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THE HARDNESS OF RECOGNISING POORLY MATCHABLE GRAPHS AND THE HUNTING OF THE d-SNARK *

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Abstract. An r-graph is an r-regular graph G on an even number of vertices where every odd set $X \subseteq V(G)$ is connected by at least r edges to its complement $V(G) \setminus X$. Every r-graph has a perfect matching and in a poorly matchable r-graph every pair of perfect matchings intersect, which implies that poorly matchable r-graphs are not r-edge-colourable. We prove, for each fixed $r \geq 3$, that poorly matchable r-graph recognition is coNP-complete, an indication that, for each odd $d \geq 3$, it may be a hard problem to recognise d-regular $(d-1)$-edge-connected non-d-edge-colourable graphs, referred to as d-snarks in this paper. We show how to construct, for every fixed odd $d \geq 5$, an infinite family of d-snarks. These families provide a natural extension to the well-known Loupekine snarks. We also discuss how the hunting of the smallest d-snarks may help in strengthening and better understanding the major Overfull Conjecture on edge-colouring simple graphs.

Keywords: Graph classes, Factorisation, Graph colouring, Connectivity, Snarks, Computational complexity

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* *Heartily dedicated to Jayme L. Szwarcfiter for his 80^{th} birthday*

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1. Introduction

Let r be a positive integer, an r-graph is an r-regular graph G on an *even* number of vertices such that $|\partial_G(S)| \geq r$ for every *odd*-cardinality $S \subseteq V(G)$ (we refer the reader to Sect. 2 for technical definitions and further preliminaries).

Due to their relation with the celebrated Edmonds's matching polytope theorem [1] and with important conjectures of Fulkerson and Tutte [2], r-graphs are much studied in Combinatorics [2–6]. A well-known characterisation is that a graph is an r-graph if and only if it can be expressed as a conic combination of the indicator vectors of its 1-factors [2], which implies that every edge of an r-graph G belongs to a 1-factor of G (see [2, 5, 7]). It was then conjectured by Seymour [2, 4] that, for every $r \geq 4$, it would be possible to decompose every r-graph by expressing it as the sum of an $(r-1)$-graph and a 1-factor. However, this conjecture was proved false by Rizzi [5] two decades later, wherein he showed how to construct, for every $r \geq 4$, not only an indecomposable r-graph, but also, and "more surprisingly", a *poorly matchable* r-graph, i.e. an r-graph wherein every pair of 1-factors intersect.

Concerning poorly matchable r-graphs, once conjectured not even to exist, we prove that the problem of recognising if a given r-graph is poorly matchable is coNP-complete for any fixed $r \geq 3$.

Rizzi's poorly matchable r-graphs are very large, with a number of vertices which is exponential on r (namely, $2 \cdot 5^{r-2}$ vertices for $r \geq 4$, as it can be straightforwardly checked). Knowing now that recognising poorly matchable r-graphs is a computational hard problem makes the task of finding these important graphs even more challenging.

To the best of our knowledge, our complexity result is the first one from which the following general complexity result can be derived: the problem of deciding if any given graph has two disjoint perfect matchings is NP-complete.

In contrast, recall that deciding if any given graph has a perfect matching is a problem which can be solved in polynomial time [8].

Let $\max_{PDPM}(G)$ be the maximum number of pairwise-disjoint perfect matchings (PDPMs) that a graph G has. When G is r-regular, the maximisation problem of determining $\max_{PDPM}(G)$ can be viewed as a generalisation of the problem of deciding if G is r-edge-colourable (i.e. has r PDPMs). Since the problem of deciding if any given graph has two disjoint perfect matchings is NP-complete, $\max_{PDPM}(G)$ is a hard parameter to compute and also to approximate, since even deciding if $\max_{PDPM}(G) \geq 2$ is NP-hard. We note an interesting dichotomy, since deciding if $\max_{PDPM}(G) \geq 1$ is polynomial [8].

In 1981, Holyer [9] proved that deciding if a graph G is $\Delta(G)$-edge-colourable is NP-complete even when restricted to cubic simple graphs. In 1983, Leven and Galil [10] showed that the problem remains NP-complete when restricted to r-regular simple graphs for every fixed $r \geq 3$. We suspect the following.

Conjecture 1.1. For every fixed $r \geq 3$, deciding if a given r-graph is r-edge-colourable is an NP-complete problem, even when restricted to simple graphs.

Conjecture 1.2. For every fixed $d \geq 3$, deciding if a given d-regular $(d-1)$-edge-connected graph is d-edge-colourable is an NP-complete problem, even when restricted to simple graphs.

When d is an *odd* integer, we refer to $(d-1)$-edge-connected d-regular non-d-edge-colourable simple graphs as *d-snarks*. The 3-snarks correspond to the snarks (2-edge-connected non-3-edge-colourable cubic simple graphs[1]). Remark that, according to our definition, every d-snark is a d-graph: if G is a d-snark which is not a d-graph, then some odd-cardinality set $X \subseteq V(G)$ induces a cut with exactly $d-1$ edges; but then the degree sum of the vertices in $G[X]$ would be $d|X|-d-1$, an odd integer, since d is also odd, contradicting the Handshaking Lemma.

Other snark generalisations have already appeared in the literature in the context of flow construction and others [13, 14], but they differ from ours, which emerges from the context of the Overfull Conjecture, as discussed in Sect. 6. Non-d-edge-colourable $(d-1)$-edge-connected d-regular graphs are not an unexplored subject either, as discussed in the sequel, in the subsection Further notes on related work concerning snarks and d-snarks. However, not much seems to be known about these graphs, perhaps due to their rareness. An aim of this paper is to discuss their importance and to present an infinite family of d-snarks for every fixed odd $d \geq 5$ (the reason why our snark generalisation considers only odd values for d shall also be clear in Sect. 6).

An important observation concerning Conjecture 1.2 is the following.

Observation 1.1. *If Conjecture 1.2 holds, then, for any fixed odd integer $d \geq 3$, the d-snark recognition problem is* coNP*-complete.* □

We remark that the edge-connectivity value on the definition of d-snarks seems to be tight, since, according to a recent submitted paper by Ma et al. [15], it "*surprisingly* seems" to be still an open question if there is any non-5-edge-colourable 5-edge-connected 5-regular graph (even though snarks are often assumed to be at least 3-edge-connected [11, 12]). This makes the *hunting* for d-snarks with $d \geq 5$ still more exciting, in addition to the fact that these graphs are unlikely to be small. We also remark that the Conjecture 1.2 does not necessarily imply Conjecture 1.1, since $(r-1)$-edge-connected graphs are not necessarily r-graphs. This subtle remark is the reason of the change of variables: to avoid confusion, when dealing with r-graphs, we use the letter r; when dealing with $(d-1)$-edge-connected graphs and d-snarks, we use the letter d.

This paper is organised as follows. In the remainder of this section, we make a brief remark on some papers from the literature concerning snarks, other snark generalisations, and non-d-edge-colourable d-regular graphs with high (edge-)connectivity. In Sect. 2, we present preliminaries for the concepts appearing throughout the text. In Sect. 3, we prove that the problem of recognising if a given r-graph is poorly matchable is coNP-complete. In Sect. 4, we present the *base gadget*, with

[1]Sometimes in the literature, further restrictions are imposed on the definition of snarks, such as having girth at least five and being cyclically 4-edge-connected (see e.g. [11, 12]). We do not consider these restrictions, since our main interest on snarks and d-snarks are their role in the major *Overfull Conjecture* in graph edge-colouring, as discussed in Sect. 6.

which, in Sect. 5, we construct the infinite families of d-snarks. In Sect. 6, we discuss the relation between d-snarks, the Overfull Conjecture, and the hardness of edge-colouring. In Sect. 7, we conclude with further remarks.

Further notes on related work concerning snarks and d-snarks

Playing an important role on major graph problems for almost 150 years [16–18], snarks were so called by the mid-1970s [19] after the mysterious creature hunted in the poem *The Hunting of the Snark*, by Lewis Carroll, due to their rareness and peculiar properties. The first known snark, and also the smallest one, is the Petersen graph (Fig. 1A), dating from the 1890s [20, 21]. Other snarks were found only half a century later: the two 18-vertex Blanuša snarks [22] and the 210-vertex Descartes snark [23]. The graph considered to be the fifth snark to be found is Szekeres snark, from 1973 [24], with 50 vertices. The first infinite family of snarks, the *Flower snarks* (Figs. 1B and 1C), was introduced by Isaacs in 1975 [11].

Also in 1973 (a few months earlier than Szekeres's paper), Meredith [25] showed a 3-vertex-connected non-3-edge-colourable cubic graph, therefore a snark. This graph has also 50 vertices and it is not isomorphic to the Szekeres snark. Even so, we have not found any mention in the literature on snarks referring to Meredith's graph as a snark, even less as the fifth to be found, possibly because this graph has girth four. Snarks with girth at most four are called *trivial*, since, from a non-trivial snark, infinitely many trivial snarks can be obtained (see e.g. [11, 12]). In Isaacs's paper [11] wherein the Flower snarks were introduced, he listed Petersen, Blanuša, Descartes, and Szekeres's graphs as the only five *non-trivial* snarks known by that time.

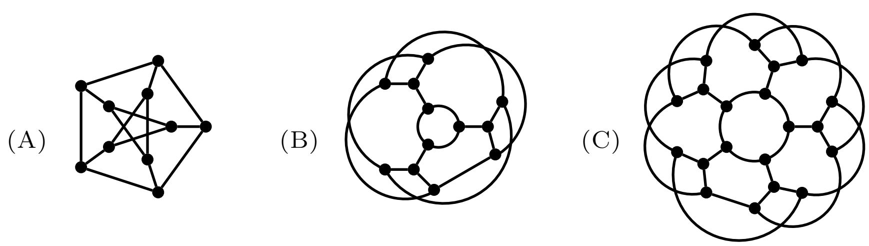


Figure 1. (A) The Petersen graph; (B) J_3 and (C) J_5, the two smallest of the Flower snarks.

