COEFICIENTES BINOMIAIS

$[r] = \{1, \dots, r\}$

$\binom{r}{k}$ = NÚMERO DE SUBCONJUNTOS DE TAMANHO k EM $[r]$

- r → ÍNDICE SUPERIOR
- k → ÍNDICE INFERIOR.

$r!$ = NÚMERO DE ORDENAÇÕES DE $[r]$

$$r^{\underline{k}} = \frac{r!}{(r-k)!} = r(r-1)(r-2)\cdots(r-k+1)$$

= NÚMERO DE SEQS COM k ELTS EM $[r]$

É ÚTIL QUE $\binom{r}{k}$ SEJA DEFINIDO P/ r REAL

$$\binom{r}{k} = \begin{cases} \dfrac{r^{\underline{k}}}{k!} & \text{SE } k \geq 0 \text{ INTEIRO} \\ 0 & \text{SE } k < 0 \end{cases}$$

→ É UM POLINÔMIO DE GRAU k EM r

$a_k r^k + a_{k-1} r^{k-1} + \dots + a_0$

Triângulo de Pascal

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

OBS: $\binom{r}{0} = 1$ e $\binom{r}{1} = r$ $\binom{r}{2} = \frac{r(r-1)}{2}$

SIMETRIA

→ AS LINHAS SÃO SIMÉTRICAS

$$\binom{r}{k} = \binom{r}{r-k}$$

→ EXCLUIR K ÍTENS

→ ESCOLHER K ÍTENS

→ VALE QUANDO r É INTEIRO

$$\binom{r}{k} = \binom{r}{r-k}$$

ABSORÇÃO

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1} \quad \text{SE } k \neq 0$$

$$\| $$

$$\frac{r^{\underline{k}}}{k!} = \frac{r}{k} \cdot \frac{(r-1)^{\underline{k-1}}}{(k-1)!} = \frac{r}{k}\binom{r-1}{k-1}$$

$$k\binom{r}{k} = r\binom{r-1}{k-1}$$

ALTERNATIVA: $k\binom{r}{k} = r\binom{r-1}{k-1}$

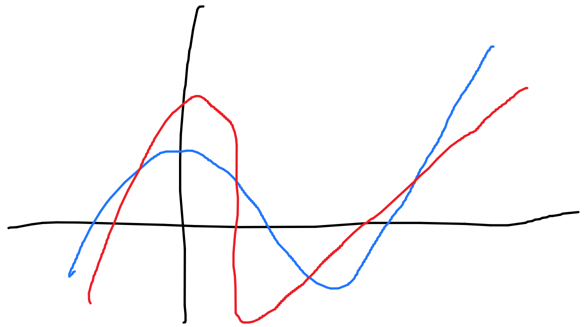
PROP: $(r-k)\binom{r}{k} = r\binom{r-1}{k}$ PARA r INTEIRO.

SIM: $\binom{r}{k} = \binom{r}{r-k}$

ABS: $k\binom{r}{k} = r\binom{r-1}{k-1}$

PROVA: $$(r-k)\binom{r}{k} \underset{\text{SIMETRIA}}{=} (r-k)\binom{r}{r-k} \underset{\text{ABSORÇÃO}}{=} r\binom{r-1}{r-1-k} \underset{\text{SIMETRIA}}{=} r\binom{r-1}{k}$$

OBS: VALE PARA TODO NÚMERO REAL



ARGUMENTO POLINOMIAL:

NOTE QUE $(r-k)\binom{r}{k}$ E $r\binom{r-1}{k}$ SÃO DOIS POLINÔMIOS DE GRAU $k+1$. LOGO $$P(r) = \underbrace{(r-k)\binom{r}{k}}_{P_1(r)} - \underbrace{r\binom{r-1}{k}}_{P_2(r)}$$ É UM POLINÔMIO DE GRAU $k+1$

$\rightarrow$ PORTANTO $P(r)$ POSSUI NO MÁXIMO $k+1$ RAÍZES OU $P(r) = 0$

MAS NOTE QUE TODO INTEIRO É RAIZ DE $P(r)$. LOGO, $P(r) = 0$

Fórmula da Adição

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1} \qquad k \text{ inteiro}$$

→ com o ovo

→ Argumento do ovo podre

→ sem o ovo

OBS:

$$\binom{r}{k} = \binom{r-1}{k} + \underline{\binom{r-1}{k-1}}$$

→ coef. de menor índice inf

$$= \binom{r-1}{k} + \binom{r-2}{k-1} + \underbrace{\binom{r-2}{k-2}}$$

$$= \binom{r-1}{k} + \binom{r-2}{k-1} + \binom{r-3}{k-2} + \binom{r-3}{k-3}$$

$$\therefore$$

$$\binom{r}{k} = \sum_{i=0}^{k} \binom{r-1-i}{k-i} \qquad \text{DIR} \to \text{ESQ}$$

$$= \sum_{i=0}^{k} \binom{r-1-k+i}{i} \qquad \text{ESQ} - \text{DIR}$$

$$r-1-k = A$$

$$\binom{A+k+1}{k} = \sum_{i=0}^{k} \binom{A+i}{i}$$

$$\binom{r+1+k}{k} = \sum_{i=0}^{k} \binom{r+i}{i}$$

→ $$\sum_{k \leq m} \binom{r+k}{k} = \binom{r+m+1}{m} \qquad m \text{ inteiro}$$

Se expandirmos o coef de maior índice inf.

