Parameterized Algorithms for Steiner Tree and Dominating Set: Bounding the Leafage by the Vertex Leafage

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Abstract. Chordal graphs are intersection graphs of subtrees of a tree, while interval graphs are intersection graphs of subpaths of a path. Undirected path graphs are an intermediate class of graphs, defined as the intersection graphs of paths of a tree. It is known that Dominating Set, Connected Dominating Set, and Steiner Tree are W[2]-hard on chordal graphs, when parameterized by the size of the solution, and are polynomial-time solvable on interval graphs. As for the undirected path graphs, all these problems are known to be NP-complete, and when parameterized by the size of the solution, no classification in the parameterized complexity theory is known apart from the trivial XP classification. We prove that Dominating Set, Connected Dominating Set, and Steiner Tree are FPT for undirected path graphs when parameterized by the size of the solution, and that they continue to be FPT for general chordal graphs when parameterized by the size of the solution plus the vertex leafage of the graph, provided a tree model with optimal vertex leafage is given. We show a relation between the parameterization of Min-LC-VSP problems by the leafage of the graph versus the vertex leafage plus the size of a solution.

Keywords: Chordal graphs · Undirected Path graphs · Dominating Set · Steiner Tree · FPT algorithms

1 Introduction

Given a graph G and a family of subsets $\mathcal{S} = \{S_u\}_{u \in V(G)}$ of a set U, we say that G is the *intersection graph of* $\mathcal{S}$ if $uv \in E(G)$ if and only if $S_u \cap S_v \neq \emptyset$, and that $(U, \mathcal{S})$ is a *model* of G. *Chordal* graphs are defined as graphs having

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no induced cycle of size bigger than three, but it is known that they are also the intersection graphs of subtrees of a characteristic tree [13]. Nested subclasses of chordal graphs are defined by putting constraints in either the characteristic tree, or the subtrees. *Interval* graphs are the intersection graphs of subpaths of a path [6]; *rooted directed path* graphs are the intersection graphs of directed paths of an out-branching [14] (an oriented rooted tree with all vertices being reachable from the root); *directed path* graphs are the intersection graphs of directed paths of an oriented tree [20]; and *undirected path* graphs are the intersection graphs of paths of a tree [12]. The cited papers give polynomial-time recognition algorithms that also provide models for these classes, called *tree models*.

A set $D \subseteq V(G)$ is *dominating* if, for every vertex $v \in V(G) \setminus D$, we have that v has a neighbor in D. Given a graph G and a positive integer κ, the Dominating Set problem consists of deciding whether G has a dominating set of size at most κ, while the Connected Dominating Set asks the same but requires additionally that $G[D]$ is connected. Given also a subset $X \subseteq V(G)$, called set of *terminals*, the Steiner Tree problem consists of deciding whether there exists a subset $S \subseteq V(G) \setminus X$, called *Steiner set*, such that $|S| \leq \kappa$ and $G[S \cup X]$ is connected—and hence $G[S \cup X]$ has a spanning tree T, called a *Steiner tree* of G for X. It is known that Connected Dominating Set and Steiner Tree have the same complexity for chordal graphs and subclasses [23]. The natural parameter of all these problems is κ.

Dominating Set is considered the canonical problem in the class W[2]-hard when parameterized by κ, which explains the great interest in it (see e.g. [15]). When restricted to chordal graphs (and even to split graphs), Dominating Set, as well as Connected Dominating Set, are still W[2]-hard when parameterized by κ [21]. However, they become polynomial-time solvable on interval graphs, and more generally on rooted directed path graphs [5,23], which brings the natural question about whether they are also polynomial-time solvable on undirected path graphs. This unfortunately is not the case, as both are NP-complete on these graphs [5,10]. Up to our knowledge, it is not known whether (Connected) Dominating Set is solvable in polynomial time on directed path graphs. Nevertheless, it could still happen that they are FPT when parameterized by κ on undirected path graphs, and indeed this is one of our results. This classification closes all the parameterized complexity open entries for undirected path graphs presented in [10].

Undirected path graphs can also be seen as intersection graphs of subtrees of a tree where each subtree has at most 2 leaves. A natural generalization therefore is to investigate intersection of subtrees with at most ℓ leaves, which leads to the definition of vertex leafage of a chordal graph. Given a tree model $\mathcal{T} = (T, \{T_u\}_{u \in V(G)})$ of a chordal G, the *vertex leafage of* $\mathcal{T}$ is the maximum number $v\ell(\mathcal{T})$ of leaves in a subtree T_u, while the *vertex leafage of* G is the minimum vertex leafage over all of its tree models [8]; we denote the parameter by $v\ell(G)$. Undirected path graphs are exactly the chordal graphs with vertex leafage 2. Recall that Dominating Set and Connected Dominating Set are NP-complete on undirected path graphs [5,10], which gives us that they are NP-complete on chordal graphs with vertex leafage k for every fixed $k \geq 2$.

