A computational study of $f$-reversible processes on graphs

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**Abstract**

An $f$-reversible process on a graph $G = (V, E)$ is a graph dynamical system on $V(G)$ defined as follows. Given a function $f : V(G) \to \mathbb{N}$ and an initial vertex labeling $c_0 : V(G) \to \{0, 1\}$, every vertex $v$ changes its label if and only if at least $f(v)$ of its neighbors have the opposite state, synchronously in discrete-time. For such processes, we present a new nondecreasing time function similar to the monotonically decreasing energy functions used to study threshold networks, which leads to a periodic behavior after a transient phase. Using this new function, we provide a tight upper bound on the transient length of $f$-reversible processes. Furthermore, we prove that it is equal to $n - 3$ for trees with $n \geq 4$ vertices and $\text{Im}(f) = \{2\}$. Moreover, we present an algorithm that generates all the initial configurations attaining this bound and we prove that the size of such configurations is $O(n)$. We also consider the problem of determining the smallest number $r_f(G)$ of vertices with initial label 1 for which all the vertices eventually reach label 1 after the complete evolution of the dynamics, which models consensus problems on networks. We prove that it is $NP$-hard to compute $r_f(G)$ even for bipartite graphs with $\Delta(G) \leq 3$ and $\text{Im}(f) = \{1, 2, 3\}$. Finally, we prove that $\beta(G) \leq r_f(G) \leq \beta(G) + 1$, when $f(v) = d(v)$ for all $v \in V(G)$, where $\beta(G)$ is the size of a minimum vertex cover of $G$.

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1. Introduction

Let $G = (V, E)$ be a simple, undirected, finite graph with $n$ vertices and $m$ edges. A discrete dynamical process on $G$ is an infinite sequence $P = (c_t)_{t \in \mathbb{N}} = (c_0, c_1, \ldots)$ of configurations $c_t : V(G) \to \{0, 1\}$. We say that $c_0$ is the initial configuration of $P$ and $c_t(v)$ denotes the state of $v$ at time $t \in \mathbb{N}$. The transition from $c_t$ to $c_{t+1}$ takes place under the same predetermined rule at each time step. This approach is employed on a wide variety of areas such as social influence [9,17,28,40–42], gene expression networks [27], immune systems [2], cellular automata [3], statistical mechanics [1,4], marketing strategies [11,13,28], finite discrete dynamical systems [5,37,44], astrophysics and physics [20,39,43], opinion and disease [10,32] dissemination, simulations of biological cell populations [30], modeling of chemical systems [29], neural networks [22], local interaction games [33,34] and distributed computing [15,16,25,31,35,38].

In this work we consider a special kind of iterative processes on simple undirected graphs that generalizes the majority voting approach studied by Peleg [38]. Formally, the update rule is such that $c_{t+1}$ is obtained from $c_t$ according to a threshold
function \( f : V(G) \to \mathbb{N} \) applied to each vertex \( v \in V(G) \) and at each time \( t \in \mathbb{N} \). Denoting the neighborhood of \( v \) by \( N(v) \), the update rule is defined as

\[
c_{t+1}(v) = \begin{cases} 
1 - c_t(v), & \text{if } |\{u \in N(v) : c_t(u) \neq c_t(v)\}| \geq f(v); \\
c_t(v), & \text{otherwise.}
\end{cases}
\]

In other words, every vertex \( v \) changes its state if and only if it has at least \( f(v) \) neighbors with the opposite state, at each time step \( t \in \mathbb{N} \). Moreover, the state updates are done synchronously.

We say that \( f(v) \) is the threshold value of \( v \), which does not change during the process. Given an initial configuration \( c_0 \), we call such an iterative process following Eq. (1), an \( f \)-reversible process on \( G \), denoted by \( R_f(G, c_0) = (c_t)_{t \in \mathbb{N}} \), or simply \( R_f(G, c_0) \).

If \( f \) is a \( k \)-constant function, it is denoted by \( R_k(G, c_0) \) and called \( k \)-reversible process on \( G \), \( k \in \mathbb{N} \).

An \( f \)-reversible process is analogous to the majority voting approach studied by Mustafa and Pekeč [35] with \( f(v) = \left\lfloor \frac{d(v)}{2} \right\rfloor + 1 \), where \( d(v) \) denotes the degree of \( v \). In fact, several works consider iterative processes on graphs where changes of states are allowed at most once during the process [7,8,13,15]. These approaches are used, for instance, in influence diffusion in social networks in which some individuals are “active” to influence their neighbors for a limited number of time steps, once they were influenced by a sufficient number of neighbors. However, if a vertex has been influenced then it remains in this state forever. An \( f \)-reversible process represents the extremal case in which the elements do not have “memory”. Thus, the vertices do not keep themselves influenced if they have the required amount of neighbors to change their states and each one needs to be convinced by a subset of neighbors to change its opinion, at each time step.

We deal with two problems regarding \( f \)-reversible processes. The first one concerns its periodic behavior, where we present the following results:

1. We show that \( f \)-reversible processes are particular cases of threshold networks, which have at most two configurations in their periodic phase. This result is presented in Theorem 1;
2. Using a so-called energy function, Goleu et al. [24] provided an upper bound on the number of time steps needed to achieve the periodic behavior (maximum transient length). In this work we present a more intuitive function which provides a tight upper bound based on \( G, f \) and \( c_0 \). This result is given in Theorem 7;
3. We also analyze the maximum transient length of \( 2 \)-reversible processes on trees with at least 4 vertices, where we prove that it is at most \( n - 3 \) in Theorem 8. The proof yields an algorithm (Algorithm 1) that generates all initial configurations that allow the process to reach \( n - 3 \) time steps. Moreover, we prove that the number of such configurations is \( O(n) \).

The second problem studied refers to finding the minimum cardinality of a vertex subset which allows all the vertices of \( G \) to reach same state 1. This problem was proved to be \( NP \)-hard even for \( 2 \)-reversible processes on general graphs [12]. In the same work, this problem was left open for paths and cycles, where \( \text{Im}(f) = \{1, 2\} \). Dreyer [14] determined in his thesis the exact value for some specific graph classes, where \( f \) is constant. Moreover he proved that such a problem is \( NP \)-complete for \( k \)-regular graphs, where the function is \( k \)-constant. Regarding this problem, we obtain the following results:

1. We prove that it is \( NP \)-hard to determine such a parameter for bipartite graphs whose maximum degree is at most 3 and \( \text{Im}(f) = \{1, 2, 3\} \). This result is presented in Theorem 12;
2. Denoting the size of a minimum vertex cover of \( G \) by \( \beta(G) \), we prove that such a parameter is equal to \( \beta(G) \) or \( \beta(G) + 1 \), when \( f(v) = d(v) \) for all \( v \in V(G) \). Theorem 13 presents this result. As a corollary, if \( G \) is not a bipartite graph then \( \beta(G) \) is the exact number of vertices required.

2. Definitions, properties, and main results

We refer to [6] for definitions and concepts related to the theory of graphs. It is clear that an \( f \)-reversible process is uniquely determined by \( G, f \) and \( c_0 \). Furthermore, due to Eq. (1), every vertex \( v \) depends only on its own state and on the states of its neighbors in the same time step to define its next one. Thus, we will consider only connected graphs in the remaining of the paper. Moreover, if \( f(v) > d(v) \) for some vertex \( v \) then its initial state does not change during the whole process. Therefore we can assume \( f(v) = d(v) + 1 \) in this case. Hence, only threshold functions \( f \) satisfying \( \text{Im}(f) \subseteq \{0, \ldots, \Delta(G) + 1\} \) will be considered, where \( \Delta(G) \) denotes the maximum degree of \( G \). Note that if \( f(v) = 0 \) then the state of \( v \) changes at every time step \( t \in \mathbb{N} \).

2.1. The periodic behavior

Let \( R_f(G, c_0) \) be an \( f \)-reversible process. Since \( G \) is finite and \( c_{t+1} \) is obtained deterministically from \( c_t \), according to Eq. (1), the number of possible configurations is equal to \( 2^n \). Hence, there must exist a finite time step in which the process enters a periodic phase, that is, a finite and continuous subsequence of \( (c_t)_{t \in \mathbb{N}} \) that starts and finishes at the same configuration. The finite and continuous subsequence of \( (c_t)_{t \in \mathbb{N}} \) preceding the periodic phase and starting at \( c_0 \) is called transient, and its length (number of configurations) is denoted by \( r(c_0) \geq 0 \). The length of the periodic phase is called period, denoted by \( p(c_0) \geq 1 \).

A periodic configuration is one that occurs in the periodic phase and note that the first one is reached at time \( r(c_0) \). Formally, the period and transient lengths satisfy the following conditions:
Fig. 1. Example of the dynamic of an $f$-reversible process.

Fig. 2. The initial configuration of a 2-reversible process on a tree whose gray vertices have state 1 and the others have state 0.

- $c_{t+p(c_0)} = c_t$, for all $t \geq \tau(c_0)$;
- $c_{t+q} \neq c_t$, for all $(t < \tau(c_0))$ and $q \geq 1$ or $(t \geq \tau(c_0)$ and $1 \leq q < p(c_0)$).

Theorem 1 represents an example of the dynamics of an $f$-reversible process on a $C_4$. Note that $c_1 = c_3 \neq c_2$, which means that $\tau(c_0) = 1$ and $p(c_0) = 2$. In this case the transient is composed only by $c_0$ and the periodic phase by $c_1$ and $c_2$.

