Números extremais

$$ex(n,H) = \max\{e(G) : G \text{ é } H\text{-livre e possui } n \text{ vértices}\}$$

Ex: $ex(n, K_k) = \left(1-\frac{1}{k-1}\right)\frac{n^2}{2}$

$$ex(n, K_{s,t}) = O\left(n^{2-1/s}\right)$$

$$ex(n, C_{2k}) = O\left(n^{1+1/k}\right) \iff \text{Existe } C \text{ t.q. } \underline{ex(n, C_{2k}) \leq C \cdot n^{1+1/k}}$$

$$ex(n, \underset{\substack{\uparrow \\ \text{árvore}}}{T}) = O(n)$$

$n^{1+\frac{1}{k}}$

$C \cdot n^2$

Teorema: Para $k \geq 3$, temos

$$ex(n, C_k) \geq n^{1+\frac{1}{k}}$$

$$ex(n, C_{2k}) \geq \underline{n^{1+1/2k}}$$

para n suficientemente grande

Teorema: Para $k \geq 3$, temos $ex(n, C_k) \geq n^{1+\frac{1}{k}}$ para n suficientemente grande

Prova: Tome $p = \underbrace{8}_{2^3} \cdot n^{-1+1/k}$ e seja X o número de cópias de C_k em $G(n,p)$.

→ Há no máximo n^k cópias de C_k em K_n e cada cópia está em $G(n,p)$ com probabilidade p^k

Logo $\mathbb{E}[X] \leq p^k n^k = 2^{3k} \cdot n^{-k+1} \cdot n^k = 2^{3k} \cdot n$ e

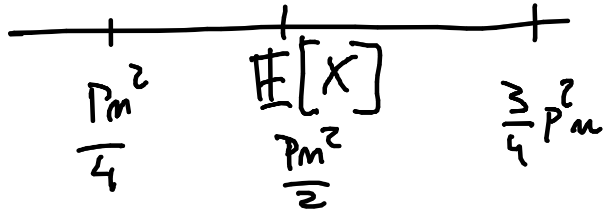
portanto, $\mathbb{P}(X \geq 2^{4k} \cdot n) \leq \frac{2^{3k} \cdot n}{2^{4k} \cdot n} = \frac{1}{2^k}$

Markov

$p\binom{n}{2} \approx p\frac{n^2}{2}$

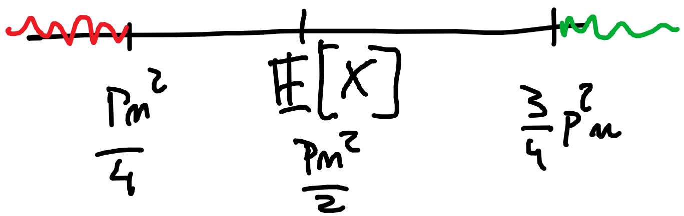
Pela des. de Chernoff, temos

$$\mathbb{P}\left(e(G(n,p)) \geq \frac{pn^2}{4}\right) \leq \frac{1}{2}$$



Proposição (Desigualdade de Chernoff): Se $\varepsilon \in (0,1]$ e X é uma variável aleatória binomial com média μ, então

$$\mathbb{P}(|X-\mu| \geq \varepsilon \cdot \mu) \leq 2 \cdot e^{-\varepsilon^2 \mu / 3}$$



Pela Des. de Chernoff, temos,

$$\mathbb{P}\left(e(G(n,p)) \leq \frac{pn^2}{4}\right) \leq \frac{1}{2}$$

$$\mathbb{P}\left(e(G(n,p)) \leq \frac{pn^2}{4}\right) \leq 2 \cdot e^{-\frac{1}{4} \frac{p\frac{n^2}{2}}{3}} = 2 \cdot e^{-\frac{1}{24} pn^2} \longrightarrow 0$$

$$pn^2 = 8 \cdot n^{-1+1/k} \cdot n^2 = 8 \cdot n^{1+1/k} \longrightarrow \infty$$

$$1 - \frac{1}{2} - \frac{1}{2^k} > 0$$

SEGUE QUE EXISTE GRAFO G' COM n VÉRTICES E PELO MENOS $\frac{pn^2}{4}$ ARESTAS COM NO MÁXIMO $2^{4k} n$ CÓPIAS DE C_k.

COMO n É SUFICIENTEMENTE GRANDE, TEMOS

$$2^{4k} \cdot n \leq n^{1+1/k} = \frac{pn^2}{8}$$

REMOVENDO UMA ARESTA DE CADA CÓPIA DE C_k, OBTEMOS UM GRAFO G C_k-LIVRE T.Q.

$$e(G) \geq \frac{pn^2}{4} - 2^{4k} \cdot n \geq \frac{pn^2}{4} - \frac{pn^2}{8} = \frac{pn^2}{8} = \frac{8 \cdot n^{-1+1/k} \cdot n^2}{8} = n^{1+1/k}$$

$\blacksquare$

ISSO IMPLICA QUE $ex(n, C_k)$ É $\Omega(n)$

<u>COROLÁRIO</u>: $ex(n, H) = O(n)$ SE E SÓ SE H É ACÍCLICOS

CONSTRUÇÕES EXPLÍCITAS PARA C_4, C_6, C_{10}.