This fact prevents the existence of FPT algorithms parameterized by the vertex leafage of chordal graphs unless P = NP.

In this work we prove that Connected Dominating Set and Dominating Set are FPT on chordal graphs when parameterized by $\kappa + v\ell(G)$, as long as a tree model with optimal vertex leafage is provided. Since a tree model with optimal vertex leafage can be computed in polynomial time for undirected path graphs [12], we get that these problems are FPT when parameterized by κ on these graphs, which is best possible by the mentioned results.

Theorem 1. *Let G be a chordal graph. If a tree model $\mathcal{T}$ such that $v\ell(\mathcal{T}) = v\ell(G)$ is provided, then* Dominating Set*,* Connected Dominating Set *and* Steiner Tree *are* FPT *when parameterized by $\kappa + v\ell(G)$. In particular, when restricted to undirected path graphs, then* Dominating Set *can always be solved in time $O^*(2^{2\kappa(1+\log\kappa)})$, while* Connected Dominating Set *and* Steiner Tree *can be solved in time $O^*(4^\kappa)$.*

A closely related parameter is the *leafage of G*, denoted by $\ell(G)$, which is the minimum number of leaves $\ell(\mathcal{T})$ in the tree of a tree model $\mathcal{T}$ of G [19]. Surprisingly enough, a tree model with $\ell(G)$ leaves can be computed in polynomial time [16]. This unfortunately is not the case for the vertex leafage parameter, as it is known [8] that it is NP-complete to decide whether a chordal graph G has vertex leafage at most 3; they also give an algorithm to compute $v\ell(G)$ in time $n^{\ell(G)}$, which is XP when parameterized by $\ell(G)$. In [11] they provide an FPT algorithm for Dominating Set when parameterized by $\ell(G)$. Since $v\ell(G)$ is a *weaker* parameter than $\ell(G)$, the algorithm provided in [11] is not readily applicable to Dominating Set parameterized by κ and $v\ell(G)$. Nevertheless, we show that positive instances of Dominating Set and Connected Dominating Set must have bounded leafage, which brought us to the question about whether the same holds for generalizations of Dominating Set. Indeed, we have found that the broader class of problems, called Min-LC-VSP problems [7,11], have the same property. Given a graph G and subsets $\sigma, \rho \subseteq \{0, \cdots, n-1\}$, a subset $S \subseteq V(G)$ is a *(σ,ρ)-set* if: $|N(v) \cap S| \in \sigma$ for every $v \in S$, and $|N(v) \cap S| \in \rho$ for every $v \in V(G) \setminus S$. Fixing σ, ρ, and given a graph G and an integer κ, the Min-LC-VSP$_{\sigma,\rho}$ problem consists in deciding whether there exists a (σ,ρ)-set S of size at most κ. Observe that if $0 \in \rho$, then the answer is always yes since taking the empty set satisfies the constraints; this is why we suppose $0 \notin \rho$ in what follows. Min-LC-VSP problems generalize a number of optimization problems, as e.g. Dominating Set, d-Dominating Set, Total Dominating Set, Induced d-regular subgraph, etc. [7]. We state our result and its corollary obtained from $v\ell(G) \leq \ell(G)$.

Theorem 2. *Let $\sigma, \rho \subseteq \{0, \cdots, n-1\}$ be such that $0 \notin \rho$, G a chordal graph and κ a positive integer. If (G, κ) is a* Yes *instance of* Min-LC-VSP$_{\sigma,\rho}$*, then $\ell(G) \leq \kappa \cdot v\ell(G)$.*

Corollary 1. *Let $\sigma, \rho \subseteq \{0, \cdots, n-1\}$, G be a chordal graph and κ be a positive integer. If* Min-LC-VSP$_{\sigma,\rho}$ *is* FPT *when parameterized by $v\ell(G)$, then* Min-LC-VSP$_{\sigma,\rho}$ *is also* FPT *when parameterized by $\ell(G)$. And if* Min-LC-VSP$_{\sigma,\rho}$

is FPT *when parameterized by* $\ell(G)$ *and a tree model* $\mathcal{T}$ *with* $v\ell(\mathcal{T}) = v\ell(G)$ *is provided, then* Min-LC-VSP$_{\sigma,\rho}$ *is also* FPT *when parameterized by* $\kappa + v\ell(G)$.

We mention that Max-LC-VSP$_{\sigma,\rho}$ can also be defined (in this case, the problem consists in deciding whether there exists a (σ, ρ)-set S such that $|S| \geq \kappa$), but our proof cannot be applied to these problems. Nevertheless, many of the Max-LC-VSP$_{\sigma,\rho}$ problems cited in [7] are known to be polynomial-time solvable in chordal graphs, e.g. Independent Set, Maximum Induced Matching, Maximum Efficient Edge Dominating Set and Maximum Dominating Induced Matching, Strong Stable Set, etc. (see for instance [18]).