Two natural parameters arise from the above definitions. Given a threshold function $f$ and a graph $G$, denote the largest transient length over all initial configurations by $\tau_f(G)$ (resp. $\tau_f(G)$ if $f$ is a $k$-constant function). Analogously, let $p_f(G)$ (resp. $p_f(G)$ if $f$ is a $k$-constant function) be the largest period over all initial configurations.

Oliveira, Barbosa and Protti [36] studied the problem of determining if a configuration has a predecessor configuration, which is one that reaches it within exactly one time step. They have dealt only with $k$-reversible processes, where they proved that determining if there exists a predecessor configuration of one given is $NP$-complete for bipartite graphs, but linear-time solvable for trees. Moreover they also presented a linear-time algorithm for $2$-reversible processes on graphs whose maximum degree is at most 3. They also dealt with the problem of counting the number of predecessor configurations, for which they showed an $O(n^2)$-time algorithm for trees.

Dreyer [14] proved that $\tau(G)$ is $O(m + n^2)$ and $p(G) \leq 2$, where these results are based on reductions from the so-called threshold networks [22,23]. It is known that the maximum period of threshold networks is at most 2 [24,40]. An intuitive approach to prove this result is based on a monotonic function called energy function. Its definition is very similar to that of the energy function associated with Hopfield networks [26]. It is a Lyapunov function and it can be used to prove several results associated with the period and transient lengths of threshold and majority networks. This function dates back to Ref. [24] (page 269, inside the proof of Proposition 2) and it was later reproduced in Ref. [21] (page 70, Eq. 3.3).

2.2. The potential function

Let us consider an $f$-reversible process $R_f(G, c_0)$ on a graph $G$, which will be referred to only by $R$ here. For all $t \in \mathbb{N}$, let $S_c(R, t)$ and $S_u(R, t)$ be a bipartition of $V(G)$, where $S_c(R, t)$ denotes the set of vertices which change their states at time $t$, and $S_u(R, t)$ the set of those that do not. Given a vertex $v$, we denote the number of neighbours with the opposite state of $v$ at time $t$ by $op_t(v)$. Thus, it follows that

$$S_c(R, t) = \{v : op_t(v) \geq f(v)\}$$

and

$$S_u(R, t) = \{v : op_t(v) < f(v)\}.$$

We define a nonnegative function that we call potential function $P(t)$ for $f$-reversible processes as

$$P(t) = \sum_{v \in S_c(R, t)} (op_t(v) - f(v)) + \sum_{v \in S_u(R, t)} (f(v) - op_t(v)). \tag{2}$$

Fig. 3 contains plots of the potential and energy functions against time. One of them (filled line) is the energy function of Ref. [24] (page 269, inside the proof of Proposition 2), later reproduced in Ref. [21] (page 70, Eq. 3.3). The other (dotted line) refers to the potential function of Eq. (2). Data in the plots correspond to the tree depicted in Fig. 2. Both plots refer to a 2-reversible process and, in the case of the energy function of Ref. [24], to $b_1 = 1.5$ as the additional required parameter.
for every vertex $v_i$. The initial configuration has 0 at vertices $v_2$–$v_{11}$ and 1 at all others. The two functions differ markedly and no simple reduction seems to exist to transform one into the other. In particular, the potential function is nondecreasing (rather than monotonically decreasing), as illustrated by Fig. 3.

2.3. The minimum $f$-conversion set problem

We say that an $f$-reversible process $R_f(G, c_0)$ is uplifting if all vertices achieve state 1 and $p(c_0) = 1$. An $f$-conversion set of $G$ is a subset of vertices whose initial states are equal to 1 and for which the process is uplifting. For this end, we cannot consider any threshold values equal to 0. Furthermore $V(G)$ is a trivial $f$-conversion set of $G$. We also study the problem of finding the minimum cardinality of an $f$-conversion set of $G$, denoted by $r_f(G)$ if $f$ is a $k$-constant function, $k \geq 1$. For $k = 1$, the trivial solution given by $V(G)$ is unique. We consider the following decision problem in this work:

\begin{itemize}
  \item **Input:** A finite graph $G$, a function $f : V(G) \to \mathbb{N}$, and an integer $q \geq 1$.
  \item **Question:** Is $r_f(G) \leq q$?
\end{itemize}

In his thesis, Dreyer [14] proved that $f$-CONVERSION SET is NP-complete for $k$-reversible processes on $k$-regular graphs, $k \geq 3$. Moreover he gave exact values of $r_f(G)$ for some specific graph classes. Dourado et al. [12] showed the NP-hardness of determining $r_f(G)$ for general graphs $G$. They also stated an algorithm based on dynamic programming which computes $r_f(P_n)$ for specific paths $P_n$ with $n$ vertices. They consider both threshold values 1 and 2, where every $v$ with $f(v) = 1$ has a neighbor $u$ with $f(u) = 1$. Thus, computing $r_f(G)$ even for paths and cycles remains open.

The remainder of the text is organized as follows. In Section 3 we prove that $f$-reversible processes are particular cases of threshold networks whose matrices are symmetric, proving that $p_f(G) \leq 2$. We also present a sharp upper bound for $r_f(G)$, based on the potential function given by Eq. (2). To this end we prove that such a function is monotonically non-decreasing. In Section 4 we show that $r_2(T) \leq n - 3$, for trees $T$ with $n \geq 4$. Moreover, we present all the initial configurations reaching such a bound. We also present an algorithm that generate all the initial configurations achieving $n - 3$ time steps until the uplifting is reached and the number of such configurations. In Section 5 we prove that $f$-CONVERSION SET is NP-complete for bipartite graphs $G$ with $\Delta(G) \leq 3$ and $\text{Im}(f) = \{1, 2, 3\}$. It also contains the proof that $r_f(G) \in \{\beta(G), \beta(G) + 1\}$, when $f(v) = d(v)$ for all $v \in V(G)$. Section 6 contains our conclusions.

3. The period and the transient length

In this section we prove that the potential function given by Eq. (2) is nondecreasing. Thus, we show how it can be used to give a tight upper bound for the maximum transient length. Next we consider the maximum transient length of 2-reversible processes on trees. But first, we concern ourselves with the maximum period of $f$-reversible processes.
3.1. The maximum period

We prove that \( p_f(G) \leq 2 \), where this bound occurs for example in an 1-reversible process on \( K_2 \) whose vertices have opposite states. To this end, we prove that \( f \)-reversible processes are particular cases of threshold networks \((A, b)\), which are defined by a square matrix \( A \) and a threshold vector \( b \). The vertex states are also determined by two different values, for example \([0, 1]\), a configuration \( c_t \) is obtained from \( c_{t+1} \), defining an iterative process such that: \( c_t(v) = 1 \) if and only if \( \sum_{j=1}^{n} A_{ij} c_j(v) - b_i \geq 0 \). It was established in [24] that \( p_f(G) \leq 2 \) if \( A \) is symmetric. Hence, our approach consists of finding appropriate symmetric matrix and threshold vector. In fact, we do it for \((f_1, f_2)\)-reversible processes, in which there are two threshold functions \( f_1 \) and \( f_2 \), where \( v \) changes its state from 0 to 1 according to \( f_1 \) and from 1 to 0 according to \( f_2 \). If \( f_1 = f_2 \) then we obtain an \( f \)-reversible process, with \( f = f_1 = f_2 \).

**Theorem 1.** An \((f_1, f_2)\)-reversible process on \( G \) is a threshold network \((A, f_1)\) for which

\[
A_{ij} = \begin{cases} 
1, & \text{if } v_i v_j \in E(G) \text{ and } i \neq j; \\
0, & \text{if } v_i v_j \notin E(G) \text{ and } i \neq j; \\
f_1(v_i) + f_2(v_i) - d(v_i) - 1, & \text{if } i = j.
\end{cases}
\]

**Proof.** Let \( n_1^i(v_i) \) and \( n_0^i(v_i) \) be the number of neighbors of \( v_i \) with states 1 and 0 at time \( t \), respectively. It is enough to show that \( c_{t+1}(v_i) \neq c_t(v_i) \) if and only if \( v_i \) has at least \( f_1(v_i) \) neighbors with state 1 or \( v_i \) has at least \( f_2(v_i) \) neighbors with state 0 at time \( t \), if \( v \) has states 0 or 1, respectively.

- \( c_t(v_i) = 1 \): we get that \( c_{t+1}(v_i) = 0 \) if and only if \( \sum_{j=1}^{n} A_{ij} c_j(v) - b_i < 0 \). Hence:

\[
\sum_{j=1}^{n} A_{ij} c_j(v_i) < f_1(v_i); \]

\[
(f_1(v_i) + f_2(v_i) - d(v_i) - 1)c_t(v_i) + n_1^i(v_i) < f_1(v_i); \]

\[
f_1(v_i) + f_2(v_i) - d(v_i) - 1 + (d(v_i) - n_0^i(v_i)) < f_1(v_i); \]

\[
f_2(v_i) - n_0^i(v_i) - 1 \leq 0; \]

\[
f_2(v_i) \leq n_0^i(v_i) + 1; \]

\[
f_2(v_i) \leq n_0^i(v_i).
\]

In this way, \( v_i \) changes its state from 1 to 0 if and only if \( f_2(v_i) \leq n_0^i(v_i) \), as \((f_1, f_2)\)-reversible processes.