Another important infinite family, originally defined by Loupekine and first presented by Isaacs [26] in 1976, are the *Loupekine snarks* (Fig. 2), whose *base block* was, in 1981, used as the base gadget in Holyer's proof of the NP-completeness of the edge-colouring problem [9].

In Meredith's paper of 1973 [25], he presented a family of graphs G_d, one for each $d \geq 3$, such that G_d is d-regular, d-vertex-connected (therefore d-edge-connected),

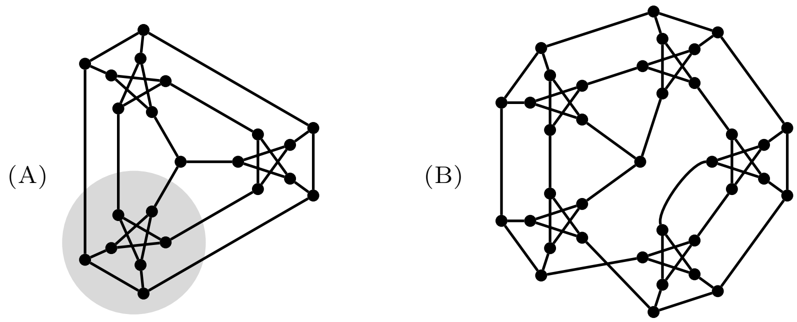


FIGURE 2. (A) A smallest Loupekine snark, with three *base blocks*, one of them highlighted, and (B) a Loupekine snark with five base blocks. A Loupekine snark can be constructed from a cycle of any odd number of base blocks by: connecting each pair of consecutive blocks in the cycle with either a *parallel* or a *cross link* (a cross link is depicted in (B)); then gathering the upper half-edges of the blocks either in groups of two, identifying both half-edges of each group, or in groups of three, joining the half-edges of each group to a new vertex.

and non-d-edge-colourable if and only if $d \equiv 2, 3, 4 \pmod 6$. Meredith also presented another family G'_d, only for $d \equiv 0, 1, 5 \pmod 6$, which are always non-d-edge-colourable. However, the graph G'_d is d-vertex-connected only for $d \geq 11$. One can verify that $\lambda(G'_5) = \lambda(G'_6) = \lambda(G'_7) = 4$, which implies that, amongst all non-d-edge-colourable graphs with odd d presented by Meredith, the graph G'_7 is the only one which is not a d-snark. Fig. 3 depicts G'_5, a 5-snark with 90 vertices. In fact, for every $d \geq 5$, both the graphs G_d and G'_d (the latter when defined) have $20d - 10$ vertices.

For $d = 3$, it is already known that the number of snarks on n vertices is at least $2^{(n-84)/18}$ for sufficiently large *even*[2] n [27]. Nevertheless, we can still regard d-snarks as *rare* graphs. In fact, for any fixed $d \geq 3$, the proportion of non-d-edge-colourable d-regular graphs on $2k$ vertices (a superset of the d-snarks on $2k$ vertices when d is odd) over all d-regular graphs on $2k$ vertices goes to 0 as k goes to ∞ [28].

In 1999, Rizzi [5] showed how to construct, for every $r \geq 4$, examples of r-graphs that are poorly matchable (thus not r-edge-colourable). All these graphs are 4-edge-connected and have a 4-edge cut, so only for $r \leq 5$ they are $(r-1)$-edge-connected. Hence, Rizzi's poorly matchable 5-graphs are examples of 5-snarks. There are finitely many choices for Rizzi's procedure to construct these 5-graphs from the Petersen graph, all of them having 250 vertices. Differently from Meredith's paper [25], from which we have a *single* d-snark for every odd

[2]Recall that, by the *Handshaking Lemma*, there cannot be a d-regular n-vertex graph with both d and n odd.

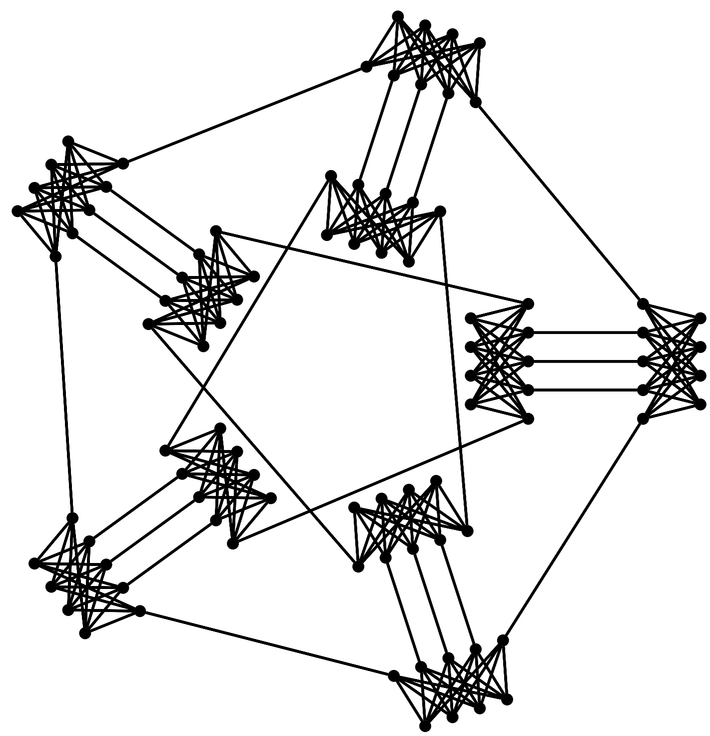


FIGURE 3. The graph G'_5, defined by Meredith [25], is to the best of our knowledge the smallest known 5-snark.

$d \geq 5$, $d \neq 7$, and differently from Rizzi's paper [5], from which we have *finitely* many 5-snarks, we present in Sect. 5, for each fixed odd $d \geq 5$, an *infinite* family of d-snarks.

In the recent submitted paper by Ma et al. [15], the authors show that for all even $d > 2$, there are d-regular d-edge-connected graphs with $\max_{PDPM}(G) \leq d-3$. These graphs, however, are not d-snarks according to our definition, because d is not odd.

2. Definitions and technical preliminaries

We use the term *graph*[3] to refer to an undirected loopless graph, which may be a simple graph or a multigraph. The set of all edges between the same pair of vertices in a multigraph is referred to as a *multiple* edge e, in contrast to the μ_e *parallel* edges of which e consists. The set of vertices and the (multi)set of edges of a graph G are denoted $V(G)$ and $E(G)$, respectively. The set of neighbours of a vertex u in

[3]Graph-theoretical definitions not present in this text follow their usual meanings (such as in [29].).

G and the (multi)set of edges incident with u in G are denoted $N_G(u)$ and $\partial_G(u)$, respectively. For $X \subseteq V(G)$, we define $\partial_G(X) \coloneqq \{uv \in E(G) : u \in X \text{ and } v \notin X\}$, referred to as the *cut* induced by X in G if $\emptyset \neq X \neq V(G)$. The *degree* of u in G is $d_G(u) \coloneqq |\partial_G(u)|$. If $d_G(u) = d$ for all $u \in V(G)$ and some integer d, then G is said to be *d-regular*, in which case we say that d is the degree of G. The d-regular complete bipartite graph is denoted $K_{d,d}$. The maximum (minimum) degree of G is denoted $\Delta(G)$ ($\delta(G)$). A *cubic* graph is a 3-regular graph. A *subcubic* graph is a (not necessarily regular) graph G with $\Delta(G) \leq 3$. When free of ambiguity, we may omit G from the notation, writing simply $d(u)$, $\partial(X)$, Δ etc.

We say that a graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $\Delta(H) = \Delta(G)$, then H is said to be a *Δ-subgraph* of G. If $V(H) = V(G)$, then H is said to be a *spanning* subgraph of G. Let $U \subseteq V(G)$ and $F \subseteq E(G)$, the subgraph of G *induced* by U, denoted $G[U]$, and the subgraph of G induced by F, denoted $G[F]$, are the graphs defined by:

$$\begin{aligned} V(G[U]) &\coloneqq U \quad \text{and} \quad E(G[U]) \coloneqq \{uv \in E(G) : u, v \in U\}\,; \\ E(G[F]) &\coloneqq F \quad \text{and} \quad V(G[F]) \coloneqq \{u \in V(G) : uv \in F \text{ for some } v \in V(G)\}\,. \end{aligned}$$

Let $u \in V(G)$ and $e \in E(G)$, we define $G - u \coloneqq G[V(G) \setminus \{u\}]$ and $G - e \coloneqq G[E(G) \setminus \{e\}]$. Analogously we define $G - U$ for $U \subseteq V(G)$, and $G - F$ for $F \subseteq E(G)$.

An *r-factor* of G is a (not necessarily induced) subgraph H of G which is r-regular and spanning. A set $M \subseteq E(G)$ is said to be a *matching* if it induces a 1-regular subgraph of G. A *perfect matching* is the edge set of a *1-factor*, but often these terms are used as synonyms, by abuse.

The *multiplicity* of an edge e of a graph G is denoted $\mu_G(e)$, being $\mu(uv)_G \coloneqq 0$ for all $u, v \in V(G)$ such that $uv \notin E(G)$. The *sum* $G_1 + G_2$ of two graphs G_1 and G_2 on the same vertex set V is the graph on V such that $\mu_{G_1+G_2}(uv) = \mu_{G_1}(uv) + \mu_{G_2}(uv)$ for all $u, v \in V$. Let $F \coloneqq E(G_2)$, the graph $G_1 + G_2$ is also denoted $G_1 + F$. If G is an r-graph and M is a 1-factor of the complete graph on $V(G)$, then clearly $G + M$ is an $(r+1)$-graph, since for every odd-cardinality $S \subseteq V(G)$ we have $M \cap \partial_G(S) \neq \emptyset$ and thus $|\partial_{G+M}(S)| > |\partial_G(S)|$.