Another parameter of interest is the mim-width of G [22], since many problems can be solved in XP time when parameterized by mim-width [2,7,17], and rooted directed path graphs have mim-width 1 [17]. One could therefore ask whether undirected path graphs also have bounded mim-width. Up to our knowledge, no explicit construction of undirected path graphs with unbounded mim-width is known, but the fact that LC-VSP problems can be solved in polynomial time on graphs with bounded mim-width [7], combined with the NP-hardness of Dominating Set on undirected path graphs, give evidence that undirected path graphs do not have bounded mim-width, unless P = NP.

2 Preliminaries

A *parameterized problem* is a language $\Pi \subseteq \Sigma \times \mathbb{N}$, where Σ is a fixed finite alphabet. A pair $(I, \kappa) \in \Sigma \times \mathbb{N}$ is called an *instance* of Π with *parameter* κ, and we say that it is a Yes *instance* if $(I, \kappa) \in \Pi$. Given instances $(I, \kappa), (I', \kappa')$ of the same parameterized problem Π, it is said that they are *equivalent* if (I, κ) is a Yes instance of Π if and only if so does (I', κ'). A *reduction rule* for Π is a polynomial-time computable function that maps an instance (I, κ) to another instance (I', κ'). It is *safe* if (I, κ) and (I', κ') are equivalent and $\kappa' \leq g(\kappa)$, where g is a computable function. We refer the reader to [9] for further background on parameterized complexity.

We denote by $\mathcal{T} = (T, \{T_u\}_{u \in V(G)})$ a tree model of G. Given a node $t \in V(T)$, we denote by V_t the set $\{u \in V(G) \colon t \in V(T_u)\}$. We say that $u \in V(G)$ is a *leafy vertex* of G (with respect to $\mathcal{T}$) if $V(T_u) = \{\ell_u\}$ and ℓ_u is a leaf in T; denote by $\mathcal{L}(G, \mathcal{T})$ the set of leafy vertices of G with respect to $\mathcal{T}$, and for each $u \in \mathcal{L}(G, \mathcal{T})$, denote by ℓ_u the unique node in T_u. We omit $(G, \mathcal{T})$ when it is clear from the context.

A tree model $(T, \{T_u\}_{u \in V(G)})$ of G is said to be *minimal* if there are no two adjacent nodes $t, t' \in V(T)$ such that $V_t \subseteq V_{t'}$. It is known that such a tree model can be computed in polynomial time [12]. Even though obtaining a minimal tree model, given a tree model of G, is a standard operation, we prove it explicitly in the appendix in order to show that also the vertex leafage does not increase.

Proposition 1 ([12]). *Let G be a chordal graph, and $\mathcal{T} = (T, \{T_u\}_{u \in V(G)})$ be a tree model of G. Then, a minimal tree model $\mathcal{T}' = (T', \{T'_u\}_{u \in V(G)})$ of G with $v\ell(\mathcal{T}') \leq v\ell(\mathcal{T})$ and $\ell(T') \leq \ell(T)$ can be computed in polynomial time.*

The following lemma directly implies Theorem 2 and will also be useful in the following sections.

Lemma 1. *Let G be a chordal graph, $\mathcal{T} = (T, \{T_u\}_{u \in V(G)})$ a minimal tree model of G such that $v\ell(\mathcal{T}) = v\ell(G)$, κ a positive integer and $S \subseteq V(G)$ such that $N[u] \cap S \neq \emptyset$ for every leafy vertex $u \in \mathcal{L}$. If $|S| \leq \kappa$, then $\ell(G) \leq \kappa \cdot v\ell(G)$.*

Proof. By contradiction, let $\ell_1, \ldots, \ell_k$ be the leaves of T, with $k \geq \kappa \cdot v\ell(G) + 1$. Since $\mathcal{T}$ is minimal, for each $i \in \{1, \ldots, k\}$, there exists $v_i \in V_{\ell_i}$ such that $V(T_{v_i}) = \{\ell_i\}$, as otherwise we would have $V_{\ell_i} \subseteq V_{t_i}$ where t_i is the neighbor of ℓ_i in T. For each $u \in S$, let $D_u = \{v_i \mid u \in N[v_i]\}$. Observe that if $v_i \in S$, then $D_{v_i} = \{v_i\}$ since $\ell_1, \ldots, \ell_k$ are all distinct leaves of T (i.e., $\{v_1, \ldots, v_k\}$ is an independent set). Note also that if $u \in S \setminus \{v_1, \ldots, v_k\}$, then $|D_u| \leq v\ell(G)$. By assumption, we know that $N[v_i] \cap S \neq \emptyset$ for every $v_i \in \{v_1, \ldots, v_k\} \setminus S$, which means that $\bigcup_{u \in S} D_u = \{v_1, \ldots, v_k\}$. However, we know that $|\bigcup_{u \in S} D_u| \leq \sum_{u \in S} |D_u| \leq |S| \cdot v\ell(G) \leq \kappa \cdot v\ell(G)$, a contradiction since $k > \kappa \cdot v\ell(G)$. □