- \( c_t(v_i) = 0 \): we get that \( c_{t+1}(v_i) = 1 \) if and only if \( \sum_{j=1}^{n} A_{ij} c_j(v) - b_i \geq 0 \). Hence:

\[
\sum_{j=1}^{n} A_{ij} c_j(v_i) \geq f_1(v_i); \]

\[
(f_1(v_i) + f_2(v_i) - d(v_i) - 1)c_t(v_i) + n_1^i(v_i) \geq f_1(v_i); \]

\[
n_1^i(v_i) \geq f_1(v_i).
\]

Analogously, \( v_i \) changes its state from 0 to 1 if and only if \( f_1(v_i) \leq n_1^i(v_i) \), as \((f_1, f_2)\)-reversible processes. \( \square \)

Next we obtain the bound on the period from Proposition 2 of Ref. [24]:

**Theorem 2 ([24]).** Let \( F = (A, b) \) be a threshold network such that \( A \) is a symmetric matrix. Then the period of \( F \) is at most 2, for any initial configuration.

**Corollary 3.** The period of \((f_1, f_2)\)-reversible processes is at most 2.

3.2. Monotonicity of the potential function

First, we will prove that the potential function given by Eq. (2) is equivalent to

\[
P^*(t) = \sum_{v \in S_+(R, t)} (op_{t+1}(v) - f(v)) + \sum_{v \in S_-(R, t)} (f(v) - op_{t+1}(v)). \tag{3}
\]

To this end, we consider a partition of the edges whose ends have opposite states at time \( t \) as follows:

- \( E_1(t) = \{(u, v) \in E(G) : u \in S_1(R, t), v \in S_2(R, t), c_t(u) \neq c_t(v)\} \);
- \( E_2(t) = \{(u, v) \in E(G) : u \in S_2(R, t), v \in S_1(R, t)\} \);
- \( E_0(t) = \{(u, v) \in E(G) \setminus (E_1(t) \cup E_2(t)) : c_t(u) \neq c_t(v)\} \).

Note that each edge of \( E'(t) \) does not have both ends in the same set \( S_1(R, t) \) or \( S_2(R, t) \). Hence, we get that

\[
\sum_{v \in S_1(R, t)} op_t(v) = 2|E_1(t)| + |E'(t)| \quad \text{and} \quad \sum_{v \in S_2(R, t)} op_t(v) = 2|E_2(t)| + |E'(t)|.
\]
Therefore,
\[ \sum_{v \in S_t(R,t)} \text{op}_t(v) - \sum_{v \in S_{t+1}(R,t)} \text{op}_{t+1}(v) = 2(|E_t(t)| - |E_{t+1}(t)|). \] (4)

**Lemma 4.** For \( t \geq 0 \), \( P(t) = P'(t) \).

**Proof.** \( P(t) \) and \( P'(t) \) can be rewritten as
\[
P(t) = \left( \sum_{v \in S_t(R,t)} \text{op}_t(v) - \sum_{v \in S_{t+1}(R,t)} \text{op}_{t+1}(v) \right) + \left( \sum_{v \in S_{t+1}(R,t)} f(v) - \sum_{v \in S_t(R,t)} f(v) \right),
\]
\[
P'(t) = \left( \sum_{v \in S_t(R,t)} \text{op}_{t+1}(v) - \sum_{v \in S_{t+1}(R,t)} \text{op}_{t+1}(v) \right) + \left( \sum_{v \in S_{t+1}(R,t)} f(v) - \sum_{v \in S_t(R,t)} f(v) \right).
\]

Thus, it is enough to prove that
\[
\sum_{v \in S_t(R,t)} \text{op}_t(v) - \sum_{v \in S_{t+1}(R,t)} \text{op}_t(v) = \sum_{v \in S_t(R,t)} \text{op}_{t+1}(v) - \sum_{v \in S_{t+1}(R,t)} \text{op}_{t+1}(v). \] (5)

We can see that both terms of Eq. (5) are defined on the same sets, but referring to the number of neighbors with opposite states at consecutive time steps, to each vertex \( v \). We also consider similar sets to \( E_t(t) \), \( E_{t+1}(t) \) and \( E'_t \), but referring to edges whose ends have opposite states at time \( t + 1 \):

- \( E'_t(t) = \{(u, v) \in E(G) : u \in S_t(R, t), \ v \in S_{t+1}(R, t) \text{ and } c_t(u) \neq c_{t+1}(v)\}; \)
- \( E'_t(t) = \{(u, v) \in E(G) : u \in S_t(R, t), \ v \in S_{t+1}(R, t) \text{ and } c_t(u) \neq c_{t+1}(v)\}; \)
- \( E'_t(t) = \{(u, v) \in E(G) \setminus (E_t(t) \cup E'_{t+1}(t)) \text{ and } c_t(u) \neq c_{t+1}(v)\}. \)

Since all vertices in \( S_t(R, t) \) change their states and all vertices in \( S_{t+1}(R, t) \) do not, all edges of \( E_t(t) \) appear in \( E'_t \). The same holds for all edges of \( E_{t+1}(t) \) in \( E'_{t+1}(t) \). Moreover, since \( E_t(t) \) and \( E'_{t+1}(t) \) are defined over edges whose ends are in the same set \( S_t(R, t) \), we get that \( E_t(t) = E'_t \). It is analogous for \( E_{t+1}(t) \) and \( E'_{t+1}(t) \) with respect to \( S_{t+1}(R, t) \). Therefore,
\[
\sum_{v \in S_t(R,t)} \text{op}_{t+1}(v) - \sum_{v \in S_{t+1}(R,t)} \text{op}_{t+1}(v) = 2(|E'_t(t)| - |E'_{t+1}(t)|). \] (6)

Eqs. (4) and (6) show that Eq. (5) is true, completing the proof. \( \square \)

Let \( \Delta P(t) \) be the variation of the potential function from \( t \) to \( t + 1 \), i.e., \( \Delta P(t) = P(t + 1) - P(t) \). Now, we will show that the potential variation is not negative, for all \( t \in \mathbb{N} \).

**Lemma 5.** \( P(t) \) is a nondecreasing function.

**Proof.** By Lemma 4, we can rewrite the potential variation as \( \Delta P(t) = P(t + 1) - P'(t) \). Therefore,
\[
\Delta P(t) = \sum_{v \in S_t(R,t)} (\text{op}_{t+1}(v) - f(v)) + \sum_{v \in S_{t+1}(R,t)} (f(v) - \text{op}_{t+1}(v))
\]
\[
- \sum_{v \in S_t(R,t)} (\text{op}_{t+1}(v) - f(v)) - \sum_{v \in S_{t+1}(R,t)} (f(v) - \text{op}_{t+1}(v)).
\]

Now, we can describe the contribution of each vertex to \( \Delta P(t) \):

- If \( v \in S_t(R, t) \) and \( v \in S_{t+1}(R, t) \): \( \text{op}_{t+1}(v) - f(v) - \text{op}_{t+1}(v) + f(v) = 0 \);
- If \( v \in S_t(R, t) \) and \( v \in S_{t+1}(R, t) \): \( f(v) - \text{op}_{t+1}(v) - f(v) + \text{op}_{t+1}(v) = 0 \);
- If \( v \in S_t(R, t) \) and \( v \in S_{t+1}(R, t) \):
\[
f(v) - \text{op}_{t+1}(v) - f(v) + f(v) = 2(f(v) - \text{op}_{t+1}(v)) > 0, \text{ since } f(v) > \text{op}_{t+1}(v);
\]
- If \( v \in S_t(R, t) \) and \( v \in S_{t+1}(R, t) \):
\[
\text{op}_{t+1}(v) - f(v) - f(v) + \text{op}_{t+1}(v) = 2(\text{op}_{t+1}(v) - f(v)) \geq 0, \text{ since } f(v) \leq \text{op}_{t+1}(v).
\]

Since in each case the contribution is not negative, the lemma follows. \( \square \)
3.3. A new upper bound on the maximum transient length

Now we present a tight upper bound on the transient length of \( f \)-reversible processes. Lemma 6 gives an upper bound based on the potential function and the cardinality of \( S_n(R, t) \) at \( \tau(c_0) - 1 \), the last time step of the transient phase.

**Lemma 6.** For \( c_0 \) such that \( \tau(c_0) > 0 \),

\[
\tau(c_0) \leq P(\tau(c_0) - 1) - |S_n(R, \tau(c_0) - 1)| + 1.
\]

Moreover, there exists a graph \( G \), a threshold function \( f \), and an initial configuration \( c_0 \) such that this bound is attained.

**Proof.** As in the proof of Lemma 5, if the process is not in the periodic phase at time \( t \), then there must exist at least one vertex \( v \) such that either \( v \in (S_i(R, t) \cap S_n(R, t + 1)) \) or \( v \in (S_n(R, t) \cap S_i(R, t + 1)) \).

