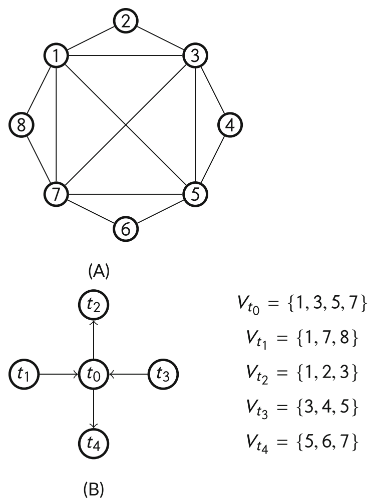

FIGURE 3 Example of in–out rooted directed path graph that is not a strongly chordal graph, and hence also not rooted directed path. (A) 4-sun. (B) Tree model of the 4-sun.

Proposition 2. *The following relations hold between the mentioned classes.*

I. In–out rooted directed path graphs is a proper subclass of directed path graphs;

II Every even sun is an in–out rooted directed path graph that is not dually chordal (and hence not strongly chordal, nor rooted directed path);

Proof. For Item I, the graph in Figure 4A was presented in [21] as an example of a directed path graph that is not rooted directed path. It turns out that the same graph is also an example of directed path graph that is not in–out rooted directed path. This can be seen from the fact that the tree model presented in the figure is the unique model for the graph using directed paths (see [21]), and that it is not an in–out branching.

For Item II, Figure 3B presents a tree model of the 4-sun where the tree is an in–out branching and each path is a directed path in such a tree; in other words, the 4-sun is an in–out rooted directed path graph. A similar trick can be applied to every even sun. For this, consider the even sun on vertices $\{v_1, \dots, v_{2q}, a_1 \dots, a_{2q}\}$ where $C = \{v_1, \dots, v_{2q}\}$ forms a clique, and a_i is adjacent to v_i and $v_{(i+1) \bmod 2q}$, for every $i \in [2q]$. Let T be such that $V(T) = \{t_0, \dots, t_{2q}\}$ and $E(T) = \{t_{2i-1}t_0, t_0t_{2i} | i \in [q]\}$; in words, the edge between t_i and t_0 points to t_0 if i is odd, and to t_i if i is even. Now, for each $i \in [2q]$, relate a_i to the subtree containing the single node t_i. Finally, for each $i \in [q]$, relate v_{2i-1} to the subtree formed by $\{t_{(2i-2) \bmod 2q}, t_0, t_{2i-1}\}$, and relate v_{2i} to the subtree formed by $\{t_{2i-1}, t_0, t_{2i}\}$. As $t_0 \in V(T_{v_i})$ for every $i \in [2q]$, we get that C is indeed a clique. Also, observe that t_i belongs exactly to the subtree of v_i and of $v_{(i+1) \bmod 2q}$ for every $i \in [2q]$. It follows that this is indeed an in–out branching model of the even sun. We then get that all even suns are in–out rooted directed path graphs. As for not being dually chordal, it is known that these graphs are exactly those admitting a maximum neighborhood ordering [6]. Because a sun does

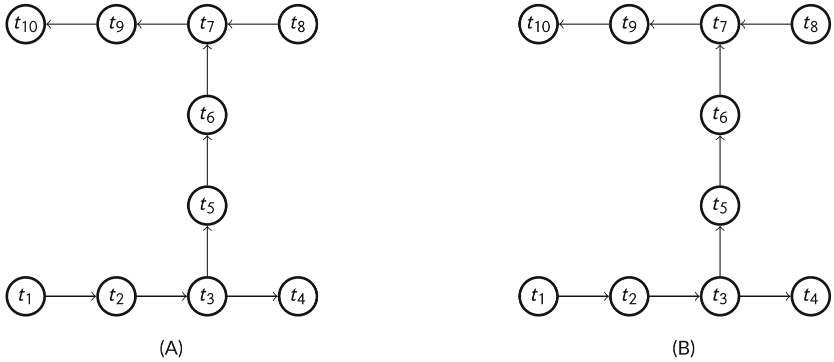

FIGURE 4 Example of a directed path graph that is not an in–out rooted directed path graph. The nodes of the tree model are related to the following subsets: $V_{t_1} = \{1,2,3\}$; $V_{t_2} = \{2,3,4,5\}$; $V_{t_3} = \{2,3,5,6,7\}$; $V_{t_4} = \{5,7,8\}$; $V_{t_5} = \{2,6,9\}$; $V_{t_6} = \{9,11,15\}$; $V_{t_7} = \{11,12,14,15,16\}$; $V_{t_8} = \{14,16,17\}$; $V_{t_9} = \{11,12,13,14\}$; $V_{t_{10}} = \{10,11,12\}$. (A) Graph G. (B) Tree model of G.

not contain a vertex with maximum neighborhood [6], it follows that they are not dually chordal. Therefore, even suns are in–out rooted directed path graphs (and hence also directed path graphs) that are not dually chordal. Since it is known that strongly chordal graphs are dually chordal [6], and that rooted directed path graphs are strongly chordal [7], the item follows. ■

2.3 | Parameterized complexity

A *parameterized problem* is a language $\Pi \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a fixed finite alphabet. A pair $(I, \kappa) \in \Sigma^* \times \mathbb{N}$ is called an *instance* of Π with *parameter* κ, and we say that it is a *Yes instance* if $(I, \kappa) \in \Pi$. Given instances $(I, \kappa), (I', \kappa')$ of the same parameterized problem Π, it is said that they are *equivalent* if (I, κ) is a Yes instance of Π if and only if so does (I', κ'). A *reduction rule* for Π is a polynomial-time computable function that maps an instance (I, κ) to another instance (I', κ'). A reduction rule is *safe* if (I, κ) and (I', κ') are equivalent and $\kappa' \leq g(\kappa)$, where g is a computable function.

Given two parameterized problems $\Pi, \Pi' \subseteq \Sigma^* \times \mathbb{N}$, a *parameterized reduction* from Π to Π' is an algorithm that, given an instance (I, κ) of Π, computes an equivalent instance (I', κ') of Π' such that $\kappa' \leq g(\kappa)$ in time $f(\kappa) \cdot |I|^{O(1)}$, where f and g are a computable functions.

We refer the reader to [10] for further background on parameterized complexity.

3 | PROOF OF PART OF THEOREM 2 AND OF THEOREM 6

Recall that Theorem 2 states that the investigated problems can be solved in time $2^{O(\kappa \cdot v\ell(G))} \cdot n^{O(1)}$, given that a tree model achieving $v\ell(G)$ is given, and that Theorem 6 states that if (G, κ) is a Yes instance of Min-LC-VSP$_{\sigma,\rho}$, then $\ell(G) \leq \kappa \cdot v\ell(G)$. We start with the proof of these two theorems because the tool lemma will also be used in the next section to prove Theorem 1. We will see that the following lemma directly implies Theorem 6 and the part of Theorem 2 that concerns Dominating Set. The part of Theorem 2 concerning Connected Dominating Set and Steiner tree then follows from Theorem 3, which is proved in Sections 5 and 6.