Since in Theorem 2 we have $0 \notin \rho$, we get directly that a solution S to Min-LC-VSP$_{\sigma,\rho}$ applied to (G, κ) must be such that $N[u] \cap S \neq \emptyset$ for every $u \in V(G)$, and in particular for every leafy vertex. Hence, Theorem 2 follows from the above lemma. Additionally, it is known that Dominating Set can be solved in time $2^{O(\ell^2)} \cdot n^{O(1)}$ on a chordal graph G, where $\ell = \ell(G)$ and $n = |V(G)|$ [11]. Since Dominating Set is equivalent to Min-LC-VSP$_{\sigma,\rho}$ with $\sigma = \{0, \ldots, n-1\}$ and $\rho = \{1, \ldots, n-1\}$, we get that Corollary 1 implies that Dominating Set can be solved in FPT time on a chordal graph G when parameterized by $\kappa + v\ell(G)$, provided the appropriate model is given. To finish the proof of Theorem 1, we need to investigate the complexity of Steiner Tree and Connected Dominating Set, and to present the claimed algorithm for Dominating Set when restricted to undirected path graphs. This is done in Sects. 3 and 4, respectively.

3 Connected Dominating Set and Steiner Tree

In this section, we present FPT algorithms for Connected Dominating Set and Steiner Tree parameterized by $\kappa + v\ell(G)$. For simplicity, in what follows we denote an instance of Steiner Tree and Connected Dominating Set parameterized by $\kappa + v\ell(G)$ simply by (G, X, κ) and (G, κ), respectively, since $v\ell(G)$ depends on G and hence appears implicitly in the notation. We start by solving Steiner Tree, and at the end of the section we prove that Connected Dominating Set is equivalent to Steiner Tree applied to $(G, \mathcal{L})$, where $\mathcal{L}$ is the set of leafy vertices in a given model of G. And to solve Steiner Tree, we apply two reduction rules that allows us to consider only instances $(G, \mathcal{L}, \kappa)$. We start by getting rid of the leafy vertices that are not in X.

Reduction Rule 1. *Let (G, X, κ) be an instance of* Steiner Tree *where G is chordal, and $\mathcal{T} = (T, \{T_u\}_{u \in V(G)})$ be a tree model of G. If there exists $v \in \mathcal{L} \setminus X$, then* delete *$v$, obtaining the instance $(G - v, X, \kappa)$.*

Proof of safeness. Removing vertices clearly cannot increase the vertex leafage; hence we just need to prove that (G, X, κ) is a YES instance if and only if $(G - v, X, \kappa)$ also is. Clearly a solution for $(G - v, X, \kappa)$ is also a solution for (G, X, κ) since $G - v \subseteq G$. Conversely, let $S \subseteq V(G)$ be a Steiner set for (G, X) such that $|S| \leq \kappa$. By definition $H = G[S \cup X]$ is connected. If $v \notin S$, then $H \subseteq G - v$, so suppose otherwise. Observe that, by definition of tree model and since $V(T_v) = \{\ell\}$ for some leaf ℓ of T, we get that $N(v)$ is a clique of G. This clearly implies that v cannot be a cut-vertex in H, i.e., that $H - v$ is still connected, which means that $S - v$ is a solution for $(G - v, X, \kappa)$. □

We now show that it is enough to consider terminal vertices that are leafy vertices. We cannot, however, simply delete the set $X \setminus \mathcal{L}$ of non-leafy terminal vertices since they might be useful to connect terminal leafy vertices, without making an impact on the size of the Steiner set. Thus we use the bypass operation to eliminate vertices in $X \setminus \mathcal{L}$ while maintaining the connectivity that is gained by including these vertices in the induced subgraph $G[S \cup X]$. The *bypass* operation of a vertex $v \in V(G)$ consists of removing v from $V(G)$, and adding uw for every pair u, w of neighbors of v (such that $uw \notin E(G)$ to avoid multiple edges). Before we apply the bypass reduction, we prove the following lemma.