Let \( T_{S_i \rightarrow S_n}(v) \), \( T_{S_n \rightarrow S_i}^{\geq 0}(v) \), and \( T_{S_n \rightarrow S_i}^{= 0}(v) \) be the number of time steps in which \( v \) passes from \( S_i(R, t) \) to \( S_n(R, t + 1) \) from \( S_n(R, t) \) to \( S_i(R, t + 1) \) with potential variation, and from \( S_n(R, t) \) to \( S_i(R, t + 1) \) without potential variation, respectively, for all \( 0 \leq t \leq \tau(c_0) - 2 \). We denote \( \sum_{v \in V(G)} T_{S_i \rightarrow S_n}(v) \) by \( T_{S_i \rightarrow S_n} \) and, analogously, we define \( T_{S_n \rightarrow S_i}^{\geq 0} \) and \( T_{S_n \rightarrow S_i}^{= 0} \). Thus, for any initial configuration, the maximum transient length could be given in such a way that just one vertex changes between \( S_i \) and \( S_n \), from \( t \) to \( t + 1 \). Hence, it follows that

\[
\tau(c_0) \leq T_{S_i \rightarrow S_n} + T_{S_n \rightarrow S_i}^{\geq 0} + T_{S_n \rightarrow S_i}^{= 0} + 1. \tag{7}
\]

Let \( S_{i} \rightarrow j \) be the set of vertices \( v \) such that \( v \in S_i(0) \) and \( v \in S_j(\tau(c_0) - 1) \), for \( i, j \in [c, n] \). Now, we consider the relation between the numbers of transitions of a vertex \( v \) from time 0 to \( \tau(c_0) - 1 \):

\[
T_{S_i \rightarrow S_j}(v) + T_{S_j \rightarrow S_i}^{= 0}(v) = \begin{cases} T_{S_i \rightarrow S_j}(v), & \text{if } v \in S_i \\
T_{S_i \rightarrow S_j}(v) + 1, & \text{if } v \in S_j \\
T_{S_i \rightarrow S_j}(v) - 1, & \text{if } v \in S_j \end{cases}
\]

The above equation shows that

\[
T_{S_i \rightarrow S_j}^{= 0} + T_{S_j \rightarrow S_i}^{= 0} = T_{S_i \rightarrow S_j} + |S_{i} \rightarrow 1| - |S_{j} \rightarrow n|. \tag{8}
\]

By Eqs. (7) and (8) we get that

\[
\tau(c_0) \leq 2T_{S_i \rightarrow S_n} + |S_n \rightarrow c| - |S_c \rightarrow n| + 1. \tag{9}
\]

As in the proof of Lemma 5, for every \( v \in (S_i(R, t) \cap S_n(R, t + 1)) \) a potential increase of at least 2 is obtained. Thus \( T_{S_i \rightarrow S_n} \leq (P(\tau(c_0) - 1) - P(0))/2 \). Furthermore, by Eq. (2), we get that \( P(0) \geq |S_i(R, 0)| = |S_n \rightarrow c| + |S_n \rightarrow n| \). Hence, we can rewrite Eq. (9) and the lemma follows:

\[
\tau(c_0) \leq P(\tau(c_0) - 1) - P(0) + |S_n \rightarrow c| - |S_c \rightarrow n| + 1;
\]

\[
\leq P(\tau(c_0) - 1) - |S_n \rightarrow c| - |S_n \rightarrow n| + |S_n \rightarrow c| - |S_c \rightarrow n| + 1;
\]

\[
= P(\tau(c_0) - 1) - |S_n(R, \tau(c_0) - 1)| + 1. \quad \Box
\]

Let us consider \( V(G) \) partitioned according to time step \( \tau(c_0) - 1 \) as follows:

- \( X_c^R \) = \( \{ v \in V(G) : c_{\tau(c_0)-1}(v) = 0 \ \text{and} \ v \in S_c(R, \tau(c_0) - 1) \} \);
- \( X_n^R \) = \( \{ v \in V(G) : c_{\tau(c_0)-1}(v) = 0 \ \text{and} \ v \in S_n(R, \tau(c_0) - 1) \} \);
- \( Y_c \) = \( \{ v \in V(G) : c_{\tau(c_0)-1}(v) = 1 \ \text{and} \ v \in S_c(R, \tau(c_0) - 1) \} \);
- \( Y_n \) = \( \{ v \in V(G) : c_{\tau(c_0)-1}(v) = 1 \ \text{and} \ v \in S_n(R, \tau(c_0) - 1) \} \).

Given disjoint subsets \( V_1 \) and \( V_2 \) of \( V(G) \), we denote the set of all edges with one end in \( V_1 \) and the other in \( V_2 \) by \( [V_1, V_2] \). Let us denote the minimum threshold value by \( f_{\min} = \min_{v \in V(G)} f(v) \). Furthermore, let us denote \( S_j = \sum_{v \in V(G)} f(v) \). Thus, by Eqs. (2) and (4), it follows that

\[
P(\tau(c_0) - 1) = \sum_{v \in S_n(R, \tau(c_0) - 1)} f(v) - \sum_{v \in S_c(R, \tau(c_0) - 1)} f(v) + 2|X_c^R, Y_c^R| - 2|X_n^R, Y_n^R| \tag{10}
\]

By Lemma 6 and Eq. (10) we obtain a tight upper bound on \( \tau(c_0) \), for all \( c_0 \).
Theorem 7. For $c_0$ such that $\tau(c_0) > 0$,
\[
\tau(c_0) \leq \begin{cases} 
S_f - (n + 2f_{\text{min}} - 2) & \text{if } X^R_{c} = \emptyset \text{ or } Y^R_{c} = \emptyset; \\
S_f - (n + 2f_{\text{min}} - 1) & \text{if } p(c_0) = 1, X^R_{c} \neq \emptyset, \text{ and } Y^R_{c} \neq \emptyset; \\
S_f + 2m - 3n + 1 - \sum_{v \in S_n(R, \tau(c_0)-1)} (2f(v) - 3) & \text{if } p(c_0) = 2, X^R_{c} \neq \emptyset, \text{ and } Y^R_{c} \neq \emptyset.
\end{cases}
\]

Proof. Case 1: $X^R_{c} = \emptyset$ or $Y^R_{c} = \emptyset$;
In this case we get that $[X^R_{c}, Y^R_{c}] = \emptyset$. Thus
\[
P(\tau(c_0) - 1) \leq S_f - 2 \left( \sum_{v \in S(R, \tau(c_0)-1)} f(v) \right)
\]
and, by Lemma 6, we have
\[
\tau(c_0) \leq S_f - 2 \left( \sum_{v \in S(R, \tau(c_0)-1)} f(v) \right) - (n - |S(R, \tau(c_0) - 1)|) + 1 - 2|[X^R_{n}, Y^R_{n}]| \\
= S_f - 2 \left( \sum_{v \in S(R, \tau(c_0)-1)} f(v) \right) + |S(R, \tau(c_0) - 1)| - n + 1 - 2|[X^R_{n}, Y^R_{n}]| \\
= S_f - (n - 1) - \sum_{v \in S_n(R, \tau(c_0)-1)} (2f(v) - 1) - 2|[X^R_{n}, Y^R_{n}]| .
\]

Since $S_f - (n - 1)$ is a constant, $|S_n(R, \tau(c_0) - 1)| \geq 1$, and $2f(v) - 1 > 0$ for all $v \in V(G)$, the limit obtained by Eq. (11a) is maximum when $[X^R_{n}, Y^R_{n}] = \emptyset$ and $S_n(R, \tau(c_0) - 1) = \{v\}$, where $f(v) = f_{\text{min}}$. Hence, we obtain
\[
\tau(c_0) \leq S_f - (n - 1) - (2f_{\text{min}} - 1) = S_f - (n + 2f_{\text{min}} - 2) .
\]

Case 2: $p(c_0) = 1, X^R_{c} \neq \emptyset$, and $Y^R_{c} \neq \emptyset$;
Since every $v \in S_n(R, \tau(c_0) - 1)$ belongs to $S_n(R, \tau(c_0))$, we have that $|N_{X^R_{c}}(v)| \leq f(v) - 1$, for every $v \in X^R_{c}$ and $|N_{Y^R_{c}}(u)| \leq f(u) - 1$, for every $u \in Y^R_{c}$. Therefore
\[
2|[X^R_{c}, Y^R_{c}]| = \sum_{v \in X^R_{c}} |N_{X^R_{c}}(v)| + \sum_{u \in Y^R_{c}} |N_{Y^R_{c}}(u)| \\
\leq \sum_{v \in S_n(R, \tau(c_0)-1)} f(v) - |S_n(R, \tau(c_0) - 1)| .
\]

Thus, we can rewrite Eq. (10) as
\[
P(\tau(c_0) - 1) \leq \sum_{v \in S_n(R, \tau(c_0)-1)} f(v) - \sum_{v \in S_n(R, \tau(c_0)-1)} f(v) \\
+ \sum_{v \in S_n(R, \tau(c_0)-1)} f(v) - |S_n(R, \tau(c_0) - 1)| - 2|[X^R_{n}, Y^R_{n}]| \\
\leq \sum_{v \in S_n(R, \tau(c_0)-1)} f(v) - |S_n(R, \tau(c_0) - 1)| .
\]

By Lemma 6 and the fact that $|S_n(R, \tau(c_0) - 1)| \geq 2$, it follows that
\[
\tau(c_0) \leq \sum_{v \in S_n(R, \tau(c_0)-1)} f(v) - |S_n(R, \tau(c_0) - 1)| - |S_n(R, \tau(c_0) - 1)| + 1 \\
= \left( \sum_{v \in V(G)} f(v) - \sum_{v \in S_n(R, \tau(c_0)-1)} f(v) \right) - n + 1 \\
\leq S_f - (n + 2f_{\text{min}} - 1) .
\]