A connected graph G is said to be *k-edge-connected*, being $k \in \mathbb{Z}_{>0}$, if $|V(G)| > 1$ and $G - F$ is connected for all $F \subseteq E(G)$ such that $|F| < k$. Similarly, G is said to be *k-vertex-connected*, being $k \in \mathbb{Z}_{>0}$, if $|V(G)| > k$ and $G - U$ is connected for all $U \subseteq V(G)$ such that $|U| < k$. The greatest k for which G is k-edge-connected (k-vertex-connected) is the *edge-connectivity* (*vertex-connectivity*) of G, denoted $\lambda(G)$ ($\kappa(G)$). Clearly, $\delta(G) \geq \lambda(G) \geq \kappa(G)$.

Petersen's Theorem states that every 2-edge-connected cubic graph has a perfect matching [30]. In fact, for any (odd or even) $d \geq 3$, every $(d-1)$-edge-connected d-regular graph on an even number of vertices (and, thus, every d-snark) has a perfect matching, and any of its edges belongs to a perfect matching [31]. In the case of 2-edge-connected cubic graphs, it has already been proved that the number of perfect matchings is bounded below by an exponential on n [32].

A *k-edge-colouring* of a graph G is a function $\varphi\colon E(G) \to \mathscr{C}$ such that $\mathscr{C}$ is a set of k colours and $\varphi(e) \neq \varphi(f)$ for all distinct adjacent $e, f \in E(G)$. We also define $\varphi(F) \coloneqq \bigcup_{e\in F}\{\varphi(e)\}$ for any $F \subseteq E(G)$ (including the case wherein F is a multiple edge). The *chromatic index* of G, denoted $\chi'(G)$, is the least k for which G is *k-edge-colourable*. By *Vizing's Theorem* [33], if G is simple, then $\chi'(G)$ is either $\Delta(G)$ or $\Delta(G) + 1$, being G said to be *Class 1* in the former case, and *Class 2* in the latter. A *critical* graph is a connected *Class 2* simple graph G such that $\chi'(G - e) < \chi'(G)$ for every $e \in E(G)$. *Seymour's r-graph conjecture* [2], proved for $r \leq 11$ [34], is the statement that $\chi'(G) \leq \Delta(G) + 1$ also for every (not necessarily simple) r-graph G.

Deciding if a graph is Δ-edge-colourable is an NP-complete problem [9], even when restricted to d-regular simple graphs for any constant $d \geq 3$ [10], to perfect graphs [35], or to C_k-free simple graphs for any constant $k \geq 3$ [35, 36]. An $O(2^m m^{O(1)})$-time exact algorithm for edge-colouring graphs with m edges is yielded by the set partition algorithm by Björklund et al. [37].

In an edge-colouring $\varphi\colon E(G) \to \mathscr{C}$, a colour $\alpha \in \mathscr{C}$ is said to be *missing* at some $u \in V(G)$ if no edge at u is coloured α. The edge-colouring φ is said to be *equitable* if, for all $\alpha, \beta \in \mathscr{C}$, the number of edges coloured α differs from the number of edges coloured β by at most one. It is a well-known interesting property of edge-colouring that every graph G has an equitable k-edge-colouring for any integer $k \geq \chi'(G)$ [38].

Another interesting edge-colouring property is the *Parity Lemma*, which appears in the construction of the Blanuša, the Descartes, and the Flower snarks [11]. The Parity Lemma states that if φ is a d-edge-colouring of a d-regular graph G with $d \geq 2$, then, for any cut F in G, the number of edges in F coloured α has the same parity for any $\alpha \in \mathscr{C}$.

A graph G on n vertices is said to be *overfull* if it has more than $\Delta\lfloor n/2\rfloor$ edges, or, equivalently [39], if n is odd and $\sum_{u\in V(G)}(\Delta - d(u)) \leq \Delta - 2$. A graph G is said to be *subgraph-overfull* (shortly, *SO*) if it has an overfull Δ-subgraph. Deciding if a graph is *SO* can be done in polynomial time [39–41]. In an edge-colouring, we need at least $\Delta + 1$ colours to colour more than $\Delta\lfloor n/2\rfloor$ edges. Therefore, being *SO* is clearly a sufficient condition for a simple graph to be *Class 2*. There has been much work (e.g. [42–48]) aimed at identifying graph classes wherein being *SO* is also a necessary condition to be *Class 2*. The *Overfull Conjecture* states that this necessity holds for all n-vertex simple graphs with $\Delta > n/3$ [42, 49, 50]. This conjecture has already been settled for some specific graph classes, such as graphs with $\Delta \geq n - 3$ [51, 52], complete multipartite graphs [43], powers of cycle graphs [53], regular join graphs [44], split-interval graphs [54, 55], and split-comparability graphs [55].

Restricted to d-regular simple graphs with $d \geq n/2$, the Overfull Conjecture is also known as the 1-*Factorisation Conjecture*, being equivalent to the statement that every d-regular simple graph of even order $n \leq 2d$ is 1-*factorisable* (or, equivalently, *Class 1* and regular) [56]. The 1-Factorisation Conjecture was demonstrated to hold *asymptotically*, i.e. for any $\epsilon > 0$, there is an n_0 such that every d-regular graph on n vertices with even $n \geq n_0$ and $d \geq (1/2 + \epsilon)n$ is 1-factorisable [57].

Let EDGE-COLOURING and 2-PDPM be the problems of deciding if a given graph G satisfies $\chi'(G) = \Delta(G)$ and $\max_{PDPM}(G) \geq 2$, respectively. Let A be any decision problem and r a predicate over the instances of A, the *restriction* of A to the instances which satisfy r is denoted A(r).

3. 2-PDPM(r-GRAPH) IS NP-COMPLETE FOR EVERY FIXED $r \geq 3$

Holyer's [9] proof of the NP-completeness of EDGE-COLOURING(cubic, simple) consists of a reduction from 3SAT which, given a 3SAT instance Φ, constructs a cubic simple graph G which is 3-edge-colourable if and only if Φ is satisfiable. A 3SAT instance Φ consists of a set of variables, and a set of clauses such that each clause is a disjunction of three literals (the variable itself or the negation of the variable). We start this section by briefly presenting Holyer's reduction so that the following can be verified by inspection. The constructed graph corresponding to a 3SAT instance consists of components called gadgets which carry out specific tasks. Information is carried between gadgets by half-edges that contain just one end vertex in the gadget.

Lemma 3.1. *For any* 3SAT *instance Φ, the graph constructed by Holyer's reduction is a* 3*-graph.* □

We remark that, for the inspection, it suffices to verify that the structure of G is the same regardless of the satisfiability of Φ (this is expected, since otherwise the reduction would be *solving* 3SAT). If this holds, G cannot have a bridge, otherwise, from the Parity Lemma, it would always be non-3-edge-colourable, even when Φ is satisfiable. Also, no cubic simple graph can have a 2-cut induced by an odd-cardinality vertex set, since this cut would allow us to construct a cubic (not necessarily simple) graph on an odd number of vertices, contradicting the well-known *Handshaking Lemma*. Therefore, G must be a 3-graph.

The base gadget in Holyer's reduction is the *inverting gadget* (Fig. 4), which had been previously used as the base blocks in the construction of the Loupekine snarks [26]. The main feature, which explains the name inverting, of this gadget is transcribed in Lemma 3.2.

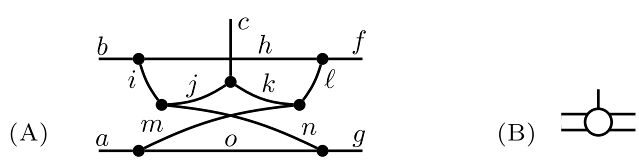


FIGURE 4. (A) the inverting gadget; (B) the representation of the inverting gadget.

Lemma 3.2 ([9])**.** *Holyer's inverting gadget (Fig. 4) is* 3*-edge-colourable and, for any copy H of this gadget in a cubic graph G and any* 3*-edge-colouring φ of G, all the following hold:*

(i) each of the three colours appears either once or three times at the five half-edges (a, b, c, f, g) of H;
(ii) either $\varphi(a) = \varphi(b)$, or $\varphi(f) = \varphi(g)$.

Moreover, any 3-edge-colouring of the half-edges satisfying (i) and (ii) can be extended to a 3-edge-colouring of H. □

In view of Lemma 3.2, a pair of half-edges (e_1, e_2) of a copy of the inverting gadget in a cubic graph G, under a 3-edge-colouring of G, is said to be *true* if $\varphi(e_1) = \varphi(e_2)$, and *false* otherwise. From Lemma 3.2, if (a, b) ((f, g)) is *true*, then any pair chosen from $\{c, f, g\}$ ($\{a, b, c\}$) is *false*.

Let Φ be a 3SAT instance wherein each variable is assumed, without loss of generality, to occur at least twice. With the inverting gadget, a copy of the *variable gadget* (Fig. 5A and 5B) is constructed for each variable x of Φ, with as many *outputs* as the occurrences of x in Φ. In a copy of the variable gadget in a 3-edge-coloured cubic graph, the outputs have all the same truth value [9]. Conversely, any assignment of 3 colours to the outputs of a variable gadget X in which the outputs have all the same truth value can be extended to a 3-edge-colouring of X [9]. For each occurrence of a variable as a negated literal in a clause, an extra inverting gadget is attached to the corresponding output, as in Fig. 5B. To complete the reduction, a copy of the *clause gadget* (Fig. 5C) is constructed for each clause of Φ, identifying each of its three *inputs* with the corresponding outputs of the variable gadgets, according to the literals of the clause. In a copy of the clause gadget in a 3-edge-coloured cubic graph, at least one of the inputs must be *true* [9]. Conversely, any assignment of three colours to the inputs of the clause gadget Ξ in which at least one of the inputs is true can be extended to a 3-edge-colouring of Ξ [9]. Hence, the graph G constructed is 3-edge-colourable if and only if all clauses of Φ are satisfiable. Remark that G has some *loose* half-edges, which are handled by duplicating G and identifying each corresponding pair of loose half-edges.