Lemma 2. *Let G be a chordal graph, $\mathcal{T} = (T, \{T_u\}_{u \in V(G)})$ be a minimal tree model of G such that $v\ell(\mathcal{T}) = v\ell(G)$, κ be a positive integer and $S \subseteq V(G)$ such that $N[u] \cap S \neq \emptyset$ for every leafy vertex $u \in \mathcal{L}(G, \mathcal{T})$. If $|S| \leq \kappa$, then $\ell(G) \leq \kappa \cdot v\ell(G)$.*

Proof. By contradiction, let $\ell_1, \ldots, \ell_k$ be the leaves of T, with $k > \kappa \cdot v\ell(G)$. Since $\mathcal{T}$ is minimal, for each $i \in \{1, \ldots, k\}$, there exists $v_i \in V_{\ell_i}$ such that $V(T_{v_i}) = \{\ell_i\}$, as otherwise we would have $V_{\ell_i} \subseteq V_{t_i}$ where t_i is the neighbor of ℓ_i in T. For each $u \in S$, let $D_u = \{v_i | u \in N[v_i]\}$. Observe that if $v_i \in S$, then $D_{v_i} = \{v_i\}$ since $\ell_1, \ldots, \ell_k$ are all distinct leaves of T (i.e., $\{v_1, \ldots, v_k\}$ is an independent set). Note also that if $u \in S \backslash \{v_1, \ldots, v_k\}$, then $|D_u| \leq v\ell(G)$. By assumption, we know that $N[v_i] \cap S \neq \emptyset$ for every $v_i \in \{v_1, \ldots, v_k\} \backslash S$, which means that $\bigcup_{u \in S} D_u = \{v_1, \ldots, v_k\}$. However, we know that $|\bigcup_{u \in S} D_u| \leq \sum_{u \in S} |D_u| \leq |S| \cdot v\ell(G) \leq \kappa \cdot v\ell(G)$, a contradiction since $k > \kappa \cdot v\ell(G)$. ■

Since in Theorem 6 we have $0 \notin \rho$, we get directly that a solution S to MIN-LC-VSP$_{\sigma,\rho}$ applied to (G,κ) must be such that $N[u] \cap S \neq \emptyset$ for every $u \in V(G)$, and in particular for every leafy vertex. Hence, Theorem 6 follows from the above lemma. Additionally, it is known that DOMINATING SET can be solved in time $2^{O(\ell)} \cdot n^{O(1)}$ on a chordal graph G, where $\ell = \ell(G)$ and $n = |V(G)|$ [16]. Since the hypothesis of Lemma 2 applies to DOMINATING SET, we get that it can be solved in time $2^{O(\kappa \cdot v\ell(G))} \cdot n^{O(1)}$ on a chordal graph, provided the appropriate model is given, that is, the part of Theorem 2 concerning DOMINATING SET follows.

4 | PROOF OF PART OF THEOREM 1

Theorem 1 states that the investigated problems can be solved in time $2^{2\kappa \log \kappa} \cdot n^{O(1)}$. We now prove the part of this theorem concerning DOMINATING SET and the remainder follows from Theorem 3, which we prove in Sections 5 and 6. Even though this complexity is worse than the one achieved by Theorem 2, we propose this algorithm for its simplicity when compared to the one presented in [16] that we used to conclude Theorem 2.

Recall that in the B-DOMINATING SET problem, we are given a graph G, a subset $B \subseteq V(G)$ and a positive integer κ, and we want to decide whether (G, B) has a B-dominating set (i.e., a set S such that $N[u] \cap S \neq \emptyset$ for every $u \in B$) of size at most κ. Therefore, DOMINATING SET is a particular case of B-DOMINATING SET where $B = V(G)$.

In this section, we present an FPT algorithm for B-DOMINATING SET parameterized by κ restricted to undirected path graphs. Clearly, solving DOMINATING SET on (G,κ) is equivalent to solving B-DOMINATING SET on $(G, V(G), \kappa)$.

From this point on, we assume that G is an undirected path graph, and that $\mathcal{T} = (T, \{P_u\}_{u \in V(G)})$ is a tree model of G where each P_u is a subpath of T (this can be computed in polynomial time [19]). We also denote by $\mathcal{L}$ the set $\mathcal{L}(G,\mathcal{T})$. We solve this problem by first applying a series of reduction rules.

Reduction Rule 1. *Let (G, B, κ) be an instance of B-DOMINATING SET.*

- *If there exists $v \in \mathcal{L} \setminus B$, then DELETE v, obtaining the instance $(G - v, B, \kappa)$.*
- *If there exists $v \in \mathcal{L} \cap B$ such that v is an isolated vertex, then DELETE v, obtaining the instance $(G-v, B-v, \kappa-1)$.*

Proof of Safeness. Deleting a vertex clearly does not increase the vertex leafage, so we just need to prove the equivalence between instances. For the first case, clearly a B-dominating set in $G - v$ is also a B-dominating set in G. So let S be a B-dominating set in G. If $v \notin S$, then there is nothing to prove. Otherwise, since v is a leafy vertex, we get that $N[v]$ is a clique, which means that any $b \in B$ dominated by v can be dominated by any $u \in N(v)$ instead. The second part is analogous, except that in such case the only vertex that can dominate v is v itself. ■

Now we can assume that every leafy vertex v of G is in B and is not isolated. The following rule allows us to bound the number of leaves in T.

Reduction Rule 2. *If $B = \emptyset$ and $\kappa = 0$, then output YES. Else if $B = \emptyset$ and $\kappa < 0$, then output NO. Else if $B \neq \emptyset$ and either $\kappa \leq 0$ or T has more than 2κ leaves, then output NO.*

Proof of Safeness. It follows from the assumption that every leafy vertex v is in B and from Lemma 2. ■

Thus, we assume that T has at most 2κ leaves. Furthermore, if $|V(T)| = 1$, then G is the complete graph and any vertex dominates B; so from now on we assume that T has at least 2 leaves. Our next operation is not a reduction rule, but a branching rule instead. More specifically, we create a number of smaller instances in order to solve the problem. The amount of instances created is bounded by a function of κ, due to the fact that T has at most 2κ leaves.

Given nodes t, t' of T, denote by $P(t,t')$ the unique t,t'-path in T. Also, given a subpath P of T, denote by V_P the set $\{u \in V(G) | P_u \subseteq P\}$. Say that $u \in V_P$ is *P-maximal* if there is no $v \in V_P$ such that P_u is a proper subpath of P_v.

Branching Rule 1. *Let $\mathcal{I} = (G, B, \kappa)$ be an instance of B-DOMINATING SET. Let $\ell \in V(T)$ be a leaf of T, and $u \in V(G)$ be such that $V(P_u) = \{\ell\}$. For each leaf $t \in V(T)$, $t \neq \ell$, do the following:*

1. *Choose $v \in V_{P(\ell,t)}$ to be a $P(\ell,t)$-maximal vertex such that $\ell \in V(P_v)$;*
2. *Define $G' = G - V_{P_v}$ and $B' = B \setminus N_G[v]$;*
3. *Create the instance $\mathcal{I}(u,t) = (G', B', \kappa - 1)$.*

We remark that $\{u, v\} \subseteq V_{P_v}$ and thus those two vertices are not in G'.

Safeness of the Branching Rule. First, observe that a minimal tree model of G' can again be obtained by applying Proposition 1 to the tree model $\mathcal{T}$ restricted to G'. Therefore, it remains to show that $\mathcal{I}$ is a YES instance of

B-DOMINATING SET if and only if there exists a leaf t of T distinct from ℓ such that the instance $\mathcal{I}(u, t)$ is also a YES instance.