Case 3: $p(c_0) = 2, X^R_{c} \neq \emptyset$, and $Y^R_{c} \neq \emptyset$;
As in Case 2, $P(\tau(c_0) - 1)$ is maximum when $[X^R_{n}, Y^R_{c}] = \emptyset$, while $|[X^R_{c}, Y^R_{c}]| \leq m - |S_n(R, \tau(c_0)) - 1|$. It means that $S_n(R, \tau(c_0) - 1)$ and $S_n(R, \tau(c_0) - 1)$ induce independent sets. Furthermore, $d(v) = 1$ for all $v \in S_n(R, \tau(c_0) - 1)$.
Hence, we can rewrite Eq. (10) as
\[
P(\tau(c_0) - 1) \leq S_f - 2 \left( \sum_{v \in S_0(R, \tau(c_0) - 1)} f(v) \right) + 2(n - |S_0(R, \tau(c_0) - 1)|)
\]
\[
= S_f + 2m - 2 \left( \sum_{v \in S_0(R, \tau(c_0) - 1)} f(v) \right) - 2|S_0(R, \tau(c_0) - 1)|.
\]

Hence, by Lemma 6 we complete the proof:
\[
\tau(c_0) \leq S_f + 2m - 2 \left( \sum_{v \in S_0(R, \tau(c_0) - 1)} f(v) \right) - 3|S_0(R, \tau(c_0) - 1)| + 1
\]
\[
= S_f + 2m + 1 - 2 \left( \sum_{v \in S_0(R, \tau(c_0) - 1)} f(v) \right) - 3(n - |S_0(R, \tau(c_0) - 1)|)
\]
\[
\leq S_f + 2m - 3n + 1 - \sum_{v \in S_0(R, \tau(c_0) - 1)} (2f(v) - 3)
\]
\[
= S_f + 2m - 3n + 1 - \sum_{v \in S_0(R, \tau(c_0) - 1)} (2f(v) - 3). \quad \square
\]

Fig. 4 shows an example illustrating that the bound on Case 1 of Theorem 7 is tight. The gray vertices have state 1 and the number above a vertex \(v_i\) represents \(f(v_i)\). We consider an \(f\)-reversible process \(R_f(C_n, c_0)\) on odd cycles \(C_n = v_1 v_2 \ldots v_n\), in which \(f(v_1) = 1\) and \(f(v_i) = 2\), for all \(i \neq 1\). Moreover \(c_0(v_1) = 0\) if and only if \(i > 1\) and odd. This process is uplifting and satisfies the following conditions:
- \(v_1 \in S_0(R, t)\), for all \(t < \tau(c_0)\);
- For every \(j > 1\), \(u_j \in S_0(R, t)\), for all \(t < (j - 2)\);
- For every \(j > 1\), \(u_j \in S_0(R, t)\), for all \(t \geq (j - 2)\).

Thus \(\tau(c_0) = n - 1\), as well as in Case 1 of Theorem 7. However, such a bound can be arbitrarily far from \(\tau(c_0)\). For instance, consider an \(f\)-reversible process on a star \(G\) with \(n\) vertices and whose central vertex is \(v\). Suppose \(f(v) = n\) and \(f(u) = 1\), for all \(u \neq v\). If \(c_0(v) = 1 - c_0(u)\), for every \(u \neq v\), then \(\tau(c_0) = 1\), although Theorem 7 results on \(\tau(c_0) \leq (n - 1)\). Notice that Lemma 6 gives the correct value.

Fig. 5(e) is an example attaining the bound on Case 2 of Theorem 7, where every vertex has threshold value 2. With respect to Case 3, we can cite an \(1\)-reversible process on a path \(P_n = \{v_1, v_2, \ldots, v_n\}\) with \(n \geq 3\), in which \(v_1\) has the opposite state from the all others. Such a process has transient length equal to \(n - 2\), as well as in Case 3.

4. The transient length of 2-reversible processes on trees

In this section we consider the maximum transient length of 2-reversible processes on trees. If \(n \leq 4\) and \(\tau(c_0) > 0\), it can be seen that \(\tau(c_0) \leq n - 2\), for all \(c_0\). Such a limit holds when \(T\) is a path with 3 vertices, whose central vertex has the opposite state to the others (Fig. 5(a)). However, all non-periodic configurations with four vertices (Fig. 5) have transient length equal to \(n - 3\). Actually, we prove that \(n - 3\) is a tight upper bound on all initial configurations on trees with \(n \geq 4\). If \(n \leq 2\) then all configurations are periodic.
Theorem 8. For a tree with \( n \geq 4 \), \( r_2(T) \leq n - 3 \).

Proof. The theorem follows directly in both Cases 2 and 3 of Theorem 7, while Case 1 shows that \( r_2(T) \leq n - 2 \). For trees \( T \) with \( f(v) = 2 \) for all \( v \in V(T) \), we resort to Eq. (10) and Lemma 6 as follows, where \( p = |S_i(R, \tau(c_0) - 1)|)\:

\[
\tau(c_0) \leq S_f - 2 \left( \sum_{i \in S_i(R, \tau(c_0) - 1)} f(v) \right) + 2|X^R_n, Y^R_n| - 2|X^R_n, Y^R_n| - 2|X^R_n, Y^R_n| + 2|X^R_n, Y^R_n| - 2|X^R_n, Y^R_n|.
\]

(12)

We split the proof into cases according to the cardinalities of the sets \( X^R \) and \( Y^R \).

Case 1: \( X^R = \emptyset \) or \( Y^R = \emptyset \):

By Eq. (12) and the fact that \( p \geq 1 \), if \( p > 1 \) or \( |X^R_n, Y^R_n| \neq \emptyset \) then \( \tau(c_0) \leq n - 4 \). Therefore, let us suppose that \( (X^R_n, Y^R_n) = \emptyset \) and \( S_f(\tau(c_0) - 1) = \emptyset \), without loss of generality. Thus, each subtree \( T_v \), rooted at \( v \), has all its vertices with the same state at \( \tau(c_0) - 1 \), for each \( i \in \{1, 2, \ldots, d(v)\} \). Since only one may change its state at time \( \tau(c_0) - 1 \), there must exist at least two subtrees \( T_v \), whose vertices have state 1 at time \( \tau(c_0) - 1 \). Furthermore there exists at most one subtree with all its vertices having state 0 at time \( \tau(c_0) - 1 \). Hence \( p(c_0) = 1 \) in this case.

Let us consider that \( T_{v_1} \) is the last subtree whose vertices reach their final state. Since the process follows concurrently in all subtrees, we split the analysis into two sub-cases based on the degree of \( v \).

Sub-case 1.1: \( d(v) = 2 \).

Notice that all vertices in \( V(T) \setminus \{v\} \) must have the same state 1 at time \( \tau(c_0) - 1 \), where the process is uplifting from the leaves toward \( v \).

Let \( t_1 \) and \( t_2 \) be the time steps in which \( T_{v_1} \) and \( T_{v_2} \) are uplifting, respectively. Since \( t_1 = \tau(c_0) - 1 \) and \( v_1 \) must keep state 0 from \( t_2 \) to \( t_1 \), if \( t_1 = t_2 \), then \( t_1 \) is maximum when \( V(T_{v_1}) \) induces a maximum path that is uplifting. Thus \( T_{v_1} \) is a path whose states of the vertices alternate. Furthermore \( T_{v_2} \) must have the same length as \( T_{v_1} \). Thus \( T \) is an odd size path whose vertices alternate, implying that \( \tau(c_0) = \frac{n-1}{2} \). Hence, if \( \tau(c_0) = n - 2 \) then \( n = 3 \), and if \( \tau(c_0) = n - 3 \) then \( n = 5 \). These cases are depicted in Figs. 5(a) and 8(b), respectively.

On the other hand, if \( t_1 > t_2 \), it means that \( v_1 \) must keep state 0 from \( t_2 \) to \( t_1 \). Thus \( t_1 - t_2 \) is equal to the length of a path \( P = \{u_{i_1}, u_{i_2} + 1, \ldots, u_{i_1}\} \), where \( u_{i_1} = v_1 \) and each \( u_i \) reaches state 1 at time \( t_i \), for all \( i \in \{t_2, t_2 + 1, \ldots, t_1\} \). Moreover, for every \( u_i \) we get that \( d(u_i) > 2 \) and each one must have at most one neighbor with opposite state, from time \( t_2 \) to time \( t_1 \). The configuration obtained at time \( t_1 \) which maximizes \( |P| \) is depicted in 6, where all vertices have state 1, unless \( v \) and those of \( V(P) \setminus \{u_{i_1}\} \).

For every tree \( T \), we can obtain a tree \( T' \) such that \( r_2(T') \geq r_2(T) \) as follows. For each pair of vertices \( w \) and \( w' \) of \( T_{v_2} \setminus \{v_2\} \), remove them from \( T_{v_2} \), add \( w \) to \( P \) and, add an edge between \( w \) and \( w' \). Moreover, assign \( c_0(w) = 0 \) and \( c_0(w') = 1 \). If \( |T_{v_2}| \) is even then there remain two vertices in \( T_{v_2} \), by the previous procedure. Thus, remove the last neighbor \( w'' \) of \( v_2 \) from \( T_{v_2} \) and add it to \( T_{v_1} \) as a neighbor of the vertex of \( P \) at maximum distance from \( v \), such that \( c_0(w'') = 1 \). Fig. 7(a) and 7(b) represent the cases in which \( |T_{v_2}| \) is odd or even, respectively. Thus path \( P \) increases, yielding a new path \( P' = \{u_1, u_2, \ldots, u_k\} \). Hence it is obtained one more time for each pair of moved vertices.

Note that \( r_2(c_0) = \lfloor |P| \rfloor \). Hence, if \( |T_{v_2}| \) is odd then \( \tau(c_0) = |P'| \leq \frac{n-3}{2} \) and it follows that:
that their initial states from \( w \) states, for all vertices of \( w \) after all of their children.