Our proof for the NP-completeness of 2-PDPM(r-graph) for every fixed $r \geq 3$ combines Holyer's reduction with Rizzi's [5] construction of poorly matchable r-graphs for every fixed $r \geq 4$. So, now we briefly describe Rizzi's construction.

Let $r \in \mathbb{Z}_{\geq 4}$ and let $\mathcal{P}(r)$ be the r-graph obtained from the Petersen graph $\mathcal{P}$ by adding $r - 3$ copies of a 1-factor M of $\mathcal{P}$ (Fig. 6A). Let $zx \in M$ and let a, b be the neighbours of z in $\mathcal{P}$ other than x. By the symmetry of $\mathcal{P}$, all choices for M, zx, a, b are equivalent under automorphisms. Let $\langle a, x, b\rangle^{(r)}$ be the component obtained by removing z from $\mathcal{P}(r)$ and leaving the r half-edges incident with a, x, b to be linked to r (not necessarily distinct) vertices in the construction of a larger r-graph (Fig. 6B).

Now comes what we call *Rizzi's r-step.* Let $G^{(r-1)}$ be an $(r-1)$-graph with $r \geq 4$, let M be a 1-factor of $G^{(r-1)}$ and let U be a minimum vertex cover of M (thus $|U| = |M|$). Then, adding a copy M' of M to $G^{(r-1)}$, we obtain an r-graph H. For each pair of parallel edges $e \in M$ and $e' \in M'$, let u be the endvertex of e in U and apply the *$\mathcal{P}(r)$-splicing operation* on (u, e, e'), i.e. replace u with a copy of $\langle a, x, b\rangle^{(r)}$ attaching e to a, e' to b, and all other edges incident with u in H to x (see Fig. 7). Let $G^{(r)}$ be the r-graph obtained after this operation is

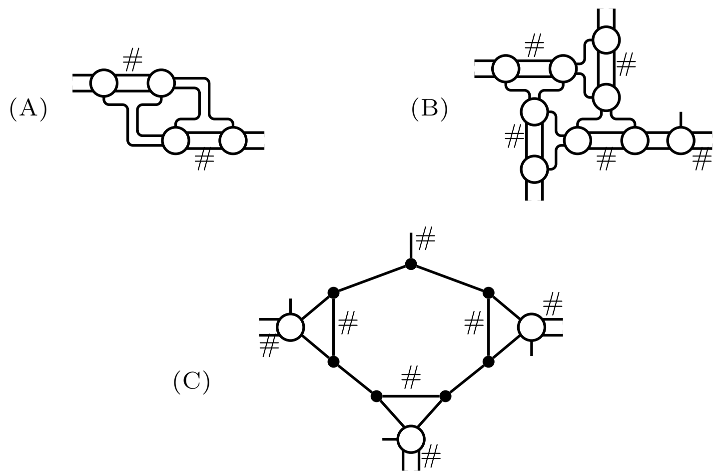


FIGURE 5. The *variable gadget* for a variable x with (A) two outputs and (B) four outputs, and (C) the *clause gadget*. In (B), an extra inverting gadget is attached to one of the outputs for an occurrence of x as a negated literal in a clause of Φ. The (half-)edges marked # in (A)–(C) are edges with multiplicity $r-2$ in Leven and Galil's reduction for r-regular graphs, as discussed in Sect. 4.

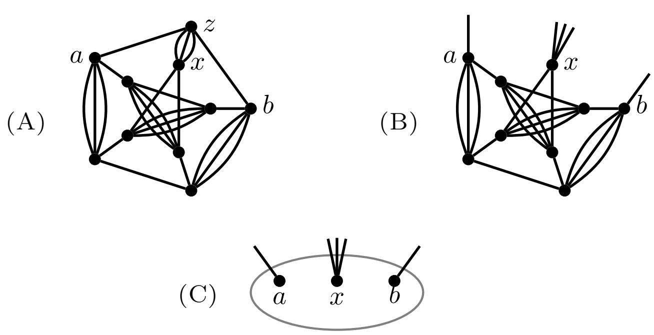


FIGURE 6. (A) the graph $\mathcal{P}(5)$; (B) the $\langle a, x, b\rangle^{(5)}$ component; (C) the representation of the component.

performed for every $e \in M$. Clearly, $|V(G^{(r)})| = 5|V(G^{(r-1)})|$, since for each of the $|V(G^{(r-1)})|/2$ in U, the $\mathcal{P}(r)$-splicing operation creates 8 new vertices, so

$$|V(G^{(r)})| = |V(G^{(r-1)})| + \frac{8|V(G^{(r-1)})|}{2} = 5|V(G^{(r-1)})|\,.$$

Rizzi [5] proved that if $G^{(r-1)}$ is poorly matchable, then so is $G^{(r)}$. Actually, the converse also holds, as we observe in Lemma 3.3.

(A) (B) (C)

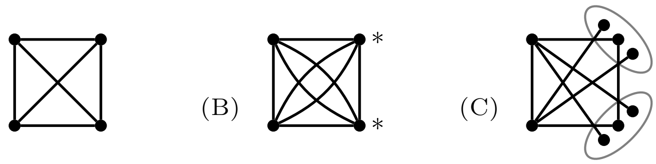


FIGURE 7. (A) K_4, which is a 3-graph, but not poorly matchable; (B) K_4 after adding a 1-factor, with the vertex cover indicated by $*$; (C) graph $G^{(4)}$ obtained by Rizzi's 4-step on $G^{(3)} := K_4$.

Lemma 3.3. *Let $r \in \mathbb{Z}_{\geq 4}$. The r-graph $G^{(r)}$ obtained from an $(r-1)$-graph $G^{(r-1)}$ by applying Rizzi's r-step is poorly matchable if and only if $G^{(r-1)}$ is poorly matchable.*

Proof. Since the necessity is by Rizzi [5], we prove only the sufficiency. Suppose that $G^{(r-1)}$ has two disjoint perfect matchings M_1, M_2. Clearly, M_1, M_2 are still disjoint matchings of $G^{(r)}$, and we show how to extend each of them by adding edges inside the $\langle a, x, b\rangle^{(r)}$ components, so that all vertices of $G^{(r)}$ are covered.

Let u be a vertex of $G^{(r-1)}$ on which a $\mathcal{P}(r)$-splicing operation was performed in the construction of $G^{(r)}$ and consider the corresponding $\langle a, x, b\rangle^{(r)}$ component. It is important that the edge set selected in the component for M_1 is disjoint from the edge set selected for M_2. Let $e \in M_1$ and $f \in M_2$ be the edges incident with u in $G^{(r-1)}$. By the construction of $G^{(r)}$, at least one of e, f must be incident with x in $G^{(r)}$, since the edge incident with b in $G^{(r)}$ was not present in $G^{(r-1)}$. We have two cases.

- *Both e and f are incident with x in $G^{(r)}$.* In this case, we extend M_1 and M_2 by selecting, from each set of parallel edges in the component (edge of multiplicity greater than one, see Fig. 8A), one edge for M_1 and other for M_2.
- *Exactly one of e, f (say e w.l.g.) is incident with a in $G^{(r)}$.* In this case, since f is incident with x, we extend M_2 also by selecting one edge from each set of parallel edges in the component. To extend M_1, we select the edges indicated in Fig. 8B. Observe that all these edges have multiplicity one in the component, except for edge y_1y_2, from whose set of parallel edges one edge is selected for M_1 and other for M_2.

The new matchings obtained are still disjoint and now each covers all vertices in the component. $\square$

Now we are ready to prove that 2-PDPM(r-graph) is NP-complete for every fixed $r \geq 3$.

Theorem 3.1. *Let r be any fixed integer not smaller than 3. The problem of deciding if a given r-graph has two disjoint perfect matchings is* NP*-complete.*

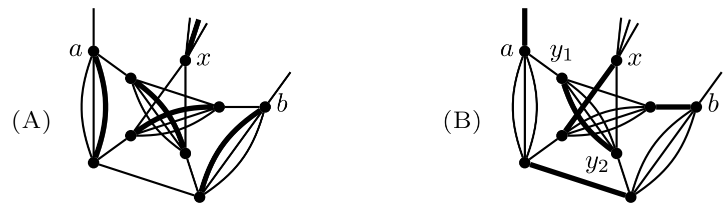


FIGURE 8. (A) a way (bold) to extend a matching in $G^{(r-1)}$ containing an edge incident with x; (B) a way (bold) to extend a matching in $G^{(r-1)}$ containing an edge incident with a.

Equivalently, the problem of recognising if a given r-graph is poorly matchable is coNP*-complete.*

Proof. We already have that 2-PDPM(3-graph) is NP-complete by Lemma 3.1, since every non-3-edge-colourable cubic graph is poorly matchable. Assume then $r \geq 4$ and, by induction, that 2-PDPM($(r-1)$-graph) is NP-complete. By Lemma 3.3, Rizzi's r-step constructs, from any $(r-1)$-graph $G^{(r-1)}$, an r-graph $G^{(r)}$ such that $\max_{PDPM}(G^{(r)}) \geq 2$ if and only if $\max_{PDPM}(G^{(r-1)}) \geq 2$. Hence, we have a reduction from 2-PDPM($(r-1)$-graph) to 2-PDPM(r-graph) which can be clearly performed in polynomial time, since it consists only of duplicating a 1-factor of $G^{(r-1)}$ and $\mathcal{P}(r)$-splicing half of its vertices. □

For a fixed $r \geq 4$, when we compose all the reduction chain of the proof of Theorem 3.1 and apply it on a 3SAT instance Φ, we obtain an r-graph whose number of vertices has a factor which is exponential on r. However, this is not a problem, since r is being regarded as a fixed constant.

From Theorem 3.1 follows the more general result below.

