SOMAS E RECORRÊNCIA

$$S_n = \sum_{i=1}^{n} a_i$$

EQUIVALENTE À RECORRÊNCIA

$$S_0 = a_0$$

$$S_n = S_{n-1} + a_n \qquad n > 0$$

EX: $a_n = \beta + \gamma n$: $\sum \beta + \gamma n$

$R_n = R_{n-1} + \beta + \gamma$

$$\left.\begin{aligned} R_0 &= \alpha \\ R_n &= R_{n-1} + \beta + \gamma n \end{aligned}\right\}$$

$0^2 = R_0 = \alpha$

$\cancel{n^2} = R_n = (n-1)^2 + \beta + \gamma n$

$\cancel{n^2} - 2n - 1$

$\gamma n = 2n$ $\gamma = 2$

$\beta = 1$

$$R_n = A(n)\,\alpha + B(n)\,\beta + C(n)\,\gamma$$

$$\left\{\begin{array}{llll} \text{TOMANDO} & R_n = 1, & \text{TEMOS} & \alpha = 1,\ \beta = 0,\ \gamma = 0 \Rightarrow A(n) = 1 \\ \text{TOMANDO} & R_n = n, & \text{TEMOS} & \alpha = 0\ \ \beta = 1\ \ \gamma = 0 \Rightarrow B(n) = n \\ \text{TOMANDO} & R_n = n^2, & \text{TEMOS} & \alpha = 0\ \ \beta = 1\ \ \gamma = 2 \Rightarrow C(n) = \dfrac{n^2 - n}{2} \end{array}\right.$$

$$R_n = \alpha + \beta n + \gamma \frac{n^2 - n}{2}$$

ASSIM, PODEMOS RESOLVER $\sum_{i=0}^{n} (\beta + \gamma n)$, USANDO A RECORRÊNCIA

$$R'_0 = \beta$$

$$R'_n = R'_{n-1} + \gamma n$$

QUE É

$$R_n = \alpha + \beta n + \gamma \frac{n^2 - n}{2} \qquad \text{COM} \qquad \alpha = \beta$$

PODEMOS RESOLVER RECORRÊNCIAS USANDO SOMAS

EX: <u>TORRE DE HANÓI</u>:

$$\begin{cases} T_0 = 0 \\ T_n = 2T_{n-1} + 1 & n > 0 \end{cases}$$

SEJA $T'_n = T_n / 2^n$

$$T'_0 = T_0 / 2^0 = T_0 = 0$$

$$T'_n = T_n / 2^n = \frac{2T_{n-1} + 1}{2^n} = \frac{T_{n-1}}{2^{n-1}} + \frac{1}{2^n} = T'_{n-1} + \frac{1}{2^n}$$

$$\Rightarrow T'_n = \sum_{i=0}^{n} \frac{1}{2^i} = 1 - \left(\frac{1}{2}\right)^n$$

$\llcorner$ VAMOS APRENDER NO FUTURO

$$\frac{T_n}{2^n} = T'_n = 1 - \left(\frac{1}{2}\right)^n$$

$$T_n = 2^n - 1$$

DIVIDIR POR 2^n FOI UMA SACADA GENIAL

ESSA IDEIA PODE SER GENERALIZADA PARA QUALQUER RECORRÊNCIA DO TIPO

$$a_n T_n = b_n T_{n-1} + c_n$$

- MULTIPLICAR OS DOIS LADOS POR UM FATOR SOMANTE s_n

$$\underline{s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n}$$

SE s_n É TAL QUE $s_n b_n = s_{n-1} \cdot a_{n-1}$

ENTÃO, DEFININDO $T'_n = s_n a_n T_n$, TEMOS

$$T'_n = s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n$$

$$= \underline{s_{n-1} a_{n-1} T_{n-1}} + s_n c_n$$

$$= T'_{n-1} + s_n c_n$$

$$T'_n = T'_{n-1} + s_n c_n$$

$$s_n b_n = s_{n-1} \cdot a_{n-1}$$

$$s_n = \frac{s_{n-1} \cdot a_{n-1}}{b_n}$$

Logo, $T'_n = s_0 a_0 T_0 + \sum_{i=1}^{n} s_i c_i$

E a original é

$$T_n = \frac{1}{s_n a_n} \left(s_0 a_0 T_0 + \sum_{i=1}^{n} s_i c_i \right)$$

Como calcular o s_n?