For the necessity part, let S be a B-dominating set of G. By our assumption that Reduction Rule 1 is not applicable, we get that $u \in B$, and $N(u) \neq \emptyset$. Note that, since $V(P_u) = \{\ell\}$ we get that $N(u)$ is a clique. This means that if $u \in S$, then $(S \backslash \{u\}) \cup \{v\}$ is also B-dominating, for any $v \in N(u)$. Therefore, we can assume that $u \notin S$. Now, let v be a neighbor of u in S. Also, let t' be the endpoint of P_v distinct from ℓ, and let t be any leaf separated from ℓ by the edge of P_v incident to t' (it might happen that $t = t'$). Then, either v is $P(\ell, t)$-maximal, or there exists $x \in V_\ell$ which is $P(\ell, t)$-maximal. If the latter occurs, we get that $P_v \subseteq P_x$, which in turn gives us that $N[v] \subseteq N[x]$ and that $(S \backslash \{v\}) \cup \{x\}$ is a B-dominating set of G. We can therefore suppose, without loss of generality, that v is $P(\ell, t)$-maximal. Now, let $\mathcal{I}(u, t)$ be the instance of B-DOMINATING SET constructed as in the statement of the Branching Rule. Observe that if v' is also $P(\ell, t)$-maximal such that $\ell \in V(P_{v'})$, then $P_{v'} = P_v$ and the constructed instance is the same, so we can suppose that indeed v is the iterated $P(\ell, t)$-maximal vertex. It remains to prove that $S' = S \backslash \{v\}$ is a B'-dominating set of $\mathcal{I}(u, t)$. For this, let $b \in B'$. By construction $b \in B \backslash N_G[v]$. Therefore, b has a neighbor in $S \backslash \{v\}$, as we wanted to show.

For the sufficiency, let $\mathcal{I}(u, t) = (G', B', \kappa - 1)$ be the instance given by the Branching Rule, and let S' be a B'-dominating set of G'. Because every $b \in B'$ is dominated by S', and $B \backslash B' = N_G[v]$, we get that $S = S' \cup \{v\}$ is a B-dominating set of G, as we wanted.

The part of Theorem 1 concerning DOMINATING SET follows by bounding the number of instances, since each instance is solved in polynomial time. As for CONNECTED DOMINATING SET and STEINER TREE, they follow from Theorem 3, proved in Sections 5 and 6.

Lemma 3. *Let G be an undirected path graph. Then DOMINATING SET can be solved in time $2^{2\kappa \log \kappa} \cdot n^{O(1)}$.*

Proof. We start by obtaining a tree model with optimal vertex leafage for G by applying the polynomial algorithm in [19]. Then, we iteratively apply Reduction Rules 1 and 2 (also applying Proposition 1 to maintain a minimal tree model), until we reach the need to apply the Branching Rule. Recall that we are assuming that T has at most 2κ leaves. The latter is then applied for every leaf of the current tree model, which generates at most $(2\kappa)^2 = 4\kappa^2$ new instances. The process then starts over on each of the generated instances. Finally, since the budget for the size of the solution decreases by 1 after applying the Branching Rule, we get that a new application of the rule would generate at most $(2\kappa - 2)^2$ new instances, and so on. Observe that this cascade can be done at most κ times, since at each application we keep one vertex in the dominating set that is being constructed. Therefore, in the worst case scenario, we get that the total number of generated instances is: $(2\kappa)^2 \cdot (2\kappa - 2)^2 \cdots (2\kappa - (2\kappa - 2))^2$ $= [2^\kappa (\kappa \cdot (\kappa - 1) \cdots 1)]^2$. Applying Stirling's approximation, we get time $\left[2^\kappa \cdot \left(\frac{\kappa}{e}\right)^\kappa \cdot \sqrt{2\pi\kappa}\right]^2 = 2^{2\kappa \log \kappa} \cdot n^{O(1)}$. Observe that if an instance eventually ends up with a non-empty set of vertices of B and a budget of 0 (base case of the branching procedure), then Reduction Rule 2 will output NO. Because the applications of Reduction Rules 1 and 2 and of Proposition 1 are done in polynomial time, we get the claimed running time. ∎

5 | REDUCING CONNECTED DOMINATING SET TO DOMINATING SET

In this section, we prove that Connected Dominating Set $\preceq_\kappa$ Dominating Set (second part of Theorem 3). Recall that we need this to conclude the part of Theorems 1 and 2 concerning CONNECTED DOMINATING SET. We also prove Theorem 4, which states that WEIGHTED CONNECTED DOMINATING SET is polynomial-time solvable on strongly chordal graphs.

Consider (G, κ) an instance of CONNECTED DOMINATING SET, where G is a chordal graph. Consider also a minimal tree model $\mathcal{T} = (T, \{T_u\}_{u \in V(G)})$ of G. We construct an instance (G', κ) of DOMINATING SET by modifying $\mathcal{T}$ and taking G' as the intersection graph of the modified model.

We give the construction step by step. Observe Figure 5 to follow the construction.

- Let T' be obtained from T by subdividing each $tt' \in V(T)$; denote the obtained node by $\eta(tt')$.
- Do the same for the subtrees related to T_u for each $u \in V(G)$ containing tt', denoting the obtained subtree of T' by T'_u.
- Finally, for each $tt' \in E(T)$, create a new vertex u and relate it to the subtree T'_u containing just the node $\eta(tt')$; denote such vertex by $v(tt')$ and let $\mathcal{V}$ be the set of all new vertices.
- Let G' be the graph whose tree model is $\mathcal{T}' = (T', \{T'_u\}_{u \in V(G) \cup \mathcal{V}})$.
- If we are dealing with the weighted version of these problems, where we are also given a weight function $\omega : V(G) \to \mathbb{R}$, then we let $\omega'(u) = \omega(u)$ if $u \in V(G)$, and $\omega'(u) = M + 1$ if $u \notin V(G)$, where $M = \max \omega(V(G))$.

