$$\square_n = \sum_{j=0}^{n} j^2 \qquad \square_n = \frac{n(n+1)(2n+1)}{6} \qquad \left(\text{PROVE POR INDUÇÃO}\right)$$

$$\square_n + (n+1)^2 = \sum_{j=0}^{n} (j+1)^2 = \sum_{j=0}^{n} \left(j^2 + 2j + 1\right)$$

$$= \sum_{j=0}^{n} j^2 + 2\sum_{j=0}^{n} j + \sum_{j=0}^{n} 1$$

$$= \square_n + 2\sum_{j=0}^{n} j + (n+1)$$

$$2\sum_{j=0}^{n} j = (n+1)^2 - (n+1) = (n+1)\cdot n$$

$$\sum_{j=0}^{n} j = \frac{n(n+1)}{2}$$

$$\boxplus_n = \sum_{J=0}^{n} J^3$$

A

$$\boxplus_n + (n+1)^3 = \sum_{J=0}^{n} (J+1)^3 = \sum_{J=0}^{n} \left(J^3 + 3J^2 + 3J + 1 \right)$$

$$= \sum_{J=0}^{n} J^3 + 3\sum_{J=0}^{n} J^2 + 3\sum_{J=0}^{n} J + \sum_{J=0}^{n} 1$$

$$= \boxplus_n + 3\,\square_n + 3\,\frac{n(n+1)}{2} + (n+1)$$

$$3\,\square_n = (n+1)^3 - 3\,\frac{n(n+1)}{2} - (n+1) = \frac{n(n+1)(2n+1)}{2}$$

$$\square_n^k = \sum_{j=0}^{n} j^k \qquad\qquad (j+1)^k = \sum_{i=0}^{k}\binom{k}{i} j^i$$

$$\square_n^k + (n+1)^k = \sum_{j=0}^{n}(j+1)^k = \sum_{j=0}^{n}\sum_{i=0}^{k}\binom{k}{i} j^i$$

$$= \sum_{i=0}^{k}\binom{k}{i}\cdot\sum_{j=0}^{n} j^i$$

$$= \underbrace{\sum_{j=0}^{n} j^k}_{\square_n^k} + \sum_{i=0}^{k-1}\binom{k}{i}\square_n^i$$

$$(n+1)^k = \sum_{i=0}^{k-1}\binom{k}{i}\square_n^i \rightsquigarrow k\,\square_n^{k-1} = (n+1)^k - \sum_{i=0}^{k-2}\binom{k}{i}\square_n^i$$

$$(k-1)\,\square_n^{k-2} = (n+1)^{k-1} -$$

$$k\,\square_m^{k-1} = (n+1)^k - \sum_{i=0}^{k-2} \binom{k}{i} \square_m^i$$

$$(k-1)\,\square_m^{k-2} = (n+1)^{k-1} - \sum_{i=0}^{k-3} \binom{k-1}{i} \square_m^i$$

$$k\,\square_m^{k-1} - (k-1)\,\square_m^{k-2} = (n+1)^k - (n+1)^{k-1} - \binom{k}{k-2} \square_m^{k-2} - \sum_{i=0}^{k-3} \left(\binom{k}{i} - \binom{k-1}{i} \right) \square_m^i$$

CÁLCULO FINITO E INFINITO

→ OPERADOR DERIVADA D:

$$Df(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

→ OPERADOR DIFERENÇA:

$$\Delta f(x) = f(x+1) - f(x) \qquad (h = 1)$$

EX: $f(x) = x^m$, $Df(x) = m \cdot x^{m-1}$

EX: O OPERADOR DIFERENÇA DÁ ALGO DIFERENTE:

$$\Delta(x^3) = (x+1)^3 - x^3 = 3x^2 \underline{+ 3x + 1}$$

POTÊNCIA

→ O OPERADOR DIFERENÇA FICA INTERESSANTE EM OUTRA CLASSE DE FUNÇÕES

DEF: POTÊNCIAS FATORIAIS DECRESCENTES / CRESCENTES.