Moreover, let \( u \) achieve its final state, and its state, say \( \sigma \), simultaneously to \( v \).

Moreover, since \( w \), we have \( d(w) = 2 \). Therefore, either \( w \) is released by at least two children, or \( w \) also depends on its parent. Hence, there exists a path \( W = \{w_x, w_{x-1}, \ldots, w_1\} \) such that \( w_i \) depends on its parent \( w_{i-1} \) to change its state, for all \( i \in \{1, \ldots, x-1\} \). Thus we get that \( d(w_i) \geq 3 \), for all \( w_i \in W \). Moreover \( w_{1} \) is the first vertex of \( W \) to be released, where the vertices in \( W \cup \{u\} \) change their initial states from \( w_1 \) to \( u \). If \( Z = N(w_1) \cap P \) does not reach its final state before \( w_{1} \), the process follows changing the states of the vertices from \( w_1 \) to \( u \) and returns to \( w_1 \). Otherwise, the process takes fewer time steps, since it would follow simultaneously to \( T \setminus T_{z_1} \). In this case, all vertices on path \( Z = \{z_1, z_2, \ldots, z_y\} \) must alternate their states until \( T_{z_1} \) is uplifting.

Let \( T'(w_i) \) be the subtree of \( w_i \) which does not intersect \( P \), for all \( i \in \{1, \ldots, x\} \). To maximize the transient length, \( w_i \) must keep state 0 from \( t = 0 \) until its parent and all of its children achieve state 1. Thus \( w_i \) waits until \( T'_{w_i} \) is uplifting (see Fig. 10). Moreover, since \( w_i \) does not affect the state of its children, the maximum number of time steps required for the uplifting

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**Fig. 7.** Representation of tree \( T' \) obtained from a tree \( T \), such that \( r_2(T') \geq r_2(T) \).

**Fig. 8.** All configurations with \( \tau(c_0) \geq n - 3 \), such that \( X^k = Y^k = \emptyset \), \( d(v) = 2 \) and \( n \geq 5 \).

**Fig. 9.** Representation of the initial configurations of trees in which \( r_2(c_0) = n - 3 \) and all non-leaf vertices reach their final states exactly one time step after all of their children.

- \( \frac{n-1}{2} = n - 2 \Rightarrow n = 3 \) and \( |P'| = 1 \) (Fig. 5(a));
- \( \frac{n-1}{2} = n - 3 \Rightarrow n = 5 \) and \( |P'| = 2 \) (Fig. 8(a)).

Finally, if \( |T_{v_2}| \) is even then \( \tau(c_0) = |P'| = \frac{n-2}{2} \) and it follows that:
- \( \frac{n-2}{2} = n - 2 \Rightarrow n = 2 \) and \( |P'| = 0 \);
- \( \frac{n-2}{2} = n - 3 \Rightarrow n = 4 \) and \( |P'| = 1 \) (Fig. 5(d)).

**Sub-case 1.2:** \( d(v) \geq 3 \).

Let us suppose that all non-leaf vertices reach their final states exactly one time step after all of their children. Let \( P = \{p_1, p_2, \ldots, p_{\ell}\} \) be a longest path from \( p_1 = v \) to a leaf \( p_\ell \). Thus, for each internal vertex \( p_i \), we get that \( p_i \) reaches its final state exactly after \( p_i+1, i < \ell \) and \( \tau(c_0) = |P|, \) where \( |P| \leq n - 3 \). Fig. 9 depicts this case, in which \( d(v) = 3 \) and the internal vertices of \( P \) must alternate their states, since they have degree equal to 2. Therefore \( v \) must have 2 neighbors with opposite states, for all \( t < \tau(c_0) - 1 \). If \( d(v) \geq 4 \) or \( d(p_i) \geq 3 \) for some internal vertex \( p_i \) then \( \tau(c_0) \leq n - 4 \), in this case.

Now, let \( u \neq v \) be a non-leaf vertex whose state does not change even if all of its children reach their final states. Moreover, let \( u \) be the farthest vertex from \( v \) satisfying this property. Since all leaves of a subtree \( T_{v_i} \) must have the same state, say \( s \in \{0, 1\} \), we have \( d(u) = 2 \). Moreover \( u \) and its parent \( w \) must have the same initial state 0, when the child of \( u \) achieves its final state, and \( u \) changes its state exactly one time step after \( w \). In other words, the uplifting of \( T_{v_i} \) follows from the leaves to \( u \), but it stops at \( u \), which is “released” by \( w \).

Therefore, either \( w \) is released by at least two children, or \( w \) also depends on its parent. Hence, there exists a path \( W = \{w_x, w_{x-1}, \ldots, w_1\} \) such that \( w_i \) depends on its parent \( w_{i-1} \) to change its state, for all \( i \in \{1, \ldots, x - 1\} \). Thus we get that \( d(w_1) \geq 3 \), for all \( w_1 \in W \). Moreover \( w_1 \) is the first vertex of \( W \) to be released, where the vertices in \( W \cup \{u\} \) change their initial states from \( w_1 \) to \( u \). If \( Z = N(w_1) \cap P \) does not reach its final state before \( w_1 \), the process follows changing the states of the vertices from \( w_1 \) to \( u \) and returns to \( w_1 \). Otherwise, the process takes fewer time steps, since it would follow simultaneously to \( T \setminus T_{z_1} \). In this case, all vertices on path \( Z = \{z_1, z_2, \ldots, z_y\} \) must alternate their states until \( T_{w_i} \) is uplifting.

Let \( T'(w_i) \) be the subtree of \( w_i \) which does not intersect \( P \), for all \( i \in \{1, \ldots, x\} \). To maximize the transient length, \( w_i \) must keep state 0 from \( t = 0 \) until its parent and all of its children achieve state 1. Thus \( w_i \) waits until \( T'_{w_i} \) is uplifting (see Fig. 10). Moreover, since \( w_i \) does not affect the state of its children, the maximum number of time steps required for the uplifting
Fig. 10. Representation of tree $T$, where $v \in S_1(t)$, for all $t \leq \tau(c_0) - 1$.

Fig. 11. Representation of a general initial configuration of trees in which $\tau(c_0) = n - 3$ and at least one non-leaf vertex reaches its final state after all of its children.

Fig. 12. Representation of trees $T$ with $\tau(c_0) = n - 3$, in which $y = 0$ and $n$ even.

Thus $\tau(c_0) = n - 3$ when $T$ is as in Fig. 11(a) and 11(b), where all subtrees $T'_{u_i}$ and $T_u$ have exactly one vertex and all vertices of $Y$ have degree 2.

Note that if $x = 0$ then the configurations obtained are equivalent to those in Fig. 9. On the other hand, if $n$ is even and $y = 0$ then $v$ "starts" the process, where its state must be equal to the states of the leaves of $P$, as well as when $n$ is odd and $y = 0$, but with initial state opposite to that of the leaves. In the latter case it is possible to keep the state of $v$, adding one neighbor to $v$ with opposite state, yielding a tree with even number of vertices (Fig. 12(a)). The effect is to extend $W$ including $v$, increasing the transient length by one time step and the number of vertices also by one, keeping the upper bound on $n - 3$. On the other hand, no other vertex can be added as a neighbor of $v$, since $v$ already has two neighbors of each state. Finally, we can obtain the configuration given in Fig. 12(b), where the additional vertex is added as a neighbor of $v_2$, in which the effect is the same as in the previous case. Moreover, any additional vertex does not increase the transient length. Hence, no other initial configuration is possible.
Case 2: \( X_n^R \neq \emptyset \) and \( Y_n^R \neq \emptyset \):

Since \( ||X_n^R, Y_n^R|| \leq p - 1 \) and \( V(S_0(R, \tau(c_0) - 1)) \) induces a forest, by Eq. (12) it follows that

\[
\tau(c_0) \leq n - 3p + 1 + 2(p - 1) - 2||X_n^R, Y_n^R||
\]
\[
= n - p - 1 - 2||X_n^R, Y_n^R||.
\]

Therefore, if \( p > 2 \) or \( ||X_n^R, Y_n^R|| = 0 \) then \( \tau(c_0) \leq n - 4 \) and the theorem follows. Let us suppose \( ||X_n^R, Y_n^R|| = 0 \) and consider \( S_0(R, \tau(c_0) - 1) = [u, v] \), such that \( (u, v) \in E(T) \) and \( c_{\tau(c_0) - 1}(v) = 1 - c_{\tau(c_0) - 1}(u) \).

Since every \( w \in S_0(R, \tau(c_0) - 1) \) has at most one neighbor with opposite state (\( u \) or \( v \)), \( w \) belongs to \( S_0(R, \tau(c_0)) \). Thus \( |S_0(R, \tau(c_0))| = 2 \) only if \( S_0(R, \tau(c_0)) = \{u, v\} \), implying that \( c_{\tau(c_0) - 1} \) is periodic, a contradiction. Therefore, either \( p(c_0) = 1 \), where \( u \) and \( v \) have opposite states, or \( S_0(R, \tau(c_0)) \) contains exactly one vertex, which is either \( u \) or \( v \).