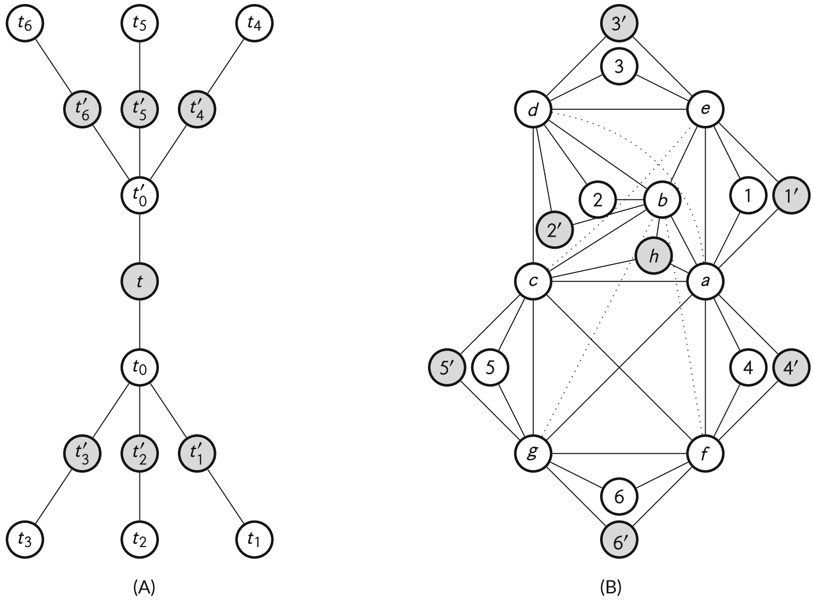

FIGURE 5 Example for the construction. The nodes of the tree are related to the following subsets. White nodes: $V_{t_0} = \{a,b,c,d,e\}$; $V_{t_1} = \{a,e,1\}$; $V_{t_2} = \{b,d,2\}$; $V_{t_3} = \{d,e,3\}$; $V_{t'_0} = \{a,b,c,f,g\}$; $V_{t_4} = \{a,f,4\}$; $V_{t_5} = \{c,g,5\}$; $V_{t_6} = \{f,g,6\}$. Gray nodes: $\boldsymbol{V_{t'_1} = \{a,e,1'\}}$; $\boldsymbol{V_{t'_2} = \{b,d,2'\}}$; $\boldsymbol{V_{t'_3} = \{d,e,3'\}}$; $\boldsymbol{V_{t'_4} = \{a,f,4'\}}$; $\boldsymbol{V_{t'_5} = \{c,g,5'\}}$; $\boldsymbol{V_{t'_6} = \{f,g,6'\}}$; $\boldsymbol{V_t = \{a,b,c,7\}}$. (A) The tree model of graph G is represented by white nodes, while the new nodes are in gray. (B) The original graph G is formed by the white vertices, while the new vertices are in gray. Some of the edges are dotted in order to make the figure visually cleaner.

Observe that the last step ensures that $u \in \mathcal{V}$ is never chosen for a minimum weight dominating set of G'. In Figure 5, the new nodes are $\{t, t'_1, \dots, t'_6\}$, while the new vertices in G' are $\{h, 1', \dots, 6'\}$. We now prove the desired reduction.

Lemma 4. *(G,κ) is a* Yes *instance of* Connected Dominating Set *if and only if (G',κ) is a* Yes *instance of* Dominating Set*. The same holds for the weighted versions of these problems.*

Proof. Suppose first that $S \subseteq V(G)$ is a connected dominating set of G such that $|S| \leq \kappa$. We want to prove that S is also a dominating set of G'. By definition and since G is an induced subgraph of G', we know that $\emptyset \neq S \cap N_G[u] \subseteq S \cap N_{G'}[u]$ for every $u \in V(G)$. It thus remains to show that S also dominates $\mathcal{V}$. For this we prove that, for every $tt' \in E(T)$, there exists $u \in S$ such that $tt' \in E(T_u)$; note that this finishes the proof by the construction of G' (i.e., the added vertices are exactly the ones related to the edges of T). Suppose otherwise and let $tt' \in E(T)$ be such $tt' \notin E(T_u)$ for every $u \in S$. Let also T_t be the component of $T - \{tt'\}$ containing t, and define $T_{t'}$ analogously. Let $G'' = G[S]$, and T'' be the subgraph of T formed by the subtrees related to S, that is, $T'' = \bigcup_{u \in S} T_u$. Since G'' is connected, we know by Lemma 1 that T' is connected and, since $tt' \notin E(T'')$, we get that T'' is either contained in T_t or in $T_{t'}$, say T_t. But now, because $\mathcal{T}$ is minimal, there is some leafy vertex u of G such that $\ell_u \in V(T_{t'})$, a contradiction as in this case u has no neighbors in S.

Now, suppose $S \subseteq V(G')$ is a dominating set of G' such that $|S| \leq \kappa$. We first show that we can suppose that $S \subseteq V(G)$. For this, consider some $tt' \in E(G)$; also, let $u = v(tt')$. Since G is connected, by Lemma 1 there exists $v \in V(G)$ such that $tt' \in E(T_v)$. By definition of tree model, we get that $N_{G'}[u] \subseteq N_{G'}[v]$. Therefore, if $u \in S$, then $(S \backslash \{u\}) \cup \{v\}$ is also a dominating set of G'. We can then suppose that $S \subseteq V(G)$, as we wanted to show. Since G is an induced subgraph of G', it follows that S is a dominating set of G. It remains to prove that $G[S]$ is also connected. Because S also dominates $\mathcal{V}$ in G', we get that for every $u \in \mathcal{V}$, there exists $v \in N_{G'}(u) \cap S$. This means that $tt' \in E(T_v)$, where tt' is such that $u = v(tt')$. In other words, for every $tt' \in E(T)$, there exists $v \in S$ such that $tt' \in E(T_v)$, and it follows that $G[S]$ is connected by Lemma 1.

Observe that the weighted version holds by the definition of the weight function. Indeed, we have that $\omega(S) = \omega'(S)$ for every $S \subseteq V(G)$. On the other hand, if $S \subseteq V(G')$ is a dominating set of G' not contained in $V(G)$, then the definition of the weight function and the above paragraph tells us there exists a dominating set S' of G' contained in $V(G)$ such that $\omega(S') < \omega(S)$. ∎

Observe that each vertex added to G' is a simplicial vertex, which is a vertex whose neighborhood is a clique. We call them *edge-simplicial* since they are adjacent exactly to the set of vertices whose trees contain a certain edge of T. Because the construction of G' can be done in polynomial time, we get the following corollary, which proves the part of Theorem 3 concerning the reduction from CONNECTED DOMINATING SET to DOMINATING SET.

Corollary 2. *If (WEIGHTED) DOMINATING SET can be solved in time $f(\kappa + v\ell(G)) \cdot n^{O(1)}$ on a chordal graph G, for every G belonging to a subclass of chordal graphs closed under inclusion of edge-simplicial vertices, then so does (WEIGHTED) CONNECTED DOMINATING SET.*

Observe also that the construction of G' preserves the vertex leafage of G. In other words, path graphs are closed under inclusion of edge-simplicial vertices. We then get that the complexities given in Theorems 1 and 2 for CONNECTED DOMINATING SET follow by applying the algorithm presented in Sections 3 and 4 to (G', κ). Additionally, it follows that if G is a rooted path graph, then the algorithm for WEIGHTED DOMINATING SET presented in [4] for strongly chordal graphs can be applied to (G', ω', κ) in order to solve WEIGHTED CONNECTED DOMINATING SET. The next lemma shows that in fact this extends to all strongly chordal graphs, as stated in Theorem 4.