$$s_n = \frac{s_{n-1} \cdot a_{n-1}}{b_n} = \frac{a_{n-1}}{b_n} \cdot \frac{a_{n-2}}{b_{n-1}} \cdot \ldots \cdot \frac{a_1}{b_2}$$

Funciona sempre que $a_i \neq 0$ e $b_i \neq 0$.

EX: QUICKSORT

$$C_0 = 0$$

$$C_n = n + 1 + \frac{2}{n} \sum_{k=0}^{n-1} C_k \qquad \text{PARA } n > 0$$

MUITO COMPLICADO!

MULTIPLICANDO OS DOIS LADOS POR n

$$n C_n = n^2 + 1 + 2 \sum_{k=0}^{n-1} C_k \qquad (1)$$

PARA $n-1$ TEMOS

$$(n-1) C_{n-1} = (n-1)^2 + 1 + 2 \sum_{k=0}^{n-2} C_k \qquad (2)$$

$(1) - (2)$

$n C_n - (n-1) C_{n-1} =$

$$n C_n = \cancel{n^2} + \cancel{n} + \cancel{2 \sum_{k=0}^{n-1} C_k} \qquad (1)$$

PARA $n-1$ TEMOS

$$(n-1) C_{n-1} = (n-1)^2 + \cancel{n-1} + \cancel{2 \sum_{k=0}^{n-2} C_k} \qquad (2)$$

$$\cancel{n^2} - 2n + \cancel{1}$$

$(1)-(2)$

$$(1)-(2):\ n C_n - (n-1) C_{n-1} = 2n - \quad + 2 C_{n-1}$$

$$\underbrace{n}_{a_n} C_n = \underbrace{(n+1)}_{b_n} C_{n-1} + \underbrace{2n}_{c_n}$$

$$S_n = \frac{S_{n-1} \cdot a_{n-1}}{b_n} = \frac{a_{n-1}}{b_n} \cdot \frac{a_{n-2}}{b_{n-1}} \cdot \ldots \cdot \frac{a_1}{b_2}$$

FATOR SOMANTE

$$S_n = \frac{\cancel{(n-1)} \ \cancel{n-1} \ 2 \cdot 1}{(n+1) \ \cancel{\cdots} \ \cancel{3}} = \frac{2}{n(n+1)} = 1 \Big/ \binom{n+1}{2}$$

$$n\,C_n = (n+1)\,C_{n-1} + 2n$$

a_n $\quad$ b_n $\quad$ c_n

$$T_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{i=1}^{n} s_i c_i\right)$$

$$s_n = \frac{2}{n(n+1)}$$

$$s_n b_n = s_{n-1} \cdot a_{n-1}$$

$$s_1 b_1 = s_0 a_0$$

$$C_n = \frac{(n+1)\cancel{n}}{\cancel{2}\;\cancel{n}}\left(\sum_{i=1}^{n} \frac{\cancel{2}}{\cancel{i}(i+1)}\, 2\cancel{i}\right)$$

$$= (n+1)\sum_{i=1}^{n} \frac{2}{i+1} = 2(n+1)\cdot\sum_{i=1}^{n} \frac{1}{i+1}$$

$$\sum_{i=2}^{n+1} \frac{1}{i} = H_n - 1 + \frac{\ }{n+1}$$

$$= 2(n+1)\left(H_n - 1 + \frac{1}{n+1}\right)$$ MISTÉRIO

$$= 2(n+1)H_n - 2n$$

NÚMERO HARMÔNICO $\quad H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{i=1}^{n} \frac{1}{k}$