DADO $m \geq 0$

$$x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1)$$ "DESCENDO"

$$x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1)$$ "SUBINDO"

CONVENÇÃO: $x^{\underline{0}} = x^{\overline{0}} = 1$

→ OPERADOR DIFERENÇA APLICADO A POTÊNCIAS DECRESCENTES

$$\Delta\left(x^{\underline{m}}\right) = (x+1)^{\underline{m}} - x^{\underline{m}}$$

IGUAIS

$$= (x+1)\cdot x\cdot(x-1)\cdots(x-m+2) - x(x-1)\cdots(x-m+1)$$

$$= x(x-1)\cdots(x-m+2)\Big(\big(\cancel{x}\cancel{+1}\big) - \big(\cancel{x} - m \cancel{+1}\big)\Big)$$

$$= m\cdot \underbrace{x(x-1)\cdots(x-m+2)}$$

$$= m\cdot x^{\underline{m-1}}$$

→ OPERADOR DIFERENÇA APLICADO A POTÊNCIAS CRESCENTES

$$\Delta\left(x^{\overline{m}}\right) = (x+1)^{\overline{m}} - x^{\overline{m}}$$

IGUAIS

$$= (x+1)\cdot(x+2)\cdots(x+m) - x(x+1)\cdots(x+m-1)$$

$$= (x+1)\cdots(x+m-1)(x+m-x)$$

$$= m(x+1)\cdots(x+m-1)$$

$$= m(x+1)^{\overline{m-1}} = m\,\frac{x^{\overline{m}}}{x}$$

MAS E A INTEGRAL?

$\Delta f(x) = f(x+1) - f(x)$

→ A INTEGRAL É O OPERADOR INVERSO DA DERIVADA

O TEO. FUND. DO CÁLCULO DIZ QUE

$$g(x) = Df(x) \quad \text{SE E SOMENTE SE} \quad \int g(x)\,dx = f(x) + C$$

→ ANALOGAMENTE, O OPERADOR Δ POSSUI INVERSO

O TEO. FUND. DO CÁLCULO FINITO DIZ

$$g(x) = \Delta f(x) \quad \text{SE E SOMENTE SE} \quad \sum g(x)\,\delta x = f(x) + C$$

→ SOMA INDEFINIDA

→ NESTE CASO, O C É UMA FUNÇÃO P T.Q. $P(x+1) = P(x)$

EX: $P(x) = C$ (CONSTANTE)

$P(x) = \text{SEN}(2\pi x)$

→ INTEGRAL DEFINIDA: SE $g(x) = Df(x)$

$$\int_a^b g(x)\,dx = f(x)\Big|_a^b = f(b) - f(a)$$

$$\sum_a^{a+1} g(x)\delta x + \sum_{a+1}^{a+2} g(x)\delta x$$
$$= (f(a+1) - f(a)) + (f(a+2) - f(a+1))$$
$$= f(a+2) - f(a) = \sum_a^{a+2} g(x)\delta x$$

→ SOMA DEFINIDA: SE $g(x) = Df(x)$

DEF: $$\sum_a^b g(x)\delta x = f(x)\Big|_a^b = f(b) - f(a)$$

OBS:

- $\sum_a^a g(x)\delta x = f(a) - f(a) = 0$
- $\sum_a^{a+1} g(x)\delta x = f(a+1) - f(a) = \Delta f(a) = g(a)$
- $\sum_a^{b+1} g(x)\delta x - \sum_a^b g(x)\delta x = (f(b+1) - \cancel{f(a)}) - (f(b) - \cancel{f(a)})$

$$= f(b+1) - f(b) = g(b)$$

APLICANDO INDUÇÃO OBTEMOS

$$\sum_a^b g(x)\,\delta x = \sum_{x=a}^{b-1} g(x) = \sum_{a\leqslant x<b} g(x)$$

$$[a,b] = \underbrace{[a,a+1]}_{g(a)} \cup \underbrace{[a+1,a+2]}_{g(a+1)} \cup \dots \cup \underbrace{[b-1,b]}_{g(b-1)}$$

$$\sum_{i=0}^{b-a-1} \sum_{a+i}^{a+i+1} g(x)\,\delta x$$

$$a+i+1 = b \iff i = b-a-1$$

$$\sum_{a\leqslant x<b} g(x) = \overbrace{\left(\cancel{f(a+1)} - f(a)\right)}^{i=0} + \overbrace{\left(\cancel{f(a+2)} - \cancel{f(a+1)}\right)}^{i=1} \cancel{+} \dots + \overbrace{\left(f(b) - \cancel{f(b-1)}\right)}^{i=b-a-1}$$

$$= f(b) - f(a)$$

• $\sum_a^{a+1} g(x)\,\delta x = g(a)$

$$\sum_a^{a+1} g(x)\,\delta x + \sum_{a+1}^{a+2} g(x)\,\delta x$$

$$= \left(f(a+1) - f(a)\right) + \left(f(a+2) - f(a+1)\right)$$

$$= f(a+2) - f(a) = \sum_a^{a+2} g(x)\,\delta x$$

TELESCÓPICA

$$g(x) = \Delta f(x) = f(x+1) - f(x)$$

$\rightsquigarrow$ O QUE ACONTECE QUANDO $b<a$?