Lemma 5. *The class of strongly chordal graphs is closed under inclusion of edge-simplicial vertices.*

Proof. Let G be a strongly chordal graph, $\mathcal{T} = (T, \{T_u\}_{u\in V(G)})$ be a minimal tree model of G, and consider $tt' \in E(T)$. Denote by N the set $\{u \in V(G) | tt' \in E(T_u)\}$, and let G' be obtained from G by adding x that is adjacent to every vertex in N. Observe that if we prove that G' is strongly chordal, then the lemma follows. Suppose otherwise. Because G' is trivially chordal, and strongly chordal graphs are exactly the sun-free chordal graphs [13], there must be a sun H in G' containing x. Denote the vertices in the clique of H by $K = \{v_1, \ldots, v_q\}$, and the vertices not in the clique of H by $x_1, \ldots, x_q$. Suppose, without loss of generality, that $x = x_1$, and that x_1 is adjacent to v_1 and v_2. By definition of sun and the fact that $N_{G'}(x) = N$, we get that $N \cap K = \{v_1, v_2\}$. Now, define T_t and $T_{t'}$ as before. Also, let G_t be the subgraph of G induced by the vertices u such that $V(T_u) \cap V(T_t) \neq \emptyset$; define $G_{t'}$ similarly. By definition of tree model, we know that $V(G_t) \cap V(G_{t'}) = N$. Since $q \geq 3$ and there must exist a node t'' for which $K \subseteq V_{t''}$, we get that $H - \{x, v_1, v_2\}$ is contained in G_t or in $G_{t'}$, say it is contained in G_t. This means that $K \cap V_{t'} = \{v_1, v_2\}$. But since $\mathcal{T}$ is minimal, there exists $x' \in V_{t'} \setminus V_t$, and because $\{v_1, v_2\} \subseteq V_t$ we know that $x' \notin \{v_1, v_2\}$. We get a contradiction as in this case x' is adjacent exactly to $\{v_1, v_2\}$ in $H - x$, that is, $H - x + x'$ is a sun in G. ∎

6 | REDUCING STEINER TREE TO CONNECTED DOMINATING SET

In this section, we make a reduction from STEINER TREE to CONNECTED DOMINATING SET that, as we will see, preserves leafage and vertex leafage. This finishes the proofs of Theorems 1–3.

Given an instance (G, X, κ) of STEINER TREE and a minimal tree model $(T, \{T_u\}_{u\in V(G)})$ of G, we construct an equivalent instance (G', κ) of CONNECTED DOMINATING SET as follows. Denote $\mathcal{L}(G, \mathcal{T})$ by $\mathcal{L}$. First, we iteratively apply the following operations.

- If $u \in \mathcal{L} \setminus X$, then remove u from G, removing also ℓ_u from the model if u is the only leafy vertex in V_{ℓ_u}.
- Do the same if $u \in \mathcal{L} \cap X$ is such that there exists $x \in (X \cap N(u)) \setminus \mathcal{L}$.
- At the end, we have an instance (G', X', κ), together with a minimal tree model $\mathcal{T}'$ of G', such that $\mathcal{L}(G', \mathcal{T}') \subseteq X'$ and $X' \cap V_\ell \subseteq \mathcal{L}(G', \mathcal{T}')$ for every leaf ℓ of the tree model $\mathcal{T}'$. We call such an instance a *clean instance*.

Now, consider a clean instance (G, X, κ), and a minimal tree model $\mathcal{T}$ of G; again denote $\mathcal{L}(G, \mathcal{T})$ by $\mathcal{L}$. We now use the bypass operation to eliminate vertices in $X \setminus \mathcal{L}$ while maintaining the connectivity that is gained by including these vertices in the induced subgraph $G[S \cup X]$. The *bypass* operation of a vertex $v \in X \setminus \mathcal{L}$ consists of:

- Removing v from $V(G)$; and
- Adding uw for every pair u, w of neighbors of v (such that $uw \notin E(G)$ to avoid multiple edges).

Denote by (G', κ) the obtained instance, that is, the instance of CONNECTED DOMINATING SET obtained from the clean instance (G, X, κ) of STEINER TREE by applying bypass to each vertex $v \in X \setminus \mathcal{L}$. We first show how to modify $\mathcal{T}$ into a tree model of G'.

Lemma 6. *Let (G, X, κ) be a clean instance of STEINER TREE where G is chordal, $\mathcal{T} = (T, \{T_u\}_{u\in V(G)})$ be a minimal tree model of G, and let G' be constructed as above. Then there exists a tree model $\mathcal{T}'$ of G' such that $v\ell(\mathcal{T}') \leq v\ell(\mathcal{T})$ and $\ell(\mathcal{T}') \leq \ell(\mathcal{T})$.*

Proof. We prove that if G' is obtained from G by bypassing $v \in V(G)$, then we can modify $\mathcal{T}$ into $\mathcal{T}'$, a tree model of G', ensuring that $v\ell(\mathcal{T}') \leq v\ell(\mathcal{T})$ and $\ell(\mathcal{T}') \leq \ell(\mathcal{T})$. By iteratively applying this operation to every $u \in X \setminus \mathcal{L}(G, \mathcal{T})$, we get the desired result. Consider $\mathcal{T}' = (T', \{T'_u\}_{u \in V(G')})$ obtained by applying the following operation.

1. T' is the tree obtained from T by contracting T_v to a single vertex, t_v; and
2. For each $u \in V(G')$, if $V(T_v) \cap V(T_u) = \emptyset$, then T_u remains the same; otherwise, T'_u is the subtree of T' containing exactly the vertices in $(V(T_u) \setminus V(T_v)) \cup \{t_v\}$.

To see that the vertex leafage and the leafage does not increase, just observe that edge contractions in trees cannot increase the number of leaves. It remains to argue that $\mathcal{T}'$ is indeed a tree model of G'. For this, we must have $uw \in E(G')$ if and only if $V(T'_u) \cap V(T'_w) \neq \emptyset$, which holds because $t_v \in V(T'_u)$ if and only if $u \in N(v)$. ∎

Now, we prove that the reduction works.

Lemma 7. *Let (G, X, κ) be a clean instance of* Steiner tree, *where G is a chordal graph, and let $\mathcal{T} = (T, \{T_u\}_{u \in V(G)})$ be a minimal tree model of G. Then, (G, X, κ) is a* Yes *instance of* Steiner tree *if and only if (G', κ) is a* Yes *instance of* Connected Dominating Set.

Proof. Let $\mathcal{T}' = (T', \{T'_u\}_{u \in V(G')})$ be the tree model of G' obtained as in Lemma 6; denote by $\mathcal{L}$ the set $\mathcal{L}(G, \mathcal{T})$. First note that, since (G, X, κ) is a clean instance, we have that $X \cap V_\ell \subseteq \mathcal{L}$ for every leaf node ℓ of $\mathcal{T}$. In other words, all the subtree contractions made to $\mathcal{T}$ in order to obtain $\mathcal{T}'$ cannot contract a leaf node; hence $\mathcal{L}(G', \mathcal{T}') = \mathcal{L}$. Now, consider a Steiner set S for (G, X) of size at most κ. We argue that S is also a connected dominating set of G'. By Lemma 1, the facts that $G[S \cup X]$ is connected, and $\mathcal{L} \subseteq X$ and every edge of T is within the unique path in T between two leaves of T, we get that:

$$\text{There exists } u \in (X \cup S) \text{ such that } tt' \in E(T_u), \text{ for every } tt' \in E(T). \tag{1}$$