→ $$\sum_{0 \leq k \leq m} \binom{k}{m} = \binom{m+1}{m+1}$$

COEFICIENTE BINOMIAL RECEBE ESTE NOME POR CAUSA

DO TEO BINOMIAL

$$(x+y)^0 = 1x^0y^0$$

$$(x+y)^1 = 1x^1y^0 + 1x^0y^1$$

$$(x+y)^2 = 1x^2y^0 + 2x^1y^1 + 1x^0y^2$$

$$(x+y)^m = \underbrace{(x+y) \quad \cdots \quad (x+y)}_{m \text{ VEZES}}$$

$$x^a y^b \qquad a+b=m$$

O COEFICIENTE DE $x^J y^{m-J}$ É O NÚMERO DE FORMAS

DE ESCOLHER J FATORES $= \binom{m}{J}$.

TEOREMA BINOMIAL: $(x+y)^r = \sum_k \binom{r}{k} x^k y^{r-k}$ r INTEIRO $r \geq 0$

COROLÁRIO: $2^r = \sum_k \binom{r}{k}$ TOMANDO $x = y = 1$ (REGRA DA LINHA)

$0^r = \sum_k \binom{r}{k} (-1)^k$ TOMANDO $x = -1$, $y = 1$

PODEMOS ESCREVER

$$\binom{r}{k} = \frac{r!}{\underbrace{k!}_{a}\underbrace{(r-k)!}_{b}} \quad \text{COMO} \quad \frac{(a+b)!}{a!\,b!} \quad \text{ONDE} \quad a=k \;\text{ E }\; b=r-k$$

SIMETRIA

$$\binom{n=a+b+c}{a} = \binom{n=a+b+c}{b+c}$$

$$\binom{n}{k} = \binom{n}{\underbrace{k,\, n-k}}$$

ANALOGAMENTE, PODEMOS ESCREVER

$$\frac{(a+b+c)!}{a!\,b!\,c!} = \frac{r!}{\underbrace{k!}_{a}\underbrace{(m-k)!}_{b}\underbrace{(r-m)!}_{c}}$$

$$\binom{n}{a}\binom{n-a}{b}$$

$$\parallel$$

TEOREMA TRINOMIAL

$$(x+y+z)^n = \sum_{\substack{0\leq a,b,c\leq n \\ a+b+c=n}} \frac{(a+b+c)!}{a!\,b!\,c!}\, x^a y^b z^c = \sum_{\substack{0\leq a,b,c\leq n \\ a+b+c=n}} \binom{a+b+c}{b+c}\binom{b+c}{c} x^a y^b z^c$$

$$\parallel$$

$$\frac{(a+b+c)!}{\cancel{(b+c)!}\,a!} \cdot \frac{\cancel{(b+c)!}}{b!\,c!}$$

TRINÔMIO:

$$\binom{a+b+c}{a,b,c} = \frac{(a+b+c)!}{a!\,b!\,c!}$$

GENERALIZANDO:

$$\binom{a_1+a_2+\cdots+a_m}{a_1,a_2,\ldots,a_m} = \frac{(a_1+\cdots+a_m)!}{a_1!\cdots a_m!}$$

CONVOLUÇÃO DE VANDERMONDE

$t = m + n$

$$\binom{r+s}{m+n} = \sum_{k} \binom{r}{m+k}\binom{s}{n-k} = \sum_{a+b=t} \binom{r}{a}\binom{s}{b}$$

DE QUANTAS FORMAS EU POSSO ESCOLHER $m+n$ PESSOAS EM UM UNIVERSO DE r HOMENS E s MULHERES?