$$\sum_a^b g(x)\,\delta x = f(b) - f(a) = -\left(f(a) - f(b)\right) = -\sum_b^a g(x)\,\delta x$$

$$\rightsquigarrow \sum_a^b + \sum_b^c = \sum_a^c$$

$$\sum_a^b g(x)\,\delta x + \sum_b^c g(x)\,\delta x = \sum_a^c g(x)\,\delta x$$

PRA QUE SERVE?

$$\text{Ex: } \sum_{0\le x<n} x^{\underline{m}} = \sum\nolimits_0^n x^{\underline{m}}\,\delta x = \frac{x^{\underline{m+1}}}{m+1}\Bigg|_0^n$$

$$= \frac{n^{\underline{m+1}}}{m+1}$$

$$D\big(\underbrace{x^{\underline{m+1}}}_{f(x)}\big) = \underbrace{(m+1)\,x^{\underline{m}}}_{g(x}$$

$$\sum\nolimits_n^0 x^{\underline{m}}\,\delta x = \underbrace{\frac{x^{\underline{m+1}}}{m+1}}\Bigg|_n^0$$

$$= -\frac{n^{\underline{m+1}}}{m+1}$$

$$\text{Ex: } \square^1_n = \sum_{0\le k<n} k^2$$

$$\square_n = \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2}$$

NOTE QUE $k^2 = k^{\underline{2}} + k^{\underline{1}}$

$$k^{\underline{2}} = k(k-1) \qquad k^2 = k(k-1) + k$$

$$\square^1_n = \sum_{0\le k<n} k^2 = \sum_{0\le k<n} k^{\underline{2}} + k^{\underline{1}} = \sum\nolimits_0^n k^{\underline{2}} + \sum\nolimits_0^n k^{\underline{1}} = \frac{k^{\underline{3}}}{3}\Bigg|_0^n + \frac{k^{\underline{2}}}{2}\Bigg|_0^n$$

$$= \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2}$$

EX: $\square'_n = \sum_{0 \leq k < n} k^3$

NOTE QUE $k^3 = k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}$

LOGO, $\square'_n = \sum_0^n k^3 = \sum_0^n k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}$

$$= \left(\frac{k^{\underline{4}}}{4} + \frac{3k^{\underline{3}}}{3} + \frac{k^{\underline{2}}}{2} \right) \Bigg|_0^n$$

$$= \frac{n^{\underline{4}}}{4} + n^{\underline{3}} + \frac{n^{\underline{2}}}{2}$$

OUTRAS PROPRIEDADES

$$(x+y)^2 = x^2 + 2xy + y^2$$

AQUI TEMOS

$$(x+y)^{\underline{2}} = (x+y)(x+y-1) = x^2 + 2xy + y^2 - x - y$$

$$= x(x-1) + 2xy + y(y-1)$$

$$= x^{\underline{2}} + 2x^{\underline{1}}y^{\underline{1}} + y^{\underline{2}}$$

EXPOENTES NEGATIVOS

$$x^{3} = x(x-1)(x-2)$$

$\div\ x-2$

$$x^{2} = x(x-1)$$

$\div\ x-1$

$$x^{1} = x$$

$\div\ x$

$$x^{0} = 1$$

$\div\ x+1$

$$x^{-1} = \frac{1}{x+1}$$

$$\vdots$$

$$x^{-m} = \frac{1}{(x+1)\cdots(x+m)} = \frac{1}{(x+1)^{\overline{m}}}$$

$$x^{m+n} = x^{m} (x-m)^{n}$$

SE $m = 2$ E $n = -3$

$$x^{m+n} = x^{2} (x-2)^{-3} = \cancel{x} \cdot \cancel{(x-1)} \cdot \frac{1}{\cancel{(x-1)}\,\cancel{x}\,(x+1)} = \frac{1}{(x+1)} = x^{-1}$$

LEMBRE-SE QUE

$$D \ln x = \frac{1}{x} \rightsquigarrow \int \frac{1}{x} = \ln x$$

O ANÁLOGO FINITO É $H_x = 1 + \frac{1}{2} + \dots + \frac{1}{x}$

$$\Delta H_x = H_{x+1} - H_x = \frac{1}{x+1} = x^{\underline{-1}}$$

$$D e^x = e^x \qquad \int e^x = e^x$$

O ANÁLOGO FINITO É UMA FUNÇÃO $f(x)$ T.Q. $\underline{\Delta f(x) = f(x)}$

$$f(x) = \Delta f(x) \overset{\text{DEF}}{=} f(x+1) - f(x) \rightsquigarrow 2 f(x) = f(x+1)$$

LOGO, $f(x) = 2^x$