We want to prove that the same holds for S and T', that is, that there exists $u \in S$ such that $tt' \in E(T'_u)$, for every $tt' \in E(T')$. Observe that this directly implies that S is a dominating set of G', and we get that it is connected by Lemma 1. So consider any $tt' \in E(T')$. If $tt' \in E(T)$, then let $u \in (S \cup X)$ be such that $tt' \in E(T_u)$. Note that, since $X \cap V_\ell \subseteq \mathcal{L}$ for every leaf ℓ of T and because every $x \in X \setminus \mathcal{L}$ has its subtree contracted, it follows that u must be in S and we are done. So now consider that $tt' \in E(T') \setminus E(T)$ and suppose, without loss of generality, that $t \in V(T') \setminus V(T)$. This means that t is a node obtained by contracting subtrees $T_{y_1}, \dots, T_{y_q}$ for some subset $Y = \{y_1, \dots, y_q\} \subseteq X \setminus \mathcal{L}$. Let T_Y be the subgraph of T formed by the union of $T_{y_1}, \dots, T_{y_q}$; in other words, the subgraph whose contraction generated t. Since only edge contractions are allowed, we know that T_Y is a subtree of T. Observe that t' could also be the product of edge contractions. We treat the cases separately. Suppose first that $t' \in V(T)$; by the definition of edge contraction, we know that t' must be adjacent to a node in T_Y, say t''. By Equation (1), there exists $u \in X \cup S$ such that $t't'' \in E(T_u)$, which in turn implies that $tt' \in E(T'_u)$. It thus remains to show that $u \in S$. Since $E(T_u) \neq \emptyset$, we know that $u \notin \mathcal{L}$. And because every node of X not in $\mathcal{L}$ got its tree contracted to a single node, while T'_u is not a single node as it contains the edge tt', we get that $u \notin X$, that is, $u \in S$, as we wanted to show. Now suppose that t' is obtained by the contraction of a subset $Y' = \{y'_1, \dots, y'_q\} \subseteq X \setminus \mathcal{L}$. Let $T_{Y'}$ be the subgraph of T formed by the union of $T_{y'_1}, \dots, T_{y'_q}$. By definition of edge contraction, there exists $h \in V(T_Y)$ and $h' \in V(T_{Y'})$ such that $hh' \in E(T)$. By Equation (1), there exists $u \in X \cup S$ such that $hh' \in E(T_u)$, which implies that $tt' \in E(T'_u)$. A similar argument can be applied to conclude that $u \in S$.

Now, let S be a connected dominating set of G' of size at most κ. We want to prove that S is a Steiner set for (G, X), that is, that $H = G[S \cup X]$ is connected. We first prove that S dominates X. Indeed, because $G'[S]$ is connected and dominates $\mathcal{L}$, by Lemma 1 we get that there exists $u \in S$ such that $tt' \in E(T'_u)$ for every $tt' \in E(T')$. It follows that S also dominates every $x \in X \setminus \mathcal{L}$, it suffices to take any $u \in S$ such that T'_u contains any edge incident to t_x, the node containing the contraction of T_x. It remains to argue that H is connected. For this let T^* be the subgraph of T formed by the union $\bigcup_{x \in X \setminus \mathcal{L}} T_x$. By construction, we know that T' is obtained from T by contracting each component of T^* to a single node. Now, if $tt' \in E(T^*)$, then by definition there exists $x \in X$ such that $tt' \in E(T_x)$. And if $tt' \in E(T) \setminus E(T^*)$, then there exists an edge of T' related to tt', in which case we know that there exists $u \in S$ such that $tt' \in E(T_u)$. We then get that there exists $u \in S \cup X$ such that $tt' \in E(T_u)$ for every $tt' \in E(T)$. It follows from Lemma 1 that H is connected. ∎

7 | PROOF OF THEOREM 5

In this section, we prove Theorem 5, namely that DOMINATING SET, CONNECTED DOMINATING SET and STEINER TREE are polynomial-time solvable on in–out rooted directed path graphs. We do this again by solving B-DOMINATING SET and applying Theorem 3, as we did in Section 4. Recall that an oriented tree T is an in–out branching rooted at r if every path between r and a leaf ℓ is oriented either towards r or towards ℓ. Also, a graph G is an in–out rooted directed path graph if G is the intersection graph of directed paths of a rooted in–out branching.

Consider an instance of B-DOMINATING SET, (G, B, κ), where G is an in–out rooted directed path graph with tree model $\mathcal{T} = (T, \{P_u\}_{u \in V(G)})$, with T rooted in $r \in V(T)$. Observe that Reduction Rules 1 and 2 still apply. In what follows, we introduce a third rule.

Let $L = \{\ell_1, \ldots, \ell_h\}$ be the set of *source leaves* of T (i.e., leaves without-degree 1) and $L' = \{\ell'_1, \ldots, \ell'_{h'}\}$ the set of *sink leaves* of T (leaves with in-degree 1). Also, for each $i \in [h]$, denote by Q_i the ℓ_i, r-path in T; similarly, for each $i \in [h']$, denote by Q'_i the r, ℓ'_i-path in T. Suppose that ℓ_i is such that every $u \in V_{\ell_i}$ has a path strictly contained in Q_i, and let $x \in V_{\ell_i}$ be such that $P_u \subseteq P_x$ for every $u \in V_{\ell_i}$; we say that x is ℓ_i*-maximal*. Let $t \in V(T)$ be the extremity of P_x different from ℓ_i. The problem is reduced by adding x to the searched solution. Recall from Section 4 that, given a path $P \subseteq T$, V_P denotes the set $\{u \in V(G) | P_u \subseteq P\}$.

Reduction Rule 3. *Let (G, B, κ) be an instance of B-DOMINATING SET, with G being an in–out rooted directed path graph, and $\mathcal{T} = (T, \{P_u\}_{u \in V})$ be a minimal tree model of G. Suppose that Reduction Rule 1 cannot be applied, and that ℓ_i is such that $P_u \subset Q_i$ for every $u \in V_{\ell_i}$. Let x be an ℓ_i-maximal vertex. Then ADD x to the solution, obtaining the instance $(G - V_{P_x}, B \backslash N_G[x], \kappa - 1)$.*

Proof of Safeness. If Reduction Rule 1 cannot be applied, then we get that there exists $v \in \mathcal{L} \cap V_{\ell_i} \cap B$. Additionally, because v is not isolated we have $V_{\ell_i} \backslash \{v\} \neq \emptyset$. Observe that there exists an ℓ_i-maximal vertex x in $V_{\ell_i} \backslash \{v\}$. We want to prove that (G, B, κ) is a YES instance of B-DOMINATING SET if and only if so is $(G - V_{P_x}, B - N[x], \kappa - 1)$. Denote $G - V_{P_x}$ by G' and $B - N_G[x]$ by B'.

First, consider a B-dominating set S for (G, B) of size at most κ. Since $v \in B$ and $N_G[v] \subseteq V_{\ell_i} \subseteq V_{P_x}$, we get that $S \cap V_{P_x} \neq \emptyset$; hence $|S'| \leq \kappa - 1$, where $S' = S \backslash V_{P_x}$. We now want to prove that S' is a B-dominating set for (G', B'). For this, let $u \in B'$. Since $u \in B$, there exists $u' \in N_G[u] \cap S$. If $u' \in S'$, then there is nothing to prove; so suppose otherwise, in which case $u' \in V_{P_x}$. By definition of V_{P_x}, this gives us that $P_{u'} \subseteq P_x$ and hence $N_G[u'] \subseteq N_G[x]$, a contradiction as in this case $u \in N_G[x]$ and should not be in B'.

Now consider a B-dominating set S' for (G', B') of size at most $\kappa - 1$. We want to prove that $S = S' \cup \{x\}$ is a B-dominating set for (G, B). For this, let $u \in B$. If $u \in B'$, then it follows since $N_{G'}[u] \cap S' \neq \emptyset$ and $G' \subseteq G$. Otherwise, $u \in N_G[x]$, and it follows trivially that $N_G[u] \cap S \neq \emptyset$ as $x \in S$. ■

Clearly, the same kind of operation can be applied if the analogous holds for any of the sink nodes. Therefore we refrain from proving safeness of the next reduction rule.