Corollary 3.1. *The problem of deciding if any given graph has two disjoint perfect matchings is* NP*-complete.* □

4. The base gadget

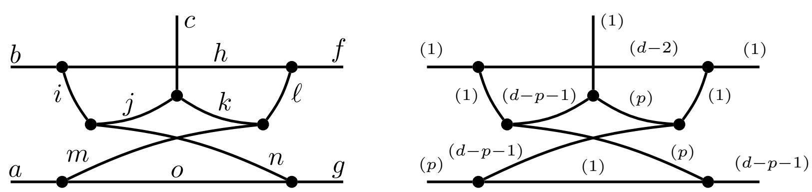
Leven and Galil [10] extended Holyer's reduction to produce, for any fixed constant $d \geq 3$, a d-regular graph which is *Class 1* if and only if a given 3SAT instance Φ is satisfiable, thus proving that EDGE-COLOURING(d-regular, simple) is NP-complete. It can be easily verified that the whole graph output by Holyer's reduction is always bridgeless. That is, the problem shown NP-complete by Holyer is the problem of deciding if a 2-edge-connected cubic graph is 3-edge-colourable or a snark. Hence, the snark recognition problem is coNP-complete.

In Leven and Galil's proof for d-regular graphs, Holyer's inverting gadget was extended by adding *multiplicities* to the edges of Holyer's original gadget. The way that these multiplicities are added to the edges is defined as in Fig. 9B for any p satisfying $1 \leq p \leq d-2$. In Leven and Galil's proof, the $(d-1)$-edge-connectedness of the graph output by the reduction is not guaranteed. We do

Table 1. The multiplicity of each edge of our inverting gadget, following Fig. 9A.

	For $d \equiv 1 \pmod 4$	For $d \equiv 3 \pmod 4$
$\mu_a = \mu_g = \mu_m = \mu_n$	$(d+3)/4$	$(d+1)/4$
$\mu_b = \mu_f = \mu_i = \mu_\ell$	$(d-1)/4$	$(d+1)/4$
μ_c	1	1
μ_h	$(d+1)/2$	$(d-1)/2$
$\mu_j = \mu_k$	$(d-1)/2$	$(d-1)/2$
μ_o	$(d-3)/2$	$(d-1)/2$

not have $(d-1)$-connectedness even inside the inverting gadget when $d > 5$ (for instance, see the cut induced by the endvertices of edge h).



(A) (B)

Figure 9. (A) Holyer's original *inverting gadget* for cubic graphs and (B) Leven and Galil's extension to d-regular graphs for $d \geq 3$ and for $1 \leq p \leq d-2$. In (B), the multiplicities are shown in parentheses.

The construction of our base gadget is presented throughout the remainder of this section, in which we show how to modify Leven and Galil's gadget for any odd $d \geq 5$, so that the graphs constructed in Sect. 5 are always $(d-1)$-edge-connected.

Our inverting gadget is also obtained from Holyer's by adding *multiplicities* to the edges. As in Leven and Galil's proof, to guarantee that no multiple edges occur, each edge uv in a set of parallel edges is replaced with a copy of the gadget obtained from $K_{d,d} - xy$, for any choice of xy, by joining x to u and joining y to v with edges. This works because every bipartite graph is *Class 1* [58] and, in any d-edge-colouring of $K_{d,d} - xy$, the colours missing at x and y must be the same, since subdividing xy in $K_{d,d}$ yields an overfull graph.

We define, for any fixed odd $d \geq 5$, an inverting gadget which depends solely on d, rather than defining a distinct gadget for each $p \in \{1, \ldots, d-2\}$. The multiplicities μ_e chosen for each edge e are defined in Table 1. We call our inverting gadget H.

The multiplicities of the edges of our inverting gadget are chosen so that, in addition to the identities in the first column of Table 1, we have Lemma 4.1, which follows by inspection on Table 1.

Lemma 4.1. *For any odd integer $d \geq 5$, let H be the inverting gadget with the multiplicities as in Table 1. Then, we have the following:*

(i) each vertex has degree d and, as in Leven and Galil's inverting gadget, there are $d+2$ half-edges in total,

(ii) the cut induced by any non-empty set of vertices in the gadget has at least $d-1$ edges. □

In Theorem 4.1, we show the properties that d-edge-colourings of our inverting gadget H must have, which explains the name inverting, when considering a d-edge-colouring of a d-regular graph containing a copy of H.

Theorem 4.1. *For any odd integer $d \geq 5$, let H be the inverting gadget with the multiplicities as in Table 1. Then, H is d-edge-colourable and, in any d-edge-colouring of a d-regular graph containing a copy of H, and*

(†) each of $d-1$ colours appears exactly once at the $d+2$ half-edges of each H, whilst the other colour α appears exactly three times.

Also,

(i) either α appears at a and b,

(ii) or α appears at f and g.

Moreover, any d-edge-colouring of the half-edges satisfying (†) and either (i) or (ii) can be extended to a d-edge-colouring of H.

Proof. Our strategy for the proof is as follows: first we prove that, if H is d-edge-colourable, then property † is satisfied for any d-edge-colouring of a d-regular graph G containing a copy of H; second we prove that any d-edge-colouring of the half-edges of H satisfying property † and either (i) or (ii) can be extended to a d-edge-colouring of H.

First, we show that, if H is d-edge-colourable, then (†) is satisfied for any d-edge-colouring $\varphi\colon E(G) \to \mathscr{C}$ of a d-regular graph G containing a copy of H. Let P be the set of the half-edges of H. By the Parity Lemma, if there are some colours in $\mathscr{C}$ not appearing at any edge in P, then all colours in $\mathscr{C}$ must appear an even number of times at the $d+2$ edges in P, which is not possible, since d is odd. Again by the Parity Lemma, as every colour appears at least once at edges in P, then some $\alpha \in \mathscr{C}$ appears exactly three times, whilst each of the other colours appears exactly once.

Now, we show that any d-edge-colouring of P satisfying (†) and either (i) or (ii) can be extended to a d-edge-colouring of H. For (i), we know that α appears at one amongst c, f, g, and the edge-colourings for these three cases are presented in Fig. 10. For (ii), the edge-colourings are symmetric to the ones presented for (i).

It remains to show that any d-edge-colouring of H satisfying (†) satisfies either (i) or (ii). For the sake of contradiction, assume that neither (i) nor (ii) holds. Then, one amongst Cases 1–3 does.

Case 1. Colour α appears at b, c, and f.

Since α cannot be missing at any vertex and it cannot appear at any amongst a, g, h, i, j, k, ℓ, it must appear at both m and n. Since $\varphi(a)$ and $\varphi(g)$ are disjoint,

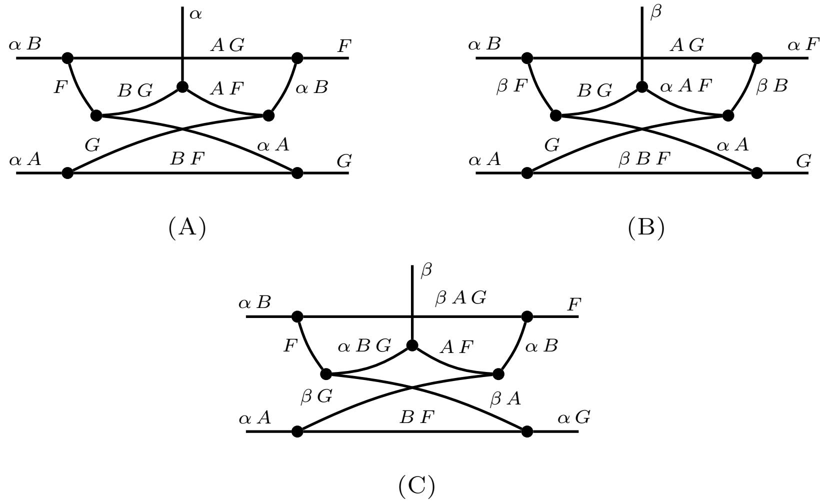


Figure 10. The '*Moreover*' part of Theorem 4.1. Here, we show the colours at each edge of H from a d-edge-colouring of the half-edges satisfying (†) and (i), with colour α appearing at edges a and b and: (A) c; (B) f; (C) g. In (B) and (C), let $\beta \coloneqq \varphi(c)$. In each of the three cases, A, B, F, G are any disjoint colour sets containing neither α nor β such that, for $e \in \{a, b, f, g\}$, the set identified by the corresponding capital letter has cardinality $\mu_e - |\{\alpha, \beta\} \cap \varphi(e)|$. By inspection on Table 1, one can verify that the edge-colourings use only d colours and each edge has exactly as many colours as its multiplicity.

every colour at a (g) must appear at n (m). Therefore, we have at least $\mu_a + 1$ ($\mu_g + 1$) colours at n (m), a contradiction, since $\mu_a = \mu_n = \mu_g = \mu_m$.

Case 2. Colour α appears at b, c, and g (at a, c, and f).

In this case, α cannot appear at any amongst i, j, n (at any amongst k, ℓ, m), a contradiction.

Case 3. Colour α appears at a, c, and g.

Analogously to Case 1. Since α cannot be missing at any vertex and it cannot appear at any amongst b, f, o, j, k, n, m, it must appear at both i and ℓ. However, any colour appearing at b (f) must also appear at ℓ (i), a contradiction, since $\mu_b = \mu_\ell = \mu_f = \mu_i$. □

With Theorem 4.1, now we can explain why the base gadget is referred to as *inverting*. Let H be a copy of our inverting gadget in a d-regular graph G, and let φ be a d-edge-colouring of H. A pair of half-edges (e_1, e_2) of H, under φ, is said to be *true* if $|\varphi(e_1) \cap \varphi(e_2)| = 1$, and *false* if $\varphi(e_1) \cap \varphi(e_2) = \emptyset$. From Theorem 4.1, if φ is part of a d-edge-colouring of G, we have that if (a, b) is *true*, then any pair chosen from $\{c, f, g\}$ is *false*. Else, if (a, b) is *false*, then (f, g) is *true*, which in turn implies that any pair chosen from $\{a, b, c\}$ is *false*. A d-edge-colouring ψ of a pair of half-edges (e_1, e_2) of H is said to be *consistent* for (e_1, e_2) if $|\psi(e_1) \cap \psi(e_2)| \leq 1$, i.e. if (e_1, e_2) is either *true* or *false*. Again by Theorem 4.1, if ψ is part of a d-edge-colouring of G, then it is consistent for any pair chosen from the half-edges of H. In Fig. 11, we introduce the representation of the inverting gadget in the construction of our infinite families of d-snarks in Sect. 5.