Suppose \( p(c_0) = 2 \), where \( S_0(R, \tau(c_0)) = \{v\} \) and \( c_{\tau(c_0) - 1}(v) = i \in \{0, 1\} \). Thus \( d(v) \geq 4 \) and \( v \) has at least one neighbor with state \( i \) and at least three state \( 1 - i \) where \( u \) is one of them, at \( \tau(c_0) - 1 \). Since \( (N(u) \setminus \{v\}) \subset S_0(R, \tau(c_0) - 1) \), it follows that the value of \( \tau(c_0) \) is at most the maximum length of a path \( P \) whose vertices alternate their states, where \( P \) is a subtree of \( u \). Hence \( \tau(c_0) \leq n - |N(u) \cup \{v\}| = n - 5 \).

Now, suppose that \( p(c_0) = 1 \). Thus \( c_{\tau(c_0) - 1}(w) = 1 - c_{\tau(c_0) - 1}(u) \), for all \( w \in N(u) \), and \( c_{\tau(c_0) - 1}(w') = 1 - c_{\tau(c_0) - 1}(v) \), for all \( w' \in N(v) \). Therefore \( \tau(c_0) \) is maximum when a subtree \( T_u \) of \( u \) has maximum transient length. Thus \( T_u \) is a path whose vertices alternate their states. Hence, we get that \( \tau(c_0) \leq n - |N(u) \cup \{v\}| \leq n - 3. Moreover, the previous limit is obtained only if \( d(u) = d(v) = 2 \), where \( \tau(c_0) = 1 \), since \( v \) is a leaf. This situation is depicted in Fig. 5(e). Finally, consider \( d(u) \geq 3 \).

Let \( \#_T(G, q) \) be the number of configurations \( c_0 \) such that \( \tau(c_0) = q \) for an \( f \)-reversible process on \( G \). We prove that \( \#_T(T, n - 3) = O(n) \) for trees \( T \) with \( n \geq 4 \).

**Corollary 9.** For \( T \) a tree with \( n \geq 4 \),

\[
\#_T(T, n - 3) = \begin{cases} 
4, & \text{if } n = 4; \\
3, & \text{if } n = 5; \\
\frac{n - 3}{2}, & \text{if } n \geq 6 \text{ and } n \text{ is even}; \\
\frac{n - 3}{2}, & \text{if } n \geq 7 \text{ and } n \text{ is odd}.
\end{cases}
\]

**Proof.** Fig. 5 presents all initial configurations when \( n = 4 \). As in the proof of Theorem 8, there are only three initial configurations attaining the bound \( n - 3 \) when \( n = 5 \), where two of them are depicted in Fig. 8 and the last one is given by Fig. 9(a). For \( n = 6 \), since \( d(v) \geq 3 \) (where \( \tau(c_0) - 1 = \{v\} \)) each subtree of \( v \) has at most three vertices. Thus, there is no configuration with \( |W| > 0 \). Hence, either \( |Z| = 0 \) or \( |Z| = 3 \). When \( |Z| = 0 \) we get that both configurations attaining the bound \( n - 3 \) are given in Figs. 12(a) and 12(b). Fig. 9(b) represents the case in which \( |Z| = 3 \).

If \( n \geq 7 \) and \( n \) is odd then \( |Z| \) must be even and there exists exactly one configuration for each even value of \( |Z| \) from 0 to \( n - 7 \). Furthermore, there exists one more for \( |Z| = n - 3 \). Hence, we get that

\[
\#_T(T, n - 3) = 1 + \sum_{i=0}^{n-7} (1) = 1 + \left( \frac{n - 5}{2} \right) = \frac{n - 3}{2}.
\]

Analogously, if \( n \geq 8 \) and \( n \) is even then for the cases in which \( |Z| > 0 \), we get that \( |Z| \) must be odd and there exists exactly one configuration for each odd value of \( |Z| \) from 1 to \( n - 7 \). Moreover, there are two configurations when \( |Z| = 0 \) (Fig. 12) and one more when \( |Z| = n - 3 \) (Fig. 9(a)). Hence, we get that

\[
\#_T(T, n - 3) = 3 + \sum_{i=1}^{n-7} (1) = 3 + \left( \frac{n - 7}{2} \right) = \frac{n - 6}{2}.
\]

**Corollary 10.** For \( T \) a tree with \( n \geq 4 \), \( \tau_2(T) = n - 3 \) if and only if \( c_0 \) is output by Algorithm 1.

**Proof.** In Algorithm 1, the trees \( T_i \) with \( n \geq 4 \) and initial configurations \( C_i \), where \( \tau_2(C_i) = n - 3 \), are as in the proof of Theorem 8. In this algorithm, lines 1–12 represent the construction of the configuration represented in Figs. 9(a) and 9(b) for \( n \) odd and even, respectively. Lines 4–7 result in the construction of the tree, while lines 8–12 represent the assignment of the states to the vertices. Thus, tree \( T_1 \) and configuration \( C_1 \) are generated. For trees with \( n = 4 \) or \( n = 5 \), lines 17–19 construct the trees and configurations given by Figs. 5(b) and 8(a) for \( n = 4 \) and \( n = 5 \), respectively. Lines 23–27 describe the construction of trees and configurations depicted by Figs. 5(d) and 8(b) for \( n = 4 \) and \( n = 5 \), respectively. Lines 31–34 yield the tree and configuration given by Fig. 5(e). For \( n \geq 6 \), lines 36–48 modify the tree and configuration generated one step before, which are respectively \( T_1 \) and \( C_1 \), for each odd \( i \) from 3 to \( n - 3 \). At each step, the graph obtained has \( |Z| \)
decreased by 1, while |W| is increased by 1. Such configurations are represented by Figs. 11(a) and 11(b) for n even and odd, respectively. If n is even and i = n − 3 then we obtain the configuration depicted in Fig. 12(a). Lines 43–48 construct the tree and configuration depicted in Fig. 12(b) when n is even. Since all configurations c0 attaining τ(c0) = n − 3 presented in Theorem 8 are generated by the algorithm and no other configuration can be obtained, the corollary follows.

Algorithm 1: Generate all trees T with n ≥ 4 and corresponding initial configurations leading to \( \tau_2(T) = n - 3 \).

**Input:** Number n of vertices.

**Output:** Trees \( T_j \) each with the corresponding initial configuration \( C_j \) such that \( \tau(C_j) = n - 3 \).

```
1 V ← {v1, v2, ..., vn};
2 E ← ∅;
3 j ← 1;
4 for i ← 1 to n − 2 do // Lines 4 -- 7 result in tree \( T_1 \).
5     E ← E \cup (v1, vn+1);
6     if i = n − 2 then
7         E ← E \cup (v1, vn+2);
8     for i ← 1 to n do // Lines 8 -- 12 result on configuration \( C_1 \).
9         if i is odd then
10             Cq(vi) ← 1;
11         else
12             Cq(vi) ← 0;
13         Tj ← G(V, E);
14     Cj ← Cq;
15     j ← j + 1;
16     if n = 4 or n = 5 then // Lines 17 -- 21 result on Figure 5b and 8a for n = 4 and n = 5, respectively.
17         Cq(v4) ← 1 − Cq(v5);
18     if n = 5 then
19         Cq(v3) ← 1 − Cq(v4);
20     Tj ← G(V, E);
21     Cj ← Cq;
22     j ← j + 1;
23     E ← E \cup (v1, v_{n−2});
24     if n = 5 then
25         Cq(v3) ← Cq(v4);
26         Cq(v4) ← 1 − Cq(v5);
27     Tj ← G(V, E);
28     Cj ← Cq;
29     if n = 4 then // Lines 31 -- 34 yield the tree and configuration given by Figure 5e.
30         j ← j + 1;
31         Cq(v4) ← 1 − Cq(v5);
32         Tj ← G(V, E);
33     Cj ← Cq;
34     else // Lines 36 -- 42 obtain Figure 11a and 11b for n even and odd, respectively.
35         if i is odd then // For n even and i = n − 3, Figure 12a is obtained.
36             E ← E \cup (v1, v_{i−1});
37             E ← E \cup (v_{i−1}, v_{i+1});
38             Tj ← G(V, E);
39             Cj ← Cq;
40             j ← j + 1;
41         if i = n − 3 and n is even then // Lines 43 -- 48 obtain Figure 12b for n even.
42             E ← E \cup (v_{i+1}, v_{i+3});
43             E ← E \cup (v_{i+1}, v_{i+3});
44             Tj ← G(V, E);
45             Cj ← Cq;
46             j ← j + 1;
```

5. NP-Completeness of f-conversion set

In this section, we prove the NP-completeness for bipartite graphs of f-CONVERSION SET. The proof is a reduction through a restriction of 3SAT, where each clause has 2 or 3 literals and each variable occurs in at most 3 clauses [18]. The formal definition is as follows:

**AM3-SAT [18]**

**Input:** A set \( C = \{c_1, c_2, ..., c_m\} \) of clauses, each one composed by Boolean variables of a set \( X = \{x_1, x_2, ..., x_n\} \), such that each clause has at most three literals, each variable appears at most three times, and each literal occurs at most twice.

**Question:** Is there a truth assignment to the variables in X satisfying C?
Note that if a literal $\ell$ occurs in three clauses, then we can consider the clauses containing $\ell$ as satisfied and thus remove them from $c$. This justifies the difference in the definition of AM3-SAT given in [18].

Given a graph $G$ and an integer $q > 0$, Dourado et al. [12] proved that determining if $r_q(G) \leq q$ is NP-hard. Denoting the neighborhood of $v$ in a vertex set $X$ by $N_X(v) = N(v) \cap X$, we say that two disjoint vertex subsets $A$ and $B$ are a bad pair of an $f$-reversible process $R_f(G, c_0)$, at any time step, if:

- each vertex of $A$ has a different state from the vertices of $B$;
- $|N_B(v)| = f(v)$, for all $v \in A$;
- $|N_A(u)| = f(u)$, for all $u \in B$.