Reduction Rule 4. *Let (G, B, κ) be an instance of B-DOMINATING SET, with G being an in–out rooted directed path graph, and $\mathcal{T} = (T, \{P_u\}_{u \in V})$ be a minimal tree model of G. Suppose that Reduction Rule 1 cannot be applied, and that ℓ'_i is such that $P_u \subset Q'_i$ for every $u \in V_{\ell'_i}$. Let x be an ℓ'_i-maximal vertex. Then ADD x to the solution, obtaining the instance $(G - V_{P_x}, B \backslash N[x], \kappa - 1)$.*

Now, for each $\ell \in L \cup L'$, let Q^*_ℓ denote Q_i in case $\ell = \ell_i \in L$ or Q'_i in case $\ell = \ell'_i \in L'$. Observe that, if Reduction Rules 3 and 4 cannot be applied, then there exists $x_\ell \in V_\ell$ such that $Q^*_\ell \subseteq P_{x_\ell}$. But note additionally that, in fact, if there exists also $u \in V_\ell \backslash \mathcal{L}$ with $P_u \subset Q^*_\ell$, then the same argument in the proof of safeness can be used to consider that u surely is not in any minimum solution; indeed such a vertex could always be replaced by x_ℓ. Therefore, from now on we assume that the following holds for every $\ell \in L \cup L'$:

$$V_\ell \backslash \mathcal{L} \neq \emptyset \text{ and } Q^*_\ell \subseteq P_u, \text{ for every } u \in V_\ell \backslash \mathcal{L}. \tag{2}$$

Now, we construct an auxiliary graph that allows us to solve the problem by applying a matching algorithm. Let $H = (L \cup L', E)$ be a bipartite graph, that is, a graph where the edges connect L to L', obtained by adding an edge $\ell_i \ell'_j$ in E whenever there exists $u \in V(G)$ whose path P_u is equal to the ℓ_i, ℓ'_j-path in T (in math terminology, $P_u = Q_i \cup Q'_j$). We make an abuse of language and write $u \in E$ in this case. An *edge cover* of H is a subset $S \subseteq E \subseteq V(G)$ such that every non-isolated vertex of H is incident to some $u \in S$. The minimum size of an edge cover of H is denoted by $\beta'(H)$ and the number of isolated vertices of H is denoted by $i(H)$. We prove that we can solve our problem by finding a minimum edge cover of H, which is known to be polynomial-time solvable (see e.g., [34]).

Lemma 8. *Let (G, B, κ) be an instance of B-Dominating Set, with G being an in–out rooted directed path graph, and $\mathcal{T} = (T, \{P_u\}_{u \in V})$ be a minimal tree model of G. Suppose that Reduction Rules 1 -4 cannot be applied, and let H be constructed as above. Then (G, B, κ) is a Yes instance if and only if $\kappa \geq \beta'(H) + i(H)$.*

Proof. First, let S be a B-dominating set for (G, B) of size at most κ. If there exists $x \in V_\ell \cap \mathcal{L}$, for some leaf ℓ, then Equation 2 tells us that there must exist $u \in V_\ell \backslash \mathcal{L}$ such that $Q^*_\ell \subseteq P_u$, or in other words, such that $N_G[x] \subseteq N_G[u]$. Therefore we could replace x by u in S. This allows us to suppose that $S \cap \mathcal{L} = \emptyset$. Also, because Reduction Rule 1 cannot be applied, we get that $S \cap V_\ell \neq \emptyset$ for every $\ell \in L \cup L'$. We prove that, additionally, we can suppose that if $\ell_i \in L$ is such that ℓ_i is not an isolated vertex in H, then $S \cap V_{\ell_i} \cap E \neq \emptyset$. Suppose otherwise and let $u \in S \cap V_{\ell_i}$. Also, let $\ell'_j \in N_H(\ell_i)$; therefore, there exists $u' \in V(G)$ such that $P_{u'}$ is equal to the ℓ_i, ℓ'_j-path in T. We want to prove that $(S\backslash\{u\}) \cup \{u'\}$ is a B-dominating set for (G, B). For this, let t be the endpoint of P_u different from ℓ_i, and denote by Q''_u the r, t-path contained in P_u. Note that Equation (2) tells us that $Q_i \subseteq P_u$ and hence P_u can be seen as the concatenation of Q_i and Q''_u. Now, let ℓ'_k be any sink leaf separated from ℓ_i by t; it exists because we suppose that $u \notin E$, that is, t is not a leaf. Now observe that Equation (2) gives us that $Q'_k \subseteq P_v$ for every $v \in V_{\ell'_k} \backslash \mathcal{L}$. Because t is within the r, ℓ'_k-path if T, we get that $Q''_u \subseteq Q'_k \subseteq P_v$ for every $v \in V_{\ell'_k} \backslash \mathcal{L}$. Since we can suppose that $S \cap (V_{\ell'_k} \backslash \mathcal{L}) \neq \emptyset$, we get that the portion of P_u not necessarily intersected by $P_{u'}$ (namely, Q''_u) is intersected by some $v \in S \cap (V_{\ell'_k} \backslash \mathcal{L})$. In other words, if $x \in B$ is such that $N_G(x) \cap S = \{u\}$ and $x \notin N_G(u')$, then $x \in N_G(v)$. It follows that $(S\backslash\{u\}) \cup \{u'\}$ is a B-dominating set for (G, B) as we wanted to prove. The analogous can be applied to the sink leaves, so from now on we suppose that:

$$\text{if } \ell \in L \cup L' \text{ is such that } \ell \text{ is not an isolated vertex in } H, \text{ then } S \cap V_\ell \cap E \neq \emptyset. \tag{3}$$

Now let $S' = S \cap E$, that is, S' is obtained from S by removing every u whose path P_u is not a path between two leaves of T. We argue first that $|S\backslash E| \geq i(H)$. Indeed, consider $\ell \in L \cup L'$ an isolated vertex in H, and let $u_\ell \in S \cap V_\ell$. Because ℓ is isolated in H, the path P_{u_ℓ} is not a path between two leaves, that is, $u_\ell \notin E$ and hence $u_\ell \in S\backslash E$. But again because ℓ is isolated in H, we get that $u_\ell \neq u_{\ell'}$ for every pair of isolated vertices ℓ, ℓ'; hence $|S\backslash E| \geq i(H)$ as claimed. Now, note that $|S'| = |S \cap E| = |S| - |S\backslash E| \leq \kappa - i(H)$. Finally note that Equation (3) also gives us that S' is an edge cover of H; hence $\beta'(H) \leq \kappa - i(H)$ as desired.