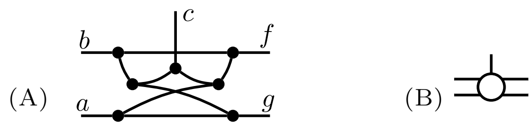


FIGURE 11. (A) The inverting gadget, having the multiplicities of the (half-)edges as defined in Table 1; (B) the representation of the inverting gadget.

5. AN INFINITE FAMILY OF d-SNARKS FOR EACH ODD d

A feature of our inverting gadget (recall Fig. 9A and Table 1) is that, differently from Leven and Galil's (Fig. 9B), the multiplicities of half-edges a and b are the same as the multiplicities of half-edges g and f, respectively. This symmetry is the key which allows us to extend the construction of the infinite family $\mathcal{L}$ of the Loupekine snarks (recall Fig. 2) to the construction of an infinite family $\mathcal{L}_d$ of d-snarks for any fixed odd $d \geq 5$. The graphs of $\mathcal{L}_d$ are the d-snarks obtained by the following procedure, illustrated in Fig. 12:

1. Take an odd integer $r \geq d$ and *any* integer solution (x, y) of the equation $2x + dy = r$. Observe that y must be odd.
2. Let H be our inverting gadget for d, take r copies of H, to which we refer as the *base blocks* of the d-snark under construction.
3. Arrange the r base blocks in a cycle, connecting each pair of consecutive blocks either with a *parallel* (i.e. half-edge f (g) of a block is identified with half-edge b (a) of the other) or a *cross* (i.e. half-edge f (g) of a block is identified with half-edge a (b) of the other) *link*. Remark that cross links are possible only when $d \equiv 3 \pmod 4$, otherwise, by Table 1, we do not have $\mu_a = \mu_f$.
4. Gather the upper half-edges c of the blocks either in groups of two, identifying both half-edges of each group, or in groups of d, joining the half-edges of each group to a new *joining* vertex, being x the amount of groups of two half-edges, and y the amount of groups of three. Observe that there are many ways of grouping these edges.

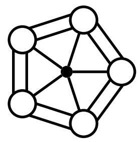
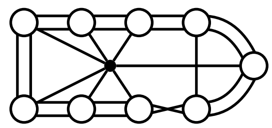
(A)  (B) 

FIGURE 12. (A) The smallest graph of $\mathcal{L}_5$; (B) a graph of $\mathcal{L}_7$ with nine base blocks and a *cross* link. Cross links are not possible for graphs of $\mathcal{L}_5$.

Theorem 5.1. *Let d, r, t be fixed odd integers with $r \geq d \geq 5$ and $1 \leq t \leq \lfloor r/d \rfloor$. Let G be a graph of $\mathcal{L}_d$ built on r base blocks and t joining vertices. Then, G is a d-snark and the number of vertices of G is:*

- *$137r + t$ if $d = 5$;*
- *$(7(d^2 + 1) - d)r + t$ if $d \geq 7$.*

Proof. First assume, for the sake of contradiction, that G is d-edge-colourable. Then, by Section 4, each link connecting two consecutive base blocks in the cycle is either *true* or *false*. Moreover, no matter how the base blocks are coloured, even though the coloured assigned to a base block may affect the coloured assigned to other base block, the truth values appear along the cycle alternately, since each base block is an *inverting* gadget. However, the number of base blocks is odd, so G must be a d-snark.

Now we count the number of vertices of G. Let H be the inverting gadget with multiplicities as defined in Table 1. The vertices in H are the seven vertices displayed in Fig. 9A, plus the $2d$ vertices of each copy of $K_{d,d}$ used in the middle of an edge for each set of at least two parallel edges. Since, in G, a pair (f, g) of a base block is the pair (a, b) of the next base block in the cycle, we shall count only the copies of $K_{d,d}$ for half-edges a and b, avoiding counting them twice in the whole graph G. Recall that half-edge c has multiplicity one.

If $d = 5$, the (half-)edges of H with multiplicity greater than one, disregarding f and g, are a, h, j, k, m, n. Since the sum of the multiplicities of these edges is 13, we count, for each of the r base blocks of G, the seven vertices displayed in Fig. 9A plus the $13(2d) = 130$ vertices for the 13 copies of $K_{d,d}$. Counting also the t joining vertices, we have that the number of vertices of G is $137r + t$ in this case.

If $d \geq 7$, then all the (half-)edges $a, b, h, i, j, k, \ell, m, n, o$ have multiplicity greater than one, and the sum of the multiplicities of these edges is, from Table 1, equal to $(7d-1)/2$ for both cases $d \equiv 1 \pmod 4$ and $d \equiv 3 \pmod 4$. Hence, for each of the r base blocks of G, we count the seven vertices displayed in Fig. 9A plus the $(2d)(7d-1)/2 = 7d^2 - d$ vertices for the $(7d-1)/2$ copies of $K_{d,d}$, yielding a total of $7d^2 - d + 7 = 7(d^2 + 1) - d$ vertices per base block. Therefore, in this case, the number of vertices of G is $(7(d^2 + 1) - d)r + t$. $\square$

Corollary 5.1. *Let $d \geq 5$ be a fixed odd integer. The number of vertices of a smallest d-snark of $\mathcal{L}_d$ is:* 686 *if $d = 5$;* $7(d^3 + d) - d^2 + 1$ *if $d \geq 7$.*

Proof. Follows from Theorem 5.1 considering a d-snark of $\mathcal{L}_d$ built on d base blocks (and thus one joining vertex, since d is odd). □

6. On d-snarks and the hardness of edge-colouring

Our discussion on d-snarks and the Overfull Conjecture starts with a useful characterisation of regular *SO* graphs of odd degree. We recall that a graph G on n vertices is said to be *overfull* if it has more than $\Delta\lfloor n/2 \rfloor$ edges, or, equivalently [39], if n is odd and $\sum_{u\in V(G)}(\Delta - d(u)) \leq \Delta - 2$. A graph G is said to be *subgraph-overfull* (shortly, *SO*) if it has an overfull Δ-subgraph.

Lemma 6.1. *Let $d \geq 3$ be a positive integer, a d-regular graph G on n vertices is* SO *if and only if n is odd or G has a cut with at most $d-2$ edges induced by some $U \subseteq V(G)$ with odd $|U|$ and $\Delta(G[U]) = d$.*

Proof. Every regular graph G of odd order is overfull, thus *SO*, since it has $\sum_{u\in V(G)}(\Delta(G) - d_G(u)) = 0$. On the other hand, if G is a regular graph of even order, then, since G cannot be overfull itself, it is *SO* if and only if it has a proper Δ-subgraph H which is overfull, which holds if and only if $|V(H)|$ is odd and

$$\sum_{u\in V(H)}(\Delta - d_H(u)) = |\partial_G(V(H))| \leq \Delta - 2\,. \qquad \square$$

Theorem 6.1 brings that d-snarks are *Class 2* graphs which are *not SO*, as it is known for snarks [49].

Theorem 6.1. *No d-snark can be* SO*.*

Proof. A d-snark is a d-regular graph with odd d and so with an even number of vertices. By Lemma 6.1, a d-snark to be *SO* must have a cut with at most $d-2$ edges, which contradicts the $(d-1)$-edge-connectivity of d-snarks. □

Theorem 6.1 brings a reason why we do not define d-snarks for even d. If we defined d-snarks for even d, then we would have d-snarks with odd order (e.g. K_{d+1}), being all overfull, but we are interested on hunting non-*SO Class 2* graphs, to better understand the limits of the Overfull Conjecture, as discussed in the sequel.

The following is another snark property which is extended to d-snarks.

Theorem 6.2. *Let G be a d-snark and u be any vertex of G. Then, $G - u$ is also a non-*SO Class 2 *graph.*

Proof. Let G be a d-snark on n vertices. Recall that n must be even by the Handshaking Lemma. Also, notice that a d-snark G cannot have universal vertices; otherwise, it would be a spanning subgraph of K_{d+1}, which is *Class 1* because d is odd. Thus, $G - u$ is a non-regular graph with $\Delta(G - u) = d$. We assume, for the sake of contradiction, that there is some $u \in V(G)$ such that $G - u$ has a d-edge-colouring. Recall that every graph G has an equitable k-edge-colouring for any integer $k \geq \chi'(G)$ [38]. Since in any equitable d-edge-colouring of $G - u$, each of the d colours must be assigned to the same number of $(n-2)/2$ edges, implying that each colour is missing at exactly one neighbour of u in G, yielding the construction of a d-edge-colouring of G. $\square$

Let P^* be the graph obtained from the Petersen graph by the removal of any vertex, in view of the symmetry of the Petersen graph. Since the Petersen graph is the smallest snark and since P^* is critical, the graph P^* is the one which maximises the ratio $\Delta(G)/|V(G)|$ amongst all known non-*SO Class 2* graphs G with $\Delta = 3$. This is one of the reasons why the Overfull Conjecture states the equivalence between *Class 2* and *SO* for n-vertex simple graphs satisfying $\Delta > n/3$. However, we believe that, for graphs with $\Delta > 3$, we could replace the lower bound $n/3$ by a smaller fraction of n, enlarging the set of graphs for which the equivalence between *SO* and *Class 2* seems to hold. In order to do so, it is important to find the order of the smallest non-*SO Class 2* graphs in function of its maximum degree. For example, if one discovers that the smallest 5-snark G has 26 vertices, then, since $G-u$ is by Theorem 6.2 also *Class 2* for any $u \in V(G)$, we may propose the following stronger form of the Overfull Conjecture: *if G is an n-vertex simple graph G satisfying $\Delta > n/3$, or $\Delta \geq 5$ and $\Delta > n/5$, then G is* Class 2 *if and only if G is* SO. To the best of our knowledge, the smallest *known* d-snarks for $d \geq 5$, $d \neq 7$, are Meredith's graphs G_d, for $d \equiv 2, 3, 4 \pmod 6$, and G'_d, for $d \equiv 0, 1, 5 \pmod 6$, both with $20d - 10$ vertices, as discussed in Sect. 1.