CONEXIDADE DE $G(n,p)$

Teorema: Para toda constante $\varepsilon > 0$, temos

$$\mathbb{P}\big(G(n,p) \text{ é conexo}\big) = \begin{cases} 0, & \text{se } p \leq \dfrac{(1-\varepsilon)\log n}{n} \\ 1, & \text{se } p \geq \dfrac{(1+\varepsilon)\log n}{n} \end{cases}$$

quando $n \to \infty$.

TEOREMA: PARA TODA CONSTANTE $\varepsilon > 0$, TEMOS

$$\mathbb{P}(G(n,p) \text{ é CONEXO}) = \begin{cases} 0, & \text{SE } p \leq \frac{(1-\varepsilon)\log n}{n} \\ 1, & \text{SE } p \geq \frac{(1+\varepsilon)\log n}{n} \end{cases}$$

QUANDO $n \to \infty$.

PROVA: DENOTE POR X O NÚMERO DE VTXS ISOLADOS.

DADO $u \in V(G)$, TEMOS $\mathbb{P}(u \text{ é ISOLADO}) = (1-p)^{n-1}$. LOGO

$$\mathbb{E}[X] = n\,(1-p)^{n-1} \geq \underbrace{\frac{n\left(e^{-p}\right)^n}{2}}_{} = \frac{\cancel{n} \cdot n^{\cancel{\varepsilon-1}}}{2} = n^{\varepsilon} \longrightarrow \infty$$

TEMOS QUE $e^{-x} = 1 - x + O(x^2)$

$$1 - x \leq e^{-x}$$

$$e^{-pn} \geq e^{-\frac{(1-\varepsilon)\log n}{\cancel{n}} \cdot \cancel{n}} = n^{-(1-\varepsilon)} = n^{\varepsilon - 1}$$

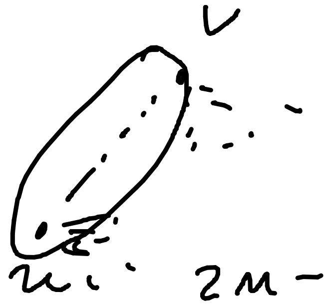
$\mathbb{P}(G(n,p) \text{ é DESCONEXO}) \geq \mathbb{P}(X \geq 1)$

$\{G(n,p) \text{ é DESCONEXO}\} \supseteq \{X \geq 1\} \to 1$

USANDO CHEBYSHEV

$$\mathbb{P}\left(|X-\mathbb{E}[X]| \geq \mathbb{E}[X]\right) \leq \frac{Var(X)}{\mathbb{E}[X]^2}$$

$\mathbb{E}[X]$



$2n-3$

$2(n-1)-1$

$\leq 2n-3$

$$\{X=0\} \subseteq \left\{|X-\mathbb{E}[X]| \geq \mathbb{E}[X]\right\} \qquad X=\sum_u \mathbb{1}_u$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$X^2 = \left(\sum_u \mathbb{1}_u\right)^2 = \underline{\sum_u \overbrace{\mathbb{1}_u \cdot \mathbb{1}_u}^{\mathbb{1}_u}} + \underline{\sum_{u\neq v} \mathbb{1}_u \mathbb{1}_v}$$

$$\mathbb{E}[X^2] = \mathbb{E}[X] + n(n-1)(1-p)^{2n-3}$$

$$\mathbb{E}[X^2] = \mathbb{E}[X] + n(n-1)(1-p)^{2n-3} = \mathbb{E}[X] + \frac{(n-1)}{(1-p)} \frac{n^2}{n} (1-p)^{2n-2}$$

$$= \mathbb{E}[X] + \frac{n-1}{n} \cdot \frac{1}{1-p} \cdot \mathbb{E}[X]^2$$

$$\mathbb{E}[X] = n(1-p)^{n-1} \longrightarrow \infty$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \mathbb{E}[X] + \frac{n-1}{n} \cdot \frac{1}{1-p} \cdot \mathbb{E}[X]^2 - \mathbb{E}[X]^2$$

$$\frac{\mathrm{Var}(X)}{\mathbb{E}[X]^2} = \underbrace{\frac{1}{\mathbb{E}[X]}}_{\to 0} + \overset{1}{\frac{n-1}{n}} \cdot \overset{1}{\frac{1}{1-p}} - 1 \longrightarrow 0$$