For the converse, assume that $\beta'(H) \leq \kappa - i(H)$, and let $S \subseteq E$ be a minimum edge cover of H and $S' \subseteq V(G)$ be obtained from S by adding, for each isolated vertex ℓ of H, some $u \in V_\ell$ such that $P_\ell \subseteq P_u$ (it exists by Equation (2)). Clearly $|S'| = |S| + i(H) = \beta'(H) + i(H) \leq \kappa$, so it remains to prove that S' is a B-dominating set for (G, B). Just observe that $V_\ell \cap S' \neq \emptyset$, for every $\ell \in L \cup L'$, and recall Equation (2) to see that $V_t \cap S' \neq \emptyset$ for every $t \in V(T)$. It thus follows that S' is a dominating set for G, and in particular, a B-dominating set for (G, B). ∎

8 | CONCLUSION

In this article, we have investigated the complexity of Dominating Set, Connected Dominating Set and Steiner tree when parameterized by the size of the solution plus the vertex leafage ($\kappa + v\ell(G)$) of a given chordal graph G. We have found that they are all FPT, provided that a tree model with optimal vertex leafage of G is given. Since such a tree model can be found in polynomial time if G is an undirected path graph (which are graphs with vertex leafage 2), we get that they are all FPT on these graphs when parameterized by the size of the solution. A question is whether the condition about the provided tree model can be lifted. Because positive instances have leafage bounded by a function of κ and $v\ell(G)$, we know that if computing $v\ell(G)$ is FPT when parameterized by $\ell(G)$, then we would have a complete fixed-parameter algorithm. Another option could be to provide a tree model which is not very far from an optimal one, that is, that has vertex leafage at most $c \cdot v\ell(G)$ for some constant c. This would increase only the constants in our complexities, and we would again have fully FPT algorithms. We ask whether this is achievable. We recall the reader that deciding $v\ell(G) \leq 3$ is NP-complete, but that the vertex leafage can be computed in time $n^{O(\ell(G))}$ [9]. We then reinforce the following question posed in [9], and add a new question.

Question 1 ([9]). Is deciding whether $v\ell(G) \leq k$ solvable in FPT time when parameterized by $\ell(G)$?

Question 2. Can $v\ell(G)$ be approximated by a constant factor in polynomial (or FPT) time?

Recall the broad class of problems defined in the Introduction, called Min-LC-VSP problems [8, 15]. Given a graph G on n vertices and subsets $\sigma, \rho \subseteq \{0, \dots, n-1\}$, a subset $S \subseteq V(G)$ is a (σ, ρ)*-set* if: $|N(v) \cap S| \in \sigma$ for every $v \in S$, and $|N(v) \cap S| \in \rho$ for every $v \in V(G)\backslash S$. Fixing σ, ρ, and given a graph G and an integer κ, the Min-LC-VSP$_{\sigma,\rho}$ problem consists in deciding whether there exists a (σ, ρ)-set S of size at most κ. The inequality $v\ell(G) \leq \ell(G)$ says that the vertex leafage of G is a more restrictive parameter, that is, that if a problem is FPT when parameterized by $v\ell(G)$, then it is also FPT when parameterized by $\ell(G)$. However, we have also seen that if some MIN-LC-VSP problem is FPT when parameterized by $\ell(G)$, then we get also parameterization by $\kappa + v\ell(G)$; this is because we can prove $\ell(G) \leq \kappa + v\ell(G)$ when $0 \notin \rho$. In [15] they provide a fixed-parameter algorithm for DOMINATING SET when parameterized by $\ell(G)$. A question is whether their result can be generalized to all MIN-LC-VSP problems. Given the complexity of the algorithm given in [15], this seems to be a very challenging problem.

Question 3. Are all MIN-LC-VSP problems FPT when parameterized by $\ell(G)$?

As mentioned above, when G is chordal and $0 \notin \rho$, then a bound on the leafage of G can be given in terms of the vertex leafage and the size of a solution to MIN-LC-VSP$_{\sigma,\rho}$. The same bound does not seem to hold for MAX-LC-VSP$_{\sigma,\rho}$ problems, which consist of deciding whether there exists a (σ, ρ)-set S such that $|S| \geq \kappa$. Nevertheless, many of the MAX-LC-VSP$_{\sigma,\rho}$ problems cited in [8] are known to be polynomial-time solvable in chordal graphs, for example, INDEPENDENT SET, MAXIMUM INDUCED MATCHING, MAXIMUM EFFICIENT EDGE DOMINATING SET and MAXIMUM DOMINATING INDUCED MATCHING, STRONG STABLE SET and so forth (see for instance [26]). We then ask the following:

Question 4. Are all MAX-LC-VSP$_{\sigma,\rho}$ problems polynomial-time solvable on chordal graphs?

In the 1985 column of the *Ongoing Guide to* NP-*completeness* by Johnson [26], the only separating problem for undirected path graphs and rooted directed path graphs was DOMINATING SET [4]. A revised version of the table [11] adds two separating problems for these classes: STEINER TREE and GRAPH ISOMORPHISM. As for problems separating rooted directed path graphs from directed paths graphs, only GRAPH ISOMORPHISM is known [1]. In fact, the complexities of the problems investigated in this article are widely open for directed path graphs, except that we prove that STEINER TREE should be the easiest among the three, while DOMINATING SET, the hardest (Theorem 3).

Question 5. What are the complexities of STEINER TREE, CONNECTED DOMINATING SET and DOMINATING SET on directed path graphs?

A byproduct of one of our reductions is that WEIGHTED CONNECTED DOMINATING SET is polynomial-time solvable on strongly chordal graphs, and hence on rooted directed path graphs. The analogous cannot hold for WEIGHTED STEINER TREE as it is NP-hard on strongly chordal graphs [35]. Additionally, we have introduced a new refinement between these classes, the in–out rooted directed path graphs, proving that the unweighted versions of the investigated problems are all polynomial-time solvable in these. Therefore, we ask:

Question 6. What is the complexity of WEIGHTED STEINER TREE on (in–out) rooted directed path graphs? What are the complexities of WEIGHTED DOMINATING SET and WEIGHTED CONNECTED DOMINATING SET on in–out rooted directed path graphs?

As for the graph classes, all the investigated classes can be recognized in polynomial time (see Section 2.2 for appropriate citations), and forbidden subgraph characterizations exist for undirected path graphs [27] and directed path graphs [30]. We then ask:

Question 7. Can we recognize in–out rooted directed path graphs in polynomial time? Is there a forbidden subgraph characterization for (in–out) rooted directed path graphs?

Finally, another parameter of interest is the mim-width of G [33]. Since many problems can be solved in XP time when parameterized by mim-width [2, 8, 25], and rooted directed path graphs have mim-width 1 [25], one could ask whether undirected path graphs also have bounded mim-width. Up to our knowledge, no explicit construction of undirected path graphs with unbounded mim-width is known, but the fact that LC-VSP problems can be solved in polynomial time on graphs with bounded mim-width [8], combined with the NP-hardness of DOMINATING SET on undirected path graphs, give evidence that undirected path graphs do not have bounded mim-width, unless P = NP. Nevertheless, we ask:

Question 8. Does there exist an explicit construction of undirected path graphs with unbounded mim-width?

FUNDING INFORMATION

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001, CNPq Grants 140399/2017-8, 407635/2018-1, and 303803/2020-7, FAPERJ Grant E-26/202.793/2017, STIC-AMSUD 88881.197438/2018-01, FUNCAP/CNPq PNE-0112-00061.01.00/16, and the Grant from French National Research Agency under JCJC program (ASSK: ANR-18-CE40-0025-01).

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

ORCID

Alexsander A. de Melo https://orcid.org/0000-0001-5268-6997
Ana Silva https://orcid.org/0000-0001-8917-0564

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How to cite this article: C. M. H. Figueiredo, R. Lopes, A. A. Melo, and A. Silva, *Parameterized algorithms for Steiner tree and (connected) dominating set on path graphs*, Networks. **84** (2024), 132–147. https://doi.org/10.1002/net.22220