It is surprising that, due to the Overfull Conjecture, when restricted to n-vertex simple graphs G with $\Delta(G)$ bounded below by a fraction of n, the edge-colouring problem (deciding if a graph is *Class 1*) would be reducible to the test of a polynomial-time verifiable property. An evidence for this is the fact that we present in Theorem 6.3 on k-EDGE-COLOURING, i.e. the problem of deciding, for some fixed integer k, if a graph is k-edge-colourable. Notice that this problem is also NP-complete, for any $k \geq 3$, from Leven and Galil's reduction [10]. We show that k-EDGE-COLOURING is polynomial whenever $\Delta(G)$ is bounded below not only by a fraction of n, but by any $\Omega(n)$ function (or, equivalently, when n is bounded above by any $O(\Delta(G))$ function). The proof is inspired in the paper by Galby et al. [59], wherein the authors use the same argument to show that k-EDGE-COLOURING is linear-time solvable, i.e. solvable in time $O(n)$, for P_t-free graphs for any fixed t. In contrast, the computational complexity of EDGE-COLOURING for P_4-free graphs (also known as *cographs*) remains open despite much effort [44–48].

Theorem 6.3. *Let $k \in \mathbb{Z}_{\geq 0}$ be a fixed constant, k-*EDGE-COLOURING *is polynomial when restricted to n-vertex graphs with n bounded above by an $O(\Delta(G))$ function $f \colon \mathbb{Z}_{\geq 0} \to \mathbb{R}$.*

Proof (inspired in [59]). Our polynomial algorithm works on a given input graph G as follows. First, find in linear time the maximum degree Δ of G. If $\Delta \neq k$, output *yes* if $\Delta < k$, or *no* if $\Delta > k$, in view of Vizing's Theorem. If $\Delta = k$, we know that $n \leq f(\Delta)$ and $|E(G)| \leq (f(\Delta))^2$ for some $O(\Delta)$ function $f \colon \mathbb{Z}_{\geq 0} \to \mathbb{R}$. Since, in this case, $\Delta = k$ is a constant, we have that $(f(\Delta))^2$ is bounded above by a constant which does not depend on the size of the input graph. Ergo, any exact edge-colouring algorithm, even a brute-force search, can determine the k-edge-colourability of G in constant time. □

For all that we have discussed in this section, it seems that the *hunting* of the smallest d-snarks may be related to the very nature of the hardness of the edge-colouring problem. An interesting graph class wherein d-snarks could be hunted is the class of the complementary prisms. Let G be any graph on a non-empty set of vertices, the *complementary prism* $G\overline{G}$ is the graph obtained from the graphs G and its complement $\overline{G}$ by connecting with an edge each vertex in G to its corresponding vertex in $\overline{G}$. The Petersen graph is the complementary prism $C_5\overline{C_5}$.

Edge-colouring complementary prisms was the subject of a previous study of ours, joint with A. Zorzi [60], in which we proved that:

- every non-regular complementary prism with maximum degree Δ is Δ-edge-colourable;
- every d-regular complementary prism on n vertices has odd d, is $(d-1)$-edge-connected, and satisfies $n = 4d - 2$.

Therefore, the Petersen graph is the only complementary prism which is 3-regular and, if some d-regular complementary prism is not d-edge-colourable, then it is a d-snark. However, we could not find yet any non-d-edge-colourable d-regular complementary prism with $d \geq 5$. In fact, we verified that all complementary prisms with $d = 5$ (and thus 18 vertices) are d-edge-colourable, as well as 10 000 randomly generated regular complementary prisms with $d = 7$ (thus 26 vertices).

7. Further remarks

As discussed in Sect. 6, finding the smallest d-snarks may be of much interest to better understand the hardness of edge-colouring and the Overfull Conjecture. However, all d-snarks with $d \geq 5$ which we know are either Meredith's graphs with $20d - 10$ vertices and $d \neq 7$, or the even larger graphs of the infinite families presented in Sect. 5. In an undergraduate final project supervised by one of the authors [61], the student conducted a computational experiment to search for 5-snarks amongst all 5-regular graphs on 16 vertices and 90 billion distinct 5-regular graphs on 18 vertices. No graph tested was a 5-snark.

We encourage future investigation on how the base gadget presented in Sect. 4 could be used to prove Conjectures 1.1 and 1.2. As already discussed in Sect. 4, these conjectures are already settled for $d = 3$ by Holyer's proof [13]. By the use of a Turing oracle reduction (instead of a more common Karp reduction), we can also settle Conjecture 1.2 for $d = 5$, as discussed in a local workshop [62] and presented below.

Proof of Conjecture 1.2 for $d = 5$. Let G be a 5-regular graph which admits a 5-edge-colouring. From the Parity Lemma follows that, if G has a cut with fewer than four edges separating two induced subgraphs H_1 and H_2, then this cut must be a matching with exactly two edges uv and xy, both coloured the same. Assuming

$u, x \in V(H_1)$ and $v, y \in V(H_2)$, let $G_1 \coloneqq H_1 + ux$ and $G_2 \coloneqq H_2 + vy$ (see Fig. 13). If ux (vy) is an edge of H_1 (H_2), we can get rid of multiple edges in G_1 (G_2) in the same manner as multiple edges have been handled in Sect. 4. Clearly, the 5-edge-colouring of G yields a 5-edge-colouring of the union of the 5-regular graphs G_1 and G_2. Conversely, to construct a 5-edge-colouring of G from a 5-edge-colouring of $G_1 \cup G_2$, it suffices to rename the colours in G_1 or in G_2 so that the colours assigned to ux and vy are the same.

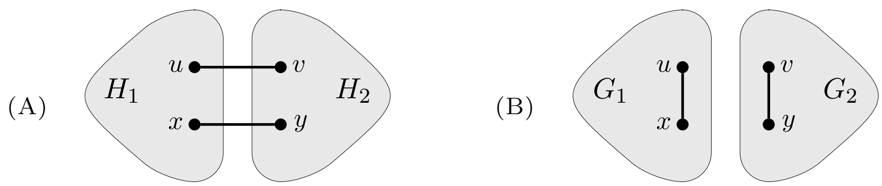


FIGURE 13. (A) Graph G with a matching $\{uv, xy\}$ separating two induced subgraphs H_1 and H_2; (B) the disjoint union of the graphs $G_1 \coloneqq H_1 + ux$ and $G_2 \coloneqq H_2 + vy$.

We have, then, a Turing oracle reduction from $\mathsf{EDGE\text{-}COLOURING}$(5-regular) to $\mathsf{EDGE\text{-}COLOURING}$(5-regular, 4-edge-connected). The reduction works on an input graph G as follows:

1. if G has some cut with fewer than four edges which is not a matching with exactly two edges, output *no*;
2. break each cut with two edges as in Fig. 13, obtaining 4-edge-connected 4-regular graphs $G_1, \ldots, G_k$, handling multiple edges as in Sect. 4;
3. call the oracle for G_i, for each $i \in \{1, \ldots, k\}$;
4. output *yes* if and only if $G_1, \ldots, G_k$ are all 5-edge-colourable. $\square$

This Turing reduction does not help much in the *hunting* of 5-snarks. If the proof had been constructed with a Karp reduction from $\mathsf{3SAT}$, then, for every non-satisfiable $\mathsf{3SAT}$ formula, even the smallest one, the Karp reduction would output a 5-snark. With the Turing reduction, on the other hand, if we get a 5-snark G_i from a non-5-edge-colourable 5-regular graph G, we already had G_i as a 4-edge-connected component of G, up to the simple operation described in Fig. 13. Therefore, in the beginning of this project, when the Turing oracle reduction was all that we had, we could prove that 5-snarks must exist, otherwise $\mathsf{P} = \mathsf{NP}$, but we could not find any of these graphs.

One can verify that all d-snarks with $d \geq 5$ appearing in this paper (Meredith's d-snarks, Rizzi's 5-snarks discussed in Sect. 1 and the d-snarks from the infinite families constructed in Sect. 5) contain snarks as subgraphs. This leads to the following.

Question 7.1. Does every d-snark with $d \geq 5$ contain a snark (as a subgraph)?

This question does not seem easy to answer. We know that every d-snark with $d \geq 5$ contains a *Class 2 subcubic* graph[4], but not necessarily cubic. Relatedly, one may wonder if *every* 5-snark can be decomposed into two disjoint perfect matchings and a (not necessarily 2-edge-connected) *Class 2* cubic graph, but this does *not* hold. Rizzi's poorly matchable 5-snarks, mentioned in Sect. 1, are clearly counterexamples, since they are graphs wherein every pair of perfect matchings intersect.