$$\binom{r+s}{t} = \sum_{a=0}^{t} \binom{r}{a}\binom{s}{t-a}$$

TRUQUE

- $$r^{\underline{k}}\left(r-\tfrac{1}{2}\right)^{\underline{k}} = r\cdot\left(r-\tfrac{1}{2}\right)\cdot(r-1)\cdot\left(r-\tfrac{3}{2}\right)\cdot(r-2)\cdot\ \cdots\ (r-k+1)\left(r-k+1-\tfrac{1}{2}\right)$$

$$= \frac{1}{2^{2k}}\,(2r)(2r-1)(2r-2)(2r-3)(2r-4)\ \cdots\ (2r-2k+2)(2r-2k+1)$$

$$= \frac{1}{2^{2k}}\cdot(2r)^{\underline{2k}}$$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}\ \Big\}$$

$$\boxed{n^{\underline{k}} = \binom{n}{k}k!}$$

DIVIDINDO POR $(k!)^2$

$$\underbrace{\frac{r^{\underline{k}}}{k!}\,\frac{\left(r-\frac{1}{2}\right)^{\underline{k}}}{k!}} = \frac{(2r)^{\underline{2k}}}{2^{2k}(k!)^2} = \frac{1}{2^{2k}}\,\frac{(2r)^{\underline{2k}}}{(2k)!}\,\frac{(2k)!}{(k!)^2} = \frac{1}{2^{2k}}\cdot\binom{2r}{2k}\binom{2k}{k}$$

$$\| $$

$$\binom{r}{k}\binom{r-\frac{1}{2}}{k}$$

$$\binom{r}{k}\binom{r-\frac{1}{2}}{k} = \frac{1}{2^{2k}}\binom{2r}{2k}\binom{2k}{k}$$

$$\binom{r}{k}\binom{r-\frac{1}{2}}{k} = \frac{1}{2^{2k}}\binom{2r}{2k}\binom{2k}{k}$$

$$\binom{n-\frac{1}{2}}{n} = \frac{\left(n-\frac{1}{2}\right)^{\underline{n}}}{n!} = (-1)^{n}\,\frac{\left(-\frac{1}{2}\right)^{\underline{n}}}{n!}$$

$$= (-1)^{n}\binom{-\frac{1}{2}}{n}$$

TOMANDO $\quad k = r = n$

$$\cancelto{1}{\binom{n}{n}}\binom{n-\frac{1}{2}}{n} = \frac{1}{2^{2n}}\cdot\cancelto{1}{\binom{2n}{2n}}\binom{2n}{n}$$

LOGO, $$\binom{n-\frac{1}{2}}{n} = \frac{1}{2^{2n}}\binom{2n}{n}$$

OBS: $$\left(n-\frac{1}{2}\right)^{\underline{n}} = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\cdots\left(\frac{1}{2}\right) = \frac{1}{2^{n}}(2n-1)(2n-3)\cdots 1$$

$$= \frac{(-1)^{n}}{2^{n}}(-2n+1)(-2n+3)\cdots(-3)\,(-1)$$

$$= (-1)^{n}\left(-n+\frac{1}{2}\right)\left(-n+\frac{3}{2}\right)\cdots\underset{-1}{\left(-\frac{3}{2}\right)}\underset{-1}{\left(-\frac{1}{2}\right)}$$

$$= (-1)^{n}\left(-\frac{1}{2}\right)^{\underline{n}}$$

EX: $\binom{4-\frac{1}{2}}{4} = \dfrac{(4-\frac{1}{2})(3-\frac{1}{2})(2-\frac{1}{2})(\frac{1}{2})}{4!} = \dfrac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}}{4!} = \binom{-1/2}{4}$

a) $\binom{n-\frac{1}{2}}{n} = \dfrac{1}{2^{2n}}\binom{2n}{n}$

b) $\binom{n-\frac{1}{2}}{n} = (-1)^n \binom{-1/2}{n}$

$\dfrac{1}{2^{2n}}\binom{2n}{n} = (-1)^n\binom{-1/2}{n}$

$\cancel{(-1)^{2n}} \to 1$

$\binom{-1/2}{n} = \dfrac{(-1)^n}{2^{2n}}\binom{2n}{n}$

$= \left(-\frac{1}{4}\right)^n\binom{2n}{m}$

$2^{2n} = 4^n$

$$\text{Ex: } \binom{4-\frac{1}{2}}{4} = \frac{\left(4-\frac{1}{2}\right)\left(3-\frac{1}{2}\right)\left(2-\frac{1}{2}\right)\left(\frac{1}{2}\right)}{4!} = \frac{\left(-\frac{7}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}{4!} = \left(-\frac{1}{2}\right)^4 \cdot \frac{1\cdot 3\cdot 5\cdot 7}{4!}$$

$$= \left(-\frac{1}{2}\right)^4 \frac{1\cdot 3\cdot 5\cdot 7 \quad \frac{2}{2}\cdot\frac{4}{2}\cdot\frac{6}{2}\cdot\frac{8}{2}}{1\cdot 2\cdot 3\cdot 4 \quad \cdot 1\cdot 2\cdot 3\cdot 4} = \underbrace{\left(-\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^4} \cdot \frac{8!}{4!\cdot 4!}$$