Lemma 2. *Let G be a chordal graph, $\mathcal{T} = (T, \{T_u\}_{u \in V(G)})$ be a tree model of G such that $\mathcal{L} \cap V_\ell \neq \emptyset$ for every leaf $\ell \in V(T)$, and $\emptyset \neq X \subseteq V(G)$ be such that $\mathcal{L} \subseteq X$. If S is a Steiner set for (G, X), then V_t contains some vertex of $S \cup X$ for every $t \in V(T)$.*

Proof. If $V(T) = \{t\}$ it follows trivially because $X \neq \emptyset$; so suppose $|V(T)| > 1$. The lemma also holds trivially for the leaves of T since $\mathcal{L} \subseteq X$ and $\mathcal{L} \cap V_\ell \neq \emptyset$ for every leaf $\ell \in V(T)$. So consider a non-leaf node t of T. Note that t must be within a path between two leaves ℓ_1 and ℓ_2 of T; let $v_1, v_2 \in V(G)$ be such that $V(T_{v_i}) = \{\ell_i\}$ for each $i \in \{1, 2\}$ (they exist by assumption). Since $G[S \cup X]$ is connected and $\{v_1, v_2\} \subseteq \mathcal{L} \subseteq X$, there is a path P in $G[S \cup X]$ between v_1, v_2. Because G is chordal, we get that V_q separates v_1 from v_2 in G for every internal node q in the ℓ_1, ℓ_2-path Q in T. Therefore, we get that P must contain a vertex of V_q for internal node q of Q, in particular it must contain a vertex of V_t. □

Reduction Rule 2. *Let (G, X, κ) be an instance of* STEINER TREE *where G is chordal, $\mathcal{T} = (T, \{T_u\}_{u \in V(G)})$ be a tree model of G such that $\mathcal{L} \cap V_\ell \neq \emptyset$ for every leaf $\ell \in V(T)$, and suppose that Reduction 1 cannot be applied. If there exists $v \in X \setminus \mathcal{L}$, then* bypass *$v$, obtaining the instance $(G', X - v, \kappa)$.*

Proof of safeness. First we show that the vertex leafage cannot increase by constructing a tree model of G' from $\mathcal{T}$. Consider $\mathcal{T}' = (T', \{T'_u\}_{u \in V(G')})$ obtained as follows.

1. T' is the tree obtained from T by contracting T_v to a single vertex, t_v; and
2. For each $u \in V(G')$, if $V(T_v) \cap V(T_u) = \emptyset$, then T_u remains the same; otherwise, T'_u is the subtree of T' containing exactly the vertices in $(V(T_u) \setminus V(T_v)) \cup \{t_v\}$.

To see that the vertex leafage does not increase, just observe that edge contractions of trees cannot increase the number of leaves. It remains to argue that $\mathcal{T}'$ is indeed a tree model of G'. For this, we must have $uw \in E(G')$ if and only if $V(T'_u) \cap V(T'_w) \neq \emptyset$. To see that it holds it suffices to observe that $t_v \in V(T'_u)$ if and only if $u \in N(v)$. Now, because the value of κ remains the same, and since the bypass operation can be clearly applied in polynomial time, it remains to show that (G, X, κ) and $(G', X - v, \kappa)$ are equivalent. Denote $X - v$ by X', and by $\mathcal{L}'$ the set $\mathcal{L}(G', \mathcal{T}')$. Note that the existence of v implies that $|\mathcal{L}| \geq 2$. Since Reduction 1 cannot be applied, we get that $\mathcal{L} \subseteq X$.

First, consider a solution S for (G, X, κ). We argue that S is also a Steiner set for (G', X'), i.e., that $S \cup X'$ induces a connected subgraph of G'. Indeed, if $u, w \in S \cup X' \subset S \cup X$, then there exists a u, w-path P in $G[S \cup X]$. If $v \notin V(P)$, then P still exists in G'; and otherwise, v is an intermediate vertex in P that is replaced by an edge in G', i.e., u, w are still connected in $G'[S \cup X']$.

Now, let S be a solution for (G', X', κ), which means that $H' = G'[S \cup X']$ is connected. We want to prove that $H = G[S \cup X]$ is also connected. For this, first observe that H' is obtained from $H - v$ by turning $N(v) \cap V(H')$ into a clique. Therefore, the only way H could be disconnected is if v is an isolated vertex in H; we show that this cannot occur. Indeed, note that contracting T_v into a single vertex t_v maintains the property that each leaf of T' must contain a leafy vertex, i.e., that $\mathcal{L}' \cap V_\ell \neq \emptyset$ for every leaf $\ell \in V(T')$. Hence, by Lemma 2 we must have $(S \cup X') \cap V_{t_v} \neq \emptyset$, i.e., v has some neighbor in H. $\square$

We are finally ready to prove the main result of this section.