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References

[1] J. Edmonds. Matching polytope and a polyhedron with 0, 1 vertices. *J. Res. Natl. Bur. of Standards B (Math. and Math. Phys.)*, 69B(1–2):125–130, 1965.
[2] P. D. Seymour. On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte. *Proc. London Math. Soc.*, 38:423–460, 1979.
[3] W. G. Brown, P. Erdős, and V. T. Sós. Some extremal problems on r-graphs. In *New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich, 1971)*, pages 53–63, New York, 1973. Academic Press.
[4] P. D. Seymour. Some unsolved problems on one-factorizations of graphs. In J. A. Bondy and U. S. R. Murty, editors, *Graph Theory and Related Topics*, pages 367–368. Academic Press, New York, 1979.
[5] R. Rizzi. Indecomposable r-graphs and some other counterexamples. *J. Graph Theory*, 32(1):1–15, 1999.
[6] J. Balogh, F. C. Clemen, and L. Mattos. Counting r-graphs without forbidden configurations. *J. Combin. Theory Ser. B*, 157:216–234, 2022.
[7] L. Lovász. Matching structure and the matching lattice. *J. Combin. Theory Ser. B*, 42:187–222, 1987.
[8] J. Edmonds. Paths, trees, and flowers. *Can. J. Math.*, 17:449–467, 1965.
[9] I. Holyer. The NP-completeness of edge-colouring. *SIAM J. Comput.*, 10(4):718–720, 1981.
[10] D. Leven and Z. Galil. NP-completeness of finding the chromatic index of regular graphs. *J. Algorithms*, 4:35–44, 1983.
[11] R. Isaacs. Infinite families of non-trivial trivalent graphs which are not Tait-colorable. *Amer. Math. Monthly*, 82:221–239, 1975.
[12] G. Brinkmann, J. Goedgebeur, J. Hägglund, and K. Markström. *Generation and properties of snarks*, volume 103. 2012.
[13] D. A. Holton and J. Sheehan. *The Petersen Graph.* Cambridge University Press, 1993.
[14] M. Kochol. Superpositions and constructions of graphs without nowhere-zero k-flows. *Europ. J. Combin.*, 23:281–306, 2002.
[15] Y. Ma, D. Mattiolo, E. Steffen, and I. H. Wolf. Pairwise disjoint perfect matchings in r-edge-connected r-regular graphs. *SIAM J. Discrete Math.*, 37(3):1548–1565, 2023.
[16] P. G. Tait. On the colouring of maps. *Proc. Roy. Soc. Edinburgh Sect. A*, 10:501–503,729, 1878–1880.

[4]Vizing [63] showed that, for every connected *Class 2* graph G, every k satisfying $2 \leq k \leq \Delta(G)$, and every $i \in \{1, 2\}$, there is a *Class i* subgraph H of G with $\Delta(H) = k$.

[17] F. Jaeger. A survey of the cycle double cover conjecture. In *Annals of Discrete Mathematics 27 — Cycles in Graphs*, volume 27 of *North-Holland Mathematics Studies*, pages 1–12, 1985.
[18] M. Chladný and M. Škoviera. Factorisation of snarks. *Electron. J. Combin.*, 17(#R32), 2010.
[19] M. Gardner. Mathematical games. *Scientific American*, 4(234):126–130, 1976.
[20] A. B. Kempe. A memoir on the theory of mathematical form. *Philos. Trans. Royal Soc.*, 177:1–70, 1886.
[21] J. Petersen. Sur le théorème de Tait. *L'Intermédiaire des Mathématiciens*, 5:225–227, 1898.
[22] D. Blanuša. Problem četiriju boja. *Glasnik. Mat. Fiz. Astr. Ser. II*, 1:31–42, 1946.
[23] B. Descartes. Network-colourings. *The Mathematical Gazette*, 32:67–69, 1948.
[24] G. Szekeres. Polyhedral decompositions of cubic graphs. *Bull. Austral. Math. Soc.*, 8(3):367–387, June 1973.
[25] G. H. J. Meredith. Regular n-valent n-connected nonHamiltonian non-n-edge-colorable graphs. *J. Comb. Theory Ser. B*, 14(1):55–60, February 1973.
[26] R. Isaacs. Loupekine's snarks: A bifamily of non-Tait-colorable graphs. Technical Report 263, The Johns Hopkins University, 1976.
[27] Z. Skupień. Exponentially many hypohamiltonian snarks. *Electron. Notes Discrete Math.*, 28:417–424, 2007.
[28] R. W. Robinson and N. C. Wormald. Almost all regular graphs are hamiltonian. *Random Struct. Algor.*, 5:363–374, 1994.
[29] R. Diestel. *Graph Theory*. Springer, 4th edition, 2010.
[30] J. Petersen. Die theorie der regulären graphs. *Acta Mathematica*, 15:193–220, 1891.
[31] D. Naddef and W. R. Pulleyblank. Matchings in regular graphs. *Discrete Math.*, 34:283–291, 1981.
[32] L. Esperet, F. Kardoš, A. D. King, D. Král, and S. Norine. Exponentially many perfect matchings in cubic graphs. *Adv. Math.*, 227:1646–1664, 2011.
[33] Vadim G. Vizing. On an estimate of the chromatic class of a p-graph (in Russian). *Diskret. Analiz.*, 3:25–30, 1964.
[34] T. Nishizeki and K. Kashiwagi. An upper bound on the chromatic index of multigraphs. In Y. Alavi, G. Chartrand, D. R. Lick, C. E. Wall, and L. Lesniak, editors, *Graph theory with applications to algorithms and computer science*, pages 595–604. Wiley, New York, 1985.
[35] L. Cai and J. A. Ellis. NP-completeness of edge-colouring some restricted graphs. *Discrete Appl. Math.*, 30:15–27, 1991.
[36] D. P. Koreas. The NP-completeness of chromatic index in triangle free graphs with maximum vertex of degree 3. *J. Appl. Math. Comput.*, 93:13–17, 1997.
[37] A. Björklund, T. Husfeldt, and M. Kolvisto. Set partitioning via inclusion-exclusion. *SIAM J. Comput.*, 39(2):546–563, 2009.
[38] J. Folkman and D. R. Fulkerson. Edge colorings in bipartite graphs. *Comb. Math. Appl.*, pages 561–577, 1969.
[39] T. Niessen. How to find overfull subgraphs in graphs with large maximum degree. *Discrete Appl. Math.*, 51:117–125, 1994.
[40] T. Niessen. How to find overfull subgraphs in graphs with large maximum degree, II. *Electron. J. Combin.*, 8, 2001.
[41] M. W. Padberg and M. R. Rao. Odd minimum cut-sets and b-matching. *Math. Oper. Res.*, 7:67–80, 1982.
[42] A. G. Chetwynd and A. J. W. Hilton. The chromatic index of graphs of even order with many edges. *J. Graph Theory*, 8:463–470, 1984.
[43] D. G. Hoffman and C. A. Rodger. The chromatic index of complete multipartite graphs. *J. Graph Theory*, 16:159–163, 1992.
[44] C. De Simone and A. Galluccio. Edge-colouring of regular graphs of large degree. *Theor. Comput. Sci.*, 389:91–99, 2007.
[45] C. De Simone and A. Galluccio. Edge-colouring of join of regular graphs. *J. Comb. Optim.*, 18:417–428, 2009.

[46] R. C. S. Machado and C. M. H. Figueiredo. Decompositions for edge-coloring join graphs and cobipartite graphs. *Discrete Appl. Math.*, 158:1336–1342, 2010.
[47] A. Zorzi and L. M. Zatesko. On the chromatic index of join graphs and triangle-free graphs with large maximum degree. *Discrete Appl. Math.*, 245:183–189, 2018.
[48] L. M. Zatesko, A. Zorzi, R. Carmo, and A. L. P. Guedes. Edge-colouring graphs with bounded local degree sums. *Discrete Appl. Math.*, 281:268–283, 2020.
[49] A. G. Chetwynd and A. J. W. Hilton. Star multigraphs with three vertices of maximum degree. *Math. Proc. Cambridge Philos. Soc.*, 100:303–317, 1986.
[50] A. J. W. Hilton and P. D. Johnson. Graphs which are vertex-critical with respect to the edge-chromatic number. *Math. Proc. Cambridge Philos. Soc.*, 102:103–112, 1987.
[51] M. J. Plantholt. The chromatic index of graphs with a spanning star. *J. Graph Theory*, 5:45–53, 1981.
[52] A. G. Chetwynd and A. J. W. Hilton. The edge-chromatic class of graphs with maximum degree at least $|V|-3$. *Ann. Discrete Math.*, 41:91–110, 1989.
[53] J. Meidanis. Edge coloring of cycle powers is easy, 1998. Unpublished manuscript. URL: `http://www.ic.unicamp.br/~meidanis/research/edge/cpowers.ps`.
[54] C. Ortiz Z., N. Maculan, and J. L. Szwarcfiter. Characterizing and edge-colouring split-indifference graphs. *Discrete Appl. Math.*, 82:209–217, 1998.
[55] L. G. S. Gonzaga, J. B. Sousa Cruz, C. N. Silva, and S. M. Almeida. The overfull conjecture on split-comparability and split-interval graphs. *Discrete Appl. Math.*, 340:228–238, 2023.
[56] A. G. Chetwynd and A. J. W. Hilton. Regular graphs of high degree are 1-factorizable. *Proc. London Math. Soc.*, 3(50):193–206, 1985.
[57] L. Perkovic and B. Reed. Edge coloring regular graphs of high degree. *Discrete Appl. Math.*, 165/166:567–578, 1997.
[58] D. Kőnig. Graphok és alkalmazásuk a determinánsok és a halmazok elméletére. *Math. Természettudományi Értesito*, 34:104–119, 1916.
[59] E. Galby, P. T. Lima, D. Paulusma, and B. Ries. Classifying k-edge colouring for H-free graphs. *Inf. Process. Lett.*, 146:39–43, 2019.
[60] L. M. Zatesko, R. Carmo, A. L. P. Guedes, A. Zorzi, R. C. S. Machado, and C. M. H. Figueiredo. On the chromatic index of complementary prisms. In *Proc. X European Conference on Combinatorics, Graph Theory and Applications (EUROCOMB '19)*, volume 88 of *Acta Math. Univ. Comenianae*, pages 1071–1077, Bratislava, 2019.
[61] Éverton A. Vieira. Grafos d-snarks, 2019. Undergraduate thesis. Federal University of Fronteira Sul.
[62] L. M. Zatesko and E. A. Vieira. P = NP or 5-snarks exist. In *Proc. 3rd Workshop de Pesquisa em Computação dos Campos Gerais (WPCCG '19)*, pages 135–140, Ponta Grossa, Brazil, 2019.
[63] V. G. Vizing. Critical graphs with a given chromatic class (in Russian). *Diskret. Analiz.*, 5:9–17, 1965.