As proved by Dourado et al. [12], every uplifting $f$-reversible process cannot contain any bad pair. Next we provide a useful lemma to prove the NP-completeness of $f$-CONVERSION SET.

**Lemma 11.** If $R_f(G, c_0)$ is uplifting and, for some $v \in V(G)$, $f(v) = d(v)$ and $f(u) = 1$ for every $u \in N(v)$, then $c_t(v) = 1$ and $c_t(u) = 1$ for some $u \in N(v)$ and every $t \geq 0$.

**Proof.** Suppose by contradiction that the lemma is false. Since $R_f(G, c_0)$ is uplifting, if there exists a time step $t_0 \geq 0$ such that $c_{t_0}(v) = 0$, then there exists a time step $t > t_0$ in which $v$ changes its state. Suppose $t$ to be the smallest. In this case, $c_{t-1}(v) = 0$ and $c_{t-1}(u) = 1$ for all $u \in N(v)$. Hence, the sets $A = \{v\}$ and $B = N(v)$ are a bad pair of $R_f(G, c_0)$, a contradiction. The argument for the existence of a vertex in $N(v)$ which has state 1 throughout the entire process is analogous.

**Theorem 12.** $f$-CONVERSION SET is NP-complete for $G$ bipartite with maximum degree 3 and $\text{Im}(f) = \{1, 2, 3\}$.

**Proof.** $f$-CONVERSION SET is in NP because, given an integer $q > 0$, we can execute the process from $c_0$, that has $q$ vertices with state 1, to $c_{t_0}(G)$. Moreover, we can obtain $c_{t+1}$ from $c_t$ in $O(n + m)$ for all $t \geq 0$. Hence, by Case 1 of Theorem 7, we can execute the whole process in $O((n + m)(S_f - n))$.

Let $F$ be an instance of AM3-SAT with $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses $c_1, c_2, \ldots, c_m$. We construct a graph $G$ as follows. For each variable $x_i$, $G$ contains three vertices denoted by $w_i, \overline{w}_i, a_i$, $1 \leq i \leq n$. For each clause $c_j$, $G$ contains one vertex denoted by $u_j$, $1 \leq j \leq m$. Each variable $x_i$ is adjacent only to $w_i$ and $\overline{w}_i$, $1 \leq i \leq n$. Moreover, edges $w_i^c u_j, \overline{w}_i \in \{w_i, \overline{w}_i\}$, are added if and only if $w_i^c \in c_j$ in $F$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

Let $A = \bigcup a_i, W = \bigcup \{w_i, \overline{w}_i\}$, and $U = \bigcup u_i$ denote the sets of vertices called auxiliary, literal, and clause vertices, respectively. Note that each vertex of $A$ has degree 2, and since each clause has 2 or 3 literals and each literal occurs in 1 or 2 clauses, the maximum degree of $G$ is $3$. Moreover, $A \cup U$ and $W$ are a bipartition of $V(G)$, where each part induces an independent set. Therefore $G$ is a bipartite graph with maximum degree 3. To complete the reduction, we assign $f(w) = 1$, for each $w \in W$ and $f(v) = d(v)$, for all $v \in V(G) \setminus \{W\}$. Fig. 13 depicts the graph, initial configuration, and threshold values obtained from $F = (x_1 \lor \overline{x}_2) \land (\overline{x}_1 \lor x_2 \lor x_3) \lor (x_2 \lor \overline{x}_3 \lor x_4) \land (\overline{x}_3 \lor \overline{x}_4)$.

Now, we prove that $F$ is satisfiable if and only if $r_f(G) = m + 2n$. By Lemma 11, we get that all vertices in $A \cup U$ and at least one of $w_i$ or $\overline{w}_i$ must have state 1 in $c_0$, for all $1 \leq i \leq n$. Thus $r_f(G) \geq m + 2n$. We can consider that $c_0(w^i) = 1$ if and only if $w^i$ has true value in $F$, $w^i \in \{w_i, \overline{w}_i\}$.

If $F$ is satisfiable, then every clause vertex must have at least one neighbor with initial state 1 and exactly one literal vertex of $\{w_i, \overline{w}_i\}$. Since its initial state equal to 1, for all $1 \leq i \leq n$. Moreover, since $r_f(G) \geq m + 2n$, then all auxiliary and clause vertices have initial state equal to 1, for any conversion set of $G$. Therefore $c_t(v) = 1$ for every $v \in V(G)$ and thus $r_f(G) = m + 2n$.

Now, if $r_f(G) = m + 2n$, then for each pair of vertices $w_i$ and $\overline{w}_i$, only one of them must have its initial state equal to 1, since all vertices $v \in A \cup U$ have their initial state equal to 1. Moreover, since such a set is a conversion set, each clause vertex $u_j$, $1 \leq j \leq m$, is adjacent to a literal vertex with initial state 1, by Lemma 11. Hence, all clauses are satisfied.

We also analyze the complexity of finding a minimum $f$-conversion set on general graphs when $f(v) = d(v)$, for all $v \in V(G)$. In this case, if an edge is such that the states of its ends are the same then these vertices will never change their states, showing that $r_f(G) \geq \beta(G)$. We establish the following result relating $r_f(G)$ and $\beta(G)$ in this situation.

**Theorem 13.** Let $R_f(G, c_0)$ be such that $f(v) = d(v)$ for all $v \in V(G). If a minimum vertex cover of $G$ exists that is not an independent set then $r_f(G) = \beta(G)$. Otherwise, $r_f(G) = \beta(G) + 1$.

**Proof.** Let $C$ be a minimum vertex cover of $G$ and let us consider that a vertex $v$ belongs to $C$ if and only if $c_0(v) = 1$. Clearly $r_f(G) \geq \beta(G)$, otherwise, there must exist at least one edge whose ends have state zero in $c_0$, meaning that the process does not uplift. Let $l_t$ be the set of vertices $v$ such that $c_t(v) = 0$, for all $0 \leq t \leq r(c_0) - 1$. Hence, $l_t$ must induce an independent set in $G$ and $r_f(G)$ is obtained taking $l_t$ to be the greatest possible.

Suppose that $C$ does not induce an independent set in $G$. Thus there is at least one edge $e = (u, v)$ in $G[C]$ and, hence, $u$ and $v$ remain with state 1 forever. Suppose by contradiction that the process is not uplifting. Let us also suppose that there exists an edge $e'$ whose ends have state zero in $c_{t(c_0)}$. Furthermore, let $t > 0$ be the first time step in which such an edge $e'$
appears. Since at time $t - 1$ there were not any such edges, all vertices $z \in I_{t-1}$ change their states. Thus, $I_t$ induces an independent set, a contradiction. Hence, if $p(c_0) = 1$, then the process is uplifting.

Thus, suppose that $p(c_0) = 2$. In this case, there must exist a bad pair $A$ and $B$. Hence $A$ and $B$ induce independent sets, and since no edge whose ends have state zero is formed through the process, we get that $\tau(c_0) = 0$ and $G[A \cup B]$ is a connected component which does not contain $u$ and $v$, a contradiction.

Finally, suppose that any minimum vertex cover $C$ of $G$ induces an independent set. Thus, we need more vertices than $\beta(G)$ for the uplifting. It is enough to show that we need only one vertex more than a minimum covering. Let $C' \cup \{v'\}$ be the vertex subset of $G$ with initial state 1, for some $v' \in V(G) \setminus C$. Hence, since $G$ is connected, there is an edge in $G[C' \cup \{v'\}]$ and the same argument of the previous case works now, completing the proof. □

**Corollary 14.** If $G$ is not bipartite and $f(v) = d(v)$ for all $v \in G$, then $r_f(G) = \beta(G)$.

**Proof.** Since there is no minimum vertex cover of $G$ inducing an independent set, the corollary follows from Theorem 13. □

6. Conclusions

We have presented a potential function to aid in the study of the periodic behavior of $f$-reversible processes and proved that they are particular threshold networks. We have shown that this function is nondecreasing and, by using this fact, we have presented a new upper bound on the transient length based on $f$, $n$ and $m$. Moreover, the potential function gave the necessary interpretation in other to obtain all initial configurations on 2-reversible processes on trees reaching at least $n - 3$ time steps, where its value is tight when $n \geq 4$. Determining such configurations based on only $G$ and $f$ is not trivial work. This can be seen in the work of Oliveira, Barbosa and Protti [36], in which determining if there exists a predecessor configuration of a given one is $NP$-complete even for bipartite graphs. In this way, we left two open problems regarding the maximum transient length:
- To characterize all initial configurations reaching the maximum transient length in terms of only $G$ and $f$;
- To count the number of such configurations.

We also have studied the problem of determining the minimum size $r_f(G)$ of a vertex subset that allows $f$-reversible processes to uplift. We have shown that determining if $r_f(G)$ is at most a constant $q > 0$ is $NP$-complete for bipartite graphs with maximum degree 3 and $\text{Im}(f) = \{1, 2, 3\}$. We also have proved that $\beta(G) \leq r_f(G) \leq \beta(G) + 1$, when $f(v) = d(v)$ for all vertices. In fact, determining $r_f(G)$ seems to be no trivial even for simple graph classes as paths and cycles [12]. In addition to this open problem, we propose the following:
- To determine a lower and an upper bound on $r_f(G)$;
- To find an approximation algorithm to compute $r_f(G)$.

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**References**