$$p \leq (1-\varepsilon) \frac{\log n}{n} \longrightarrow 0$$

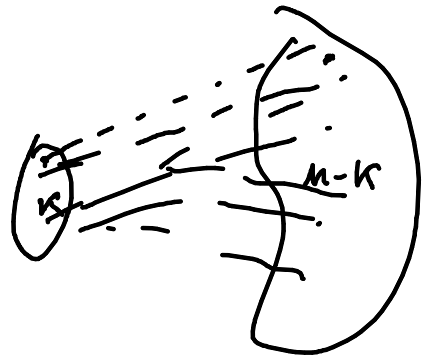
$$\mathbb{P}(X=0) \leqslant \mathbb{P}\left(|X-\mathbb{E}[X]| \geqslant \mathbb{E}[X]\right) \leqslant \frac{\mathrm{Var}(X)}{\mathbb{E}[X]^2} \longrightarrow 0$$

Logo, $\mathbb{P}(X \geqslant 1) = \mathbb{P}(X \neq 0) \longrightarrow 1.$

SUPONHA AGORA QUE $p \geq (1+\varepsilon)\frac{\log n}{n}$

PARA CADA $k \in \mathbb{N}$, DENOTE POR Y_k O NÚMERO DE COMP. DE $G(n,p)$ COM PRECISAMENTE k VTXS.

VAMOS MOSTRAR QUE $Y_k = 0$ PARA TODO $1 \leq k \leq \frac{n}{2}$, E PORTANTO $G(n,p)$ É CONEXO.



NOTE QUE $\mathbb{E}[Y_k] \leq \binom{n}{k}(1-p)^{k(n-k)}$

$1-x \leq e^{-x}$

$$\leq \left(\frac{en}{k}\right)^k \cdot e^{-pk(n-k)}$$

$$= \left(\frac{en}{k} \cdot e^{-p(n-k)}\right)^k$$

$-p \leq -(1+\varepsilon)\frac{\log n}{n}$

$$\leq \left(\frac{en}{k} \cdot e^{-(1+\varepsilon)\frac{\log n}{n}(n-k)}\right)^k$$

$$\mathbb{E}[Y_k] \leq \binom{n}{k} (1-p)^{k(n-k)}$$

$$\leq \left(\frac{en}{k}\right)^k \cdot e^{-pk(n-k)}$$

$$= \left(\frac{en}{k} \cdot e^{-p(n-k)}\right)^k$$

$$\leq \left(\frac{en}{k} \cdot e^{-(1+\varepsilon)\frac{\log n}{n}(n-k)}\right)^k$$

$$\leq \left(\frac{en}{k} \cdot e^{-(1+\varepsilon)\frac{\log n}{\cancel{n}}\cancel{\frac{n}{2}}}\right)^k \qquad k \leq \frac{n}{2}$$

$$= \left(\frac{en}{k} \cdot n^{-(1+\varepsilon)\frac{1}{2}}\right)^k$$

$$\frac{en}{k} \cdot e^{-p(n-k)}$$

$$= \frac{en}{k} \cdot e^{-pn} \cdot e^{pk}$$

$$= \frac{e^{pk+1}}{k} n \cdot e^{-pn}$$

$$\leq \frac{e^{pk+1}}{k} n \, e^{-(1+\varepsilon)\frac{\log n}{\cancel{n}} \cdot \cancel{n}}$$

$$= \frac{e^{pk+1}}{k} \cancel{n} \cdot n^{\cancel{-1}-\varepsilon}$$

$$= \frac{e^{pk+1}}{k} \cdot n^{-\varepsilon}$$

$$= \frac{e^{pk+1}}{k} \cdot n^{-\varepsilon} \leq n^{-\frac{\varepsilon}{2}} \qquad k \leq \frac{n}{2}$$

$$\mathbb{P}\left(G(n,p) \in \text{CONEXO} \right) \leq \sum_{k=1}^{n/2} \mathbb{E}[Y_k] \leq \sum_{k=1}^{n/2} n^{-\frac{\varepsilon k}{2}}$$

$$S = \sum_{k=1}^{n/2} \left(n^{-\varepsilon/2}\right)^{k}$$

$$\left(n^{-\varepsilon/2}\right) \cdot S = \sum_{k=1}^{n/2} \left(n^{-\varepsilon/2}\right)^{k+1}$$

$$\left(n^{-\varepsilon/2} - 1\right) S = \left(n^{-\varepsilon/2}\right)^{\frac{n}{2}+1} - n^{-\frac{\varepsilon}{2}}$$

$$S = \frac{n^{-\frac{\varepsilon}{2}\left(\frac{n}{2}+1\right)} - n^{-\frac{\varepsilon}{2}}}{n^{-\varepsilon/2} - 1} \to 0$$