$$= \left(-\frac{1}{4}\right)^4 \cdot \binom{8}{4}$$

LEMBRANDO : $\Delta f(x) = f(x+1) - f(x)$

APLICANDO NOVAMENTE

$$\Delta^2 f(x) = \Delta\left(f(x+1) - f(x)\right)$$

$$= f(x+2) - f(x+1) - f(x+1) + f(x)$$

$$= f(x+2) - 2f(x+1) + f(x)$$

CONTINUANDO :

$$\Delta^3 f(x) = f(x+3) - 2f(x+2) + f(x+1) - \left(f(x+2) - 2f(x+1) + f(x)\right)$$

$$= f(x+3) - 3f(x+2) + 3f(x+1) - f(x)$$

$$\Delta^3 f(x) = f(x+3) - 3f(x+2) + 3f(x+1) - f(x)$$

$$\Delta^4 f(x) = f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x)$$

O CASO GERAL FICA

$$\Delta^n f(x) = \sum_k \binom{n}{k} (-1)^{n-k} f(x+k) = \sum_k \binom{n}{k} (-1)^{n-k} E^k f(x)$$

$$\Delta^n = (E-1)^n = \sum_k \binom{n}{k} (-1)^{n-k} E^k$$

LEMBRE-SE: $Ef(x) = f(x+1)$.

PODEMOS PENSAR $\Delta f(x) = f(x+1) - f(x) = Ef(x) - f(x)$

$$\Delta = E - 1$$

FUNÇÕES GERADORAS

- GOSTARÍAMOS DE REPRESENTAR A SÉRIE INFINITA $(a_0, a_1, \dots)$ POR UMA SÉRIE DE POTÊNCIAS.

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{k \geq 0} a_i z^i$$

↳ FUNÇÃO GERADORA P/ $(a_0, a_1, \dots)$

- SE $A(z)$ É UMA SÉRIE DE POTÊNCIAS, USAMOS $[z^n] A(z) = a_n$

- SE $A(z)$ E $B(z)$ SÃO FUNÇÕES GERADORAS PARA $(a_0, a_1, \dots)$ E $(b_0, b_1, \dots)$

CONSIDERE $A(z) \cdot B(z) = (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots)$

O COEF. DE z^n EM $A(z) \cdot B(z)$ É $a_0 b_n + a_1 b_{n-1} + a_2 \cdot b_{n-2} + \cdots + a_n b_0$

$$= \sum_{k=0}^{n} a_k b_{n-k}$$

↳ CONVOLUÇÃO DAS SEQUÊNCIAS

EX: A FUNÇÃO GERADORA DE $(1, 1, 1, \ldots)$ É

$$f(z) = 1 + z + z^2 + z^3 + \cdots = 1 + z f(z)$$

LOGO, TEMOS $f(z) = \frac{1}{1-z}$

EX: QUEM É A SEQUÊNCIA GERADA POR $\left(\frac{1}{1-z}\right)^2$?

VIMOS QUE $[z^n]\, A(z) \cdot B(z) = \sum_{k=0}^{n} a_k b_{k-1}$

LOGO $[z^n] \left(\frac{1}{1-z}\right)^2 = [z^n] \left(\frac{1}{1-z}\right)\left(\frac{1}{1-z}\right) = \sum_{k=0}^{n} 1 \cdot 1 = n+1$

E PORTANTO, TEMOS $\left(\frac{1}{1-z}\right)^2 = 1 + 2z + 3z^2 + 4z^3 + \cdots$

EX: A SEQUÊNCIA GERADA POR

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} 1 \cdot z^k = \binom{n}{0} \cdot z^0 + \binom{n}{1} z^1 + \binom{n}{2} z^2 + \dots + \binom{n}{n} z^n + 0 + \cdots$$

$$\binom{n}{r}$$

É $\left(\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}, 0, 0, \dots \right)$

$(1+z)^r$ GERA $\left(\binom{r}{0}, \binom{r}{1}, \dots \right)$

$(1+z)^s$ GERA $\left(\binom{s}{0}, \binom{s}{1}, \dots \right)$

$$\sum_{k=0}^{r+s} \binom{r+s}{k} z^k = (1+z)^{r+s} = \underline{(1+z)^r}\,\underline{(1+z)^s} = \sum_{k}^{r+s} \underbrace{\left(\sum_{t=0}^{k} \binom{r}{t} \binom{s}{k-t} \right)}_{\text{CONVOLUÇÃO DE VANDERMONDE}} z^k$$

CONCLUÍMOS QUE

$$\binom{r+s}{k} = [z^k]\,(1+z)^{r+s} = \sum_{t=0}^{k} \binom{r}{t} \binom{s}{k-t}$$

EX: A SEQ. GERADA POR

$$(1-z)^n = \sum^{n} \binom{n}{k} (-1)^{n-k} z^k \quad \text{é} \quad \left(\binom{n}{0}, -\binom{n}{1}, \binom{n}{2}, -\binom{n}{3}, \dots, 0, \dots \right.$$