MANIPULAÇÃO DE SOMAS

$$c(a_1 + a_2 + \cdots + a_K) = c a_1 + \cdots + c \cdot a_K$$

- $$\sum_{k \in K} c \cdot a_k = c \sum_{k \in K} a_k$$

DISTRIBUTIVIDADE

- $$\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k$$

ASSOCIATIVIDADE

$$((a_1 + b_1) + (a_2 + b_2) \cdots)$$

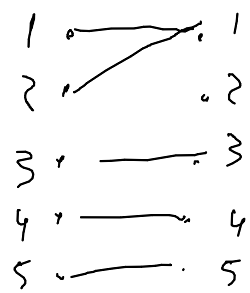
$$= (a_1 + \cdots + a_m) + (b_1 + \cdots + b_m)$$

- $$\sum_{k \in K} a_k = \sum_{k \in K} a_{P(k)}$$

EM QUE $P(K)$ É UMA PERMUTAÇÃO DE K

COMUTATIVIDADE

$$a + b = b + a$$



EX: $S = \sum_{K=0}^{n} (a + bK) \quad K = 0 \dots n$

$i = 0 \dots n$

$K = n \dots 0$

Pela COMUTATIVIDADE $\quad S = \sum_{i=n-K=0}^{n} (a + b \cdot i) = \sum_{i=n-K=0}^{n} (a + b(n-K))$

Pela ASSOCIATIVIDADE $\quad S + S = \sum_{K=0}^{n} (a + bK) + \sum_{i=n-K=0}^{n} (a + b(n-K))$

$$= \sum_{K=0}^{n} (2a + bK + b(n-K)) = \sum_{K=0}^{n} (2a + bn)$$

DISTRIBUTIVIDADE

$$= (2a + bn) \sum_{K=0}^{n} 1 = (2a + bn)(n+1)$$

- OUTRA REGRA

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A| + |B| = |A \cup B| + |A \cap B|$

$$\sum_{k \in A} a_k + \sum_{k \in B} a_k = \sum_{k \in A \cup B} a_k + \sum_{k \in A \cap B} a_k$$

EX: $$\sum_{i=0}^{n} a_k = a_0 + \sum_{k=1}^{n} a_k$$

$A = \{0\}$ $\quad B = \{1, \ldots, n\}$

EX: $$\sum_{k=0}^{m} a_k + \sum_{k=m}^{n} a_k = a_m \sum_{k=0}^{n} a_k$$

$A = \{0, \ldots, m\}$

$B = \{m, \ldots, n\}$

MÉTODO DA PERTURBAÇÃO

NOMEAR PARA CONQUISTAR

$$S_n = \sum_{k=0}^{n} a_k$$

$$S_n + a_{n+1} = \sum_{k=0}^{n+1} a_k = a_0 + \sum_{k=1}^{n+1} a_k = a_0 + \sum_{k=0}^{n} a_{k+1}$$

EX: PROG. GEOMÉTRICA

$$S_n = \sum_{0 \leq k \leq n} \underbrace{a x^k}_{a_k = a x^k}$$

PERTURBAÇÃO

$$S_n + a x^{n+1} = a + \sum_{k=0}^{n} a x^{k+1} = a + x \overbrace{\sum_{k=0}^{n} a x^k}^{S_n}$$

$$S_n + a x^{n+1} = a + x S_n \Rightarrow S_n = \frac{a x^{n+1} - a}{x - 1}$$

$$(x_1 + \dots + x_{2m})^2 = x_1^2 + \dots + x_{2m}^2 + 2\sum_{j \neq i} x_i x_j$$

$$x_1 \cdots x_{2m} \leq$$

$P(n)$

$$\underbrace{(x_1 \cdots x_m)}_{a_1} \underbrace{(x_{m+1} \cdots x_{2m})}_{a_2} \leq \left(\frac{x_1 \cdot x_2 + x_3 \cdot x_4 + \dots + x_{2m-1} x_{2m}}{m} \right)^m$$

$P(2)$

$$\left(\frac{(x_1 + x_2)^2 + \dots + (x_{2m-1} + x_{2m})^2}{2m} \right)^m$$

$=$

$$a_1 \cdot a_2 \leq \left(\frac{a_1 + a_2}{2} \right)^2$$