Theorem 3. *Let G be a chordal graph on n vertices and m edges, $X \subseteq V(G)$, and κ be a positive integer.* Steiner Tree *can be solved on (G, X, κ) in time $O^*(2^{\kappa \cdot v\ell(G)})$, provided a tree model with optimal vertex leafage is given. In particular, if G is an undirected path graph, then* Steiner Tree *can always be solved in time $O(4^\kappa n^2 + nm)$.*

Proof. Let (G, X, κ) be an instance of Steiner tree where G is a chordal graph. If $\ell(G) = 1$ or $|X| = 1$, then $G[X]$ is a complete graph and thus $S = \emptyset$ is a solution. Thus we now assume that $\ell(G) \geq 2$ and $|X| \geq 2$. First, compute a minimal tree-model of G; this can be done in polynomial time [12]. Observe that a minimal tree model satisfies the condition of Reduction Rule 2. By iteratively applying Reduction Rules 1 and 2, and Proposition 1 to maintain a minimal tree model, we obtain in polynomial time an equivalent instance (G', X', κ) such that $v\ell(G') \leq v\ell(G)$, and X' is the set of leafy vertices of G' (related to a tree model $\mathcal{T}' = (T', \{T'_u\}_{u \in V(G)})$). Now, let $S \subseteq V(G')$ be a Steiner set for (G', X') such that $|S| \leq \kappa$. The connected components of $G[X']$ are exactly the cliques $V_\ell \cap X'$, ℓ a leaf of T'. So, we get that either $N[u] = N[v]$ or $N[u] \cap N[v] = \emptyset$ for every pair of leafy vertices $u, v \in X'$. Hence, we get $N(u) \cap S \neq \emptyset$ for every $u \in X'$, and by Lemma 1 we get $\ell(G') \leq \kappa \cdot v\ell(G') \leq \kappa \cdot v\ell(G)$. We can solve $(G', \mathcal{L}', \kappa)$ in the claimed time using the algorithm given in [3] for Steiner Tree which runs in this instance in time $O(2^{\kappa \cdot v\ell(G)} n^2 + nm)$ time, and in particular if G is an undirected path graph, the starting tree model with optimal vertex leafage can be found in polynomial time [12]. □

Finally, our result for Connected Dominating Set is obtained by proving equivalence to Steiner Tree on $(G, \mathcal{L}, \kappa)$. Our proof is necessary since the complexity equivalence proved in [23] concerns only classical complexity.

Theorem 4. *Let G be a chordal graph on n vertices and m edges, and κ be a positive integer.* Connected Dominating Set *can be solved on (G, κ) in time $O^*(2^{\kappa \cdot v\ell(G)})$, provided a tree model with optimal vertex leafage is given. In particular, if G is an undirected path graph, then* Connected Dominating Set *can always be solved in time $O(4^\kappa n^2 + nm)$.*

Proof. We prove that S is a connected dominating set of G if and only if S is a Steiner set for $(G, \mathcal{L})$, where $\mathcal{L}$ denotes the set of leafy vertices in a tree model $(T, \{T_u\}_{u \in V(G)})$ of G. The theorem follows by Theorem 3.

Let S be a connected dominating set of G. So $G[S]$ is a connected subgraph of G, and since it is also dominating, we get that $N(u) \cap S \neq \emptyset$ for every $u \in \mathcal{L}$. It follows that $G[S \cup \mathcal{L}]$ is also connected, and hence S is a Steiner set for $(G, \mathcal{L})$.

On the other hand, if S is a Steiner set for $(G, \mathcal{L})$, then by Lemma 2 we know that $V_t \cap (S \cup \mathcal{L}) \neq \emptyset$ for every $t \in V(T)$, which in turn implies that every $u \in V(G)$ has a neighbor in $S \cup \mathcal{L}$. To finish the proof, just recall that if v is a leafy vertex, then $N(v)$ is a clique. Hence, if $u \in V(G)$ is adjacent to $v \in \mathcal{L}$, then u is also adjacent to $w \in S \cap N(v)$ (which exists since S is a Steiner set for $(G, \mathcal{L})$ and $\mathcal{L}$ is a collection of disjoint cliques). □

4 Dominating Set

In this section, we present an FPT algorithm for Dominating Set parameterized by κ restricted to undirected path graphs. Although we believe that our method can be extended to any chordal graph, when parameterized by $\kappa + v\ell(G)$, we remark that the expected running time of such approach is worse than simply applying the algorithm given in [11] after bounding the leafage of the input graph. Thus we refrain from discussing this extension and focus only on the particular case of undirected path graphs since, in this case, our proof is self-contained, simpler, and the $O^*(2^{2\kappa(1+\log \kappa)})$ running time beats the $2^{O(\kappa^2)} n^{O(1)}$ running time provided by applying the algorithm in [11].

In the B-Dominating Set, we are given a graph G, a positive integer κ, and a subset $B \subseteq V(G)$ (called set of black vertices), and the goal is to decide if there is a set $D \subseteq V(G)$ with $|D| \leq \kappa$ such that $N[b] \cap D \neq \emptyset$ for every $b \in B$. In other words, the goal is to find a set of at most κ vertices that dominates every black vertex of the instance. We say that such a set D is a *B-dominating set* (in G). Clearly, solving Dominating Set on (G, κ) is equivalent to solving B-Dominating Set on $(G, V(G), \kappa)$.

From this point on, we assume that G is an undirected path graph, and that $\mathcal{T} = (T, \{P_u\}_{u \in V(G)})$ is a tree model of G where each P_u is a subpath of T (this can be computed in polynomial time [12]). We also denote by $\mathcal{L}$ the set $\mathcal{L}(G, \mathcal{T})$. As in the previous section, we solve this problem by first applying a series of reduction rules, the first of which is analogous to Reduction Rule 1.

Reduction Rule 3. *Let* (G, B, κ) *be an instance of* B-Dominating Set. *If there exists* $v \in \mathcal{L} \setminus B$, *then* delete v, *obtaining the instance* $(G - v, B, \kappa)$. *And if there exists* $v \in \mathcal{L} \cap B$ *such that* v *is an isolated vertex, then* delete v, *obtaining the instance* $(G - v, B - v, \kappa - 1)$.

Proof of safeness. Deleting a vertex clearly does not increase the vertex leafage, so we just need to prove the equivalence between instances. For the first case, clearly a B-dominating set in $G - v$ is also a B-dominating set in G. So let S be a B-dominating set in G. If $v \notin S$, then there is nothing to prove. Otherwise, since v is a leafy vertex, we get that $N[v]$ is a clique, which means that any $b \in B$ dominated by v can be dominated by any $u \in N(v)$ instead. The second part is analogous. □

Now we can assume that every leafy vertex v of G is black and is not isolated. The following rule allows us to bound the number of leaves in T.

Reduction Rule 4. *If* $B = \emptyset$, *then output* Yes. *And if* $B \neq \emptyset$ *and either* $\kappa \leq 0$ *or* T *has more than* 2κ *leaves, then output* No.

Safeness. Follows from the assumption that every leafy vertex v is black and from Lemma 1. □

Thus, we assume that T has at most 2κ leaves. Furthermore, if $|V(T)| = 1$, then G is the complete graph and any vertex dominates B; so from now on we assume that T has at least 2 leaves. Our next operation is not a reduction rule, but a branching rule instead. More specifically, we create a number of smaller instances in order to solve the problem. The amount of instances created is bounded by a function of κ, thanks to the fact that T has at most 2κ leaves.

Given nodes t, t' of T, denote by $P(t, t')$ the t, t'-path in T. Also, given a subpath P of T, denote by V_P the set $\{u \in V(G) \mid P_u \subseteq P\}$. Say that $u \in V_P$ is *P-maximal* if there is no $v \in V_P$ such that P_u is a proper subpath of P_v.

Branching Rule. *Let* $\mathcal{I} = (G, B, \kappa)$ *be an instance of* B-Dominating Set. *Let* $\ell \in V(T)$ *be a leaf of* T, *and* $u \in V(G)$ *be such that* $V(P_u) = \{\ell\}$. *For each leaf* $t \in V(T)$, $t \neq \ell$, *do the following:*

1. *Choose* $v \in V_{P(\ell,t)}$ *to be a* $P(\ell, t)$*-maximal vertex such that* $\ell \in V(P_v)$;
2. *Define* $G' = G - V_{P_v}$ *and* $B' = B \setminus N_G[v]$;
3. *Create the instance* $\mathcal{I}(u, t) = (G', B', \kappa - 1)$.

We remark that $\{u, v\} \subseteq V_{P_v}$ and thus those two vertices are not in G'.

Correctness of the Branching Rule. First, observe that a minimal tree model of G' can again be obtained by applying Proposition 1 to the tree model $\mathcal{T}$ restricted to G'. Therefore, it remains to show that $\mathcal{I}$ is a Yes instance of B-Dominating

Set if and only if there exists a leaf t of T distinct from ℓ such that the instance $\mathcal{I}(u,t)$ is also a Yes instance.

For the necessity, let S be a B-dominating set of G. By our assumption that Reduction Rule 3 is not applicable, we get that $u \in B$, and $N(u) \neq \emptyset$. Note that, since $V(P_u) = \{\ell\}$ we get that $N(u)$ is a clique. This means that if $u \in S$, then $(S \setminus \{u\}) \cup \{v\}$ is also B-dominating, for any $v \in N(u)$. Therefore, we can assume that $u \notin S$. Now, let v be the neighbor of u in S. Also, let t' be the endpoint of P_v distinct from ℓ, and let t be any leaf separated from ℓ by the edge of P_v incident to t' (it might happen that $t = t'$). Then, either v is $P(\ell,t)$-maximal, or there exists $x \in V_\ell$ which is $P(\ell,t)$-maximal. If the latter occurs, we get that $P_v \subseteq P_x$, which in turn gives us that $N[v] \subseteq N[x]$ and that $(S \setminus \{v\}) \cup \{x\}$ is a B-dominating set of G. We can therefore suppose, without loss of generality, that v is $P(\ell,t)$-maximal. Now, let $\mathcal{I}(u,t)$ be the instance of B-Dominating Set constructed as in the statement of the Branching Rule. Observe that if v' is also $P(\ell,t)$-maximal such that $\ell \in V(P_{v'})$, then $P_{v'} = P_v$ and the constructed instance is the same, so we can suppose that indeed v is the iterated $P(\ell,t)$-maximal vertex. It remains to prove that $S' = S \setminus \{v\}$ is a B'-dominating set of $\mathcal{I}(u,t)$. For this, let $b \in B'$. By construction $b \in B \setminus N_G[v]$. Therefore, b has a neighbor in $S \setminus \{v\}$, as we wanted to show.

For the sufficiency, let $\mathcal{I}(u,t) = (G', B', \kappa - 1)$ be the instance given by the Branching Rule, and let S' be a B'-dominating set of G'. Because every $b \in B'$ is dominated by S', and $B \setminus B' = N_G[v]$, we get that $S = S' \cup \{v\}$ is a B-dominating set in G, as we wanted. □

The last part of Theorem 1 follows by bounding the number of instances, since each instance is solved in polynomial time.

Theorem 5. *Let G be an undirected path graph. Then* Dominating Set *can be solved in time $O^*(2^{\mathcal{O}(\kappa \log \kappa)})$.*

Proof. We start by obtaining a tree model with optimal vertex leafage for G by applying the polynomial algorithm in [12]. Then, we iteratively apply Reduction Rules 3 and 4 (also applying Proposition 1 to maintain a minimal tree model), until we reach the need to apply the Branching Rule. The latter is then applied for every leaf of the current tree model, which generates at most $(2\kappa)^2 = 4\kappa^2$ new instances. The process then starts over on each of the generated instances. Finally, since the budget for the size of the solution decreases by 1 after applying the Branching Rule, we get that a new application of the rule would generate at most $(2\kappa - 2)^2$ new instances, and so on. Observe that this cascade can be done at most κ times, since at each application we keep one vertex in the dominating set that is being constructed. Therefore, in the worst case scenario, we get that the total number of generated instances is: $(2\kappa)^2 \cdot (2\kappa - 2)^2 \cdot \cdots \cdot (2\kappa - (2\kappa - 2))^2 = [(2\kappa) \cdot (2\kappa - 2) \cdot \cdots \cdot (2)]^2 = O([(2\kappa)^\kappa]^2) = O(2^{2\kappa(\log \kappa + 1)})$. Observe that if an instance eventually ends up with a non-empty set of black vertices and a budget of 0 (base case of the branching procedure), then Reduction Rule 4 will output No. Because the applications of Reduction Rules 3 and 4 and of Proposition 1 are done in polynomial time, we get the claimed running time. □

5 Conclusion

We have investigated the complexity of Dominating Set, Connected Dominating Set and Steiner Tree when parameterized by the size of the solution plus the vertex leafage ($\kappa + v\ell(G)$) of a given chordal graph G. We have found that they are all FPT, provided that a tree model with optimal vertex leafage of G is given. Since such a tree model can be found in polynomial time if G is an undirected path graph (which are graphs with vertex leafage 2), we get that they are all FPT on these graphs when parameterized by the size of the solution. A question is whether the condition about the provided tree model can be lifted. Because positive instances have leafage bounded by a function of κ and $v\ell(G)$, we know that if computing $v\ell(G)$ is FPT when parameterized by $\ell(G)$, then we would have a complete fixed-parameter algorithm. Another option could be to provide a tree model which is not very far from an optimal one, i.e., that has vertex leafage at most $c \cdot v\ell(G)$ for some constant c. This would increase only the constants in our complexities, and we would again have complete algorithms. We ask whether this is achievable. We recall the reader that deciding $v\ell(G) \leq 3$ is NP-complete, but that the vertex leafage can be computed in time $n^{O(\ell(G))}$ [8].

The inequality $v\ell(G) \leq \ell(G)$ says that the vertex leafage of G is a weaker parameter, i.e., that if a problem is FPT when parameterized by $v\ell(G)$, then it is also FPT when parameterized by $\ell(G)$. However, we have also seen that if some Min-LC-VSP problem is FPT when parameterized by $\ell(G)$, then we get also parameterization by $\kappa + v\ell(G)$. In [11] they provide a fixed-parameter algorithm for Dominating Set when parameterized by $\ell(G)$. A question is whether their result can be generalized to all Min-LC-VSP problems. Given the complexity of the algorithm given in [11], this seems to be a very challenging problem.

Recall the definitions of undirected path, rooted directed path and directed path graphs given in the introduction. It is known that undirected path graphs and rooted directed path graphs are separated by Dominating Set, Steiner Tree, Connected Dominating Set and Graph Isomorphism [1,4,5,10,23], while directed path and rooted directed path graphs are separated by Graph Isomorphism [1]. Therefore we ask whether any of the investigated problems also separates these classes. More generally, is there a problem that separates directed path graphs from undirected path graphs?

Finally, we also leave as open the question of whether Steiner Tree and Dominating Set admit polynomial kernels with relation to the parameter κ when restricted to undirected path